FIND A BLOCKY FIT TO A GIVEN SIGNAL

Given a signal $d(t)$, here we construct a blocky version of it $m(t)$. In the ℓ_2 world, this is simply a low-pass filter. In the ℓ_1 world, the blocky signal would have a controllable amount of "total variation". What will we have in the HPF world?

$$0 \approx_{h_d} m(t) - d(t) = q(t) \tag{1}$$

$$0 \approx_{h_m} \frac{d}{dt} m(t) \tag{2}$$

By denoting the residual as q instead of r we are implying that it as well as m and d are in dimensionless form, i.e. the whole regression has been scaled by a gain that brings components to the neighborhood of the ℓ_1/ℓ_2 threshold. But wait, there may be a different gain in data space and model space. Do we need to distinguish $\approx_{h_d}$ from $\approx_{h_m}$? Maybe not. Precondition with causal integration $\mathbf{C}$ where $\mathbf{m} = \mathbf{Cn}$.

$$0 \approx_{h_d} \mathbf{Cn} - \mathbf{d} = \mathbf{q} \tag{3}$$

$$0 \approx_{h_m} \mathbf{n} \tag{4}$$

May as well have a good starting model so let $\mathbf{D} = \mathbf{C}^{-1}$ and start with $\mathbf{n}_0 = \mathbf{Dh}'(\mathbf{d})$. Notice when $h_d = \ell_2$, then $\mathbf{n}_0 = \mathbf{Dd}$, so the initial $\mathbf{q} = \mathbf{0}$ which is a good start.

We minimize

$$\sum_i h(r_i) + \epsilon\, h(n_i) \tag{5}$$

The direction to go is:

$$\Delta\mathbf{n} = \mathbf{C}^{\mathrm{T}}\mathbf{h}_d'(\mathbf{q}) + \epsilon\, \mathbf{h}_m'(\mathbf{n}) \tag{6}$$

Let us do steepest descent. Looks like the next step is to seek α

$$h(\alpha) = h(\mathbf{q} + \alpha\Delta\mathbf{n}) + \epsilon\, h(\mathbf{n} + \alpha\Delta\mathbf{n}) \tag{7}$$

by the usual Newton approach. The lead term and the epsilon term behave similarly. To avoid clutter, we omit the epsilon term for several steps, then pick it up again. We have a Taylor series.

$$\bar{h}(\alpha) = \frac{1}{N} \sum_i h_i + \alpha\Delta q_i h_i' + (\alpha\Delta q_i)^2 h_i''/2 \tag{8}$$

To find α, set $d\bar{h}/d\alpha = 0$. Then solve for α.

$$0 = \frac{d\bar{h}}{d\alpha} = \sum_i \Delta q_i h_i' + \alpha\, (\Delta q_i)^2 h_i'' \tag{9}$$

$$\alpha = -\frac{\sum_i \Delta q_i\, h_i'}{\sum_i (\Delta q_i)^2\, h_i''} \tag{10}$$

Now recall the epsilon term

$$\alpha \quad = \quad - \ \frac{\sum_i \Delta q_i h'_i \ + \ \epsilon \sum_i \Delta n_i h'_i}{\sum_i (\Delta q_i)^2 h''_i \ + \ \epsilon \sum_i (\Delta n_i)^2 h''_i} \tag{11}$$

I need to ask myself if I need separate h' for data space and model space. If they are both non-dimensional, they may be the same and the sums may be merged. Finally, update the vectors.

$$\mathbf{n} \quad \leftarrow \quad \mathbf{n} + \alpha \Delta \mathbf{n} \tag{12}$$

$$\mathbf{q} \quad \leftarrow \quad \mathbf{C}\mathbf{n} - \mathbf{d} \tag{13}$$

The Newton steps may be iterated, but need not be, because they are embedded in a larger iteration.

Then for the interesting part. The investigator must choose epsilon and the two HPF gains, or has the unit-free form helped us by scaling to a single gain? I could use help gathering my wits. I don't understand the difference between playing with gain and playing with epsilon.

This seems like a nice tool. The Galilee water surface is one test case (start from the residual of the model having ship tracks). I wonder if there are others that would interest anybody besides me?

What's boring about this project? No data. I know how parameters will affect results. What's exciting about this project? No data. I don't know how parameters will affect results. People admire Occam's razor, but have no general tools for it. This is one. This tool is mostly safe because of convexity, but there is a little danger because of the Taylor series. When does that arise? When is convergence rapid or slow? Is conjugate direction a worthwhile upgrade? When? It might interface well with my other project, Jensen inequalities to choose the best gain.