VARIATIONAL CONSTRAINTS FOR ELECTRICAL IMPEDANCE TOMOGRAPHY

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ABSTRACT

The task of electrical impedance tomography is to invert electrical boundary measurements for the conductivity distribution of a body. This inverse problem can be formulated so the primary data are the measured powers dissipated across injection electrodes. Then, since these powers are minima of the pertinent variational principles (Dirichlet's or Thomson's principle), feasibility constraints can be formulated for the nonlinear inversion problem. These constraints may also be used to stabilize iterative reconstruction algorithms where voltage differences across other electrodes are the primary data and the measured powers are treated only as secondary data. When the powers may be measured accurately, the existence of these dual variational principles implies that an exact solution (if any) must lie at a point of intersection of the two feasibility boundaries.

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Inverse problems continue to play a central role in many fields of science and engineering. For example, biomedical imaging using x-rays or ultrasound is now common practice. Similarly, geophysical imaging using seismic methods is also common. Each imaging method tends to reconstruct the contrasts in a single physical property, *i.e.*, the density for x-rays and the sound (or seismic) wave speed for ultrasound (or seismic tomography). If we are lucky, the contrasts are low so linear inversion methods may apply. If we are unlucky, the contrasts in the physical property we wish to image are high so nonlinear inversion methods must be developed.

For applications to nonlinear traveltime (seismic) tomography, it had been previously discovered[1] that Fermat's principle implies the existence of a set of constraints (called feasibility constraints) which may be used to stabilize practical algorithms for inverting traveltime data for wave speed distribution. The main result of the present work is to point out that this previous result is just a special case of a much more general result. When an inverse problem can be formulated so the data are minima of one of the variational problems of mathematical physics, feasibility constraints can be found for the nonlinear inversion problem and the corresponding reconstruction algorithm can be stabilized.

The example we will consider is electrical impedance tomography. This technique attempts to image the electrical impedance (or just the conductivity) distribution inside a body using electrical measurements on its boundary. The method has been used successfully in both biomedical[2] and geophysical[3,4] applications, but the analysis of optimal reconstruction algorithms is still progressing.[5,6] The most common application in both biomedical and geophysical applications is monitoring the influx or eflux of a conducting fluid (such as blood in the heart or brain, or brine in a porous rock) through the body whose conductivity is being imaged. Compared with radiological methods in diagnostic medicine, this approach does not have such high resolving power, but it is comparatively inexpensive and uses no ionizing radiation so it is suitable for applications requiring continuous monitoring of patients, and practical for both laboratory and field measurements monitoring fluid flow through rocks and soils.

We wish to avoid discussing the details of specific experimental arrangements and reconstruction algorithms here (since that is probably only of interest to specialists), so we will simply describe the fundamental ideas. Although other methods are in use, the data for electrical impedance tomography have most often (especially in geophysical applications) been gathered (see Fig. 1) by injecting a measured current between two electrodes while simultaneously measuring the voltage differences between pairs of other electrodes placed around the boundary of the body being imaged. This process is then repeated, injecting current between all possible (generally adjacent) pairs of electrodes and recording the set of voltage differences for each injection pair. Note that this data set (normally) does not include the voltage difference across the injection electrodes.[7]

Then, to summarize the reconstruction method: First, guess a conductivity distribution (*e.g.*, a constant) and do forward modeling to determine predicted voltage differences across the measurement electrodes for each injection pair. Second, the predicted and measured voltage differences are compared and these differences are used (in a different way specific to each algorithm) to determine an update for the conductivity distribution. Then, the process is repeated with the updated conductivity as the new guess. The method may stop after a single iteration in some methods or it may proceed for a large number of iterations in other methods. A convergence criterion may be used to terminate the algorithm based on the agreement between the predicted and measured voltage differences, or a limit may be set for the total number of iterations.

Uniqueness results have been obtained by Kohn and Vogelius[8] for piecewise analytic conductivities and by Sylvester and Uhlmann[9] for smooth conductivities assuming complete sets of current and voltage measurement pairs are available. These papers have shown that an unique solution is obtained either by inverting the data just described (but including the voltages across the injection electrodes to make a complete set) or by inverting an apparently more limited set of data. This alternative data set is obtained by measuring *only* the voltages across the injection electrodes, or equivalently the power dissipated during current injection; a complete set of such power measurements is required. It is important to recall that the normal data set described previously does *not* include the power dissipation measurements. The practical reason for this gap in the data is that the voltages across the injection electrodes are difficult to measure reliably – at least partially because a substantial contact impedance develops at the interface between the body and the electrodes when large currents are injected. Since the contact impedance is a function of current magnitude, the data collection process can itself be nonlinear. The effects of contact impedance can be eliminated to some extent by using electrodes with large surface areas;[7] however, this approach is not possible in all applications. So one difficulty that we may have to live with in practice is a substantial inaccuracy in the power dissipation measurements –

when they are available.

Having given the traditional reasons for *not* considering power dissipation measurements, we will now develop a reconstruction method based on this data. Dirichlet's principle states that, given conductivity distribution $\sigma(\vec{x})$ and potential distribution $\phi(\vec{x})$, the power dissipation p_i realized for the i-th current injection configuration is the one that minimizes the integral $\int \sigma |\nabla\phi|^2\, d^3x$, or

$$p_i(\sigma) = \min_{\phi_i^{(trial)}} \int \sigma |\nabla \phi_i^{(trial)}|^2\, d^3x \equiv \int \sigma |\nabla \phi_i^*[\sigma]|^2\, d^3x. \tag{1}$$

The trial potential field for the i-th injection pair is $\phi_i^{(trial)}(\vec{x})$ while the actual potential field $\phi_i^*[\sigma](\vec{x})$ is the one that satisfies Poisson's equation $\nabla\cdot(\sigma\nabla\phi_i^*) = 0$ within the body. Furthermore, if we define the power dissipation associated with the trial potential $\phi_i^{(trial)}$ by the integral

$$\tilde{p}_i^{(\phi_i)}(\sigma) \equiv \int \sigma |\nabla \phi_i^{(trial)}|^2\, d^3x, \tag{2}$$

then the measured powers P_i must satisfy

$$P_i = p_i(\sigma_0) \le \tilde{p}_i^{(\phi_i)}(\sigma_0), \tag{3}$$

if $\sigma_0(\vec{x})$ is the true conductivity distribution. The set of constraints which may be inferred from (3) for all i and all trial potential fields will be called the feasibility (or variational) constraints for electrical impedance tomography.

To see what these constraints are, we can follow the same reasoning used to obtain feasibility constraints in the seismic tomography problem[1]: If, at some stage of the inversion algorithm, we reconstruct a conductivity $\sigma(\vec{x})$ such that

$$\tilde{p}_i^{(\phi_i)}(\sigma) < P_i \tag{4}$$

(*i.e.*, that violates the feasibility constraints (3) for any injection pair i and/or any trial potential field), then we know that such a conductivity distribution is not feasible – it is inconsistent with the data. Conductivity distributions satisfying (4) based on the variational principle (1) belong to the Dirichlet infeasibility region. The corresponding result in traveltime inversion produced the Fermat infeasibility region.

Somewhat surprisingly, there follows a close analogy between the electrical impedance tomography problem and the seismic traveltime inversion problem[10]

– even though the physics of these two inverse problems is completely different: one being a d.c. electrical conduction problem and the other a wave propagation problem. The connection arises from the similarity of the variational formulations of these two problems. To see the connection, consider the first arrival traveltime as a function of the wave slowness $s(\vec{x})$. Fermat's principle states that

$$t_i(s) = \min_{\{paths\}} \int s\, dl_i^{(path)} \equiv \int s\, dl_i^*[s], \tag{5}$$

where $l_i^{(path)}$ is the arc length along any connected path between the source and receiver and where $l_i^*[s]$ is an arc length along a ray path that minimizes the integral of the traveltime for the i-th path and the wave slowness s. We will also define a trial traveltime by

$$\tau_i^{(path)}(s) \equiv \int s\, dl_i^{(path)}. \tag{6}$$

Then, the correspondence between the electrical impedance tomography problem and the first arrival traveltime inversion problem is determined by:

$$\sigma \rightarrow s,$$

$$p_i(\sigma) \rightarrow t_i(s),$$

$$\tilde{p}_i^{(\phi_i)}(\sigma) \rightarrow \tau_i^{(path)},$$

$$|\nabla \phi_i^{(trial)}|^2\, d^3x \rightarrow dl_i^{(path)},$$

$$|\nabla \phi_i^*[\sigma]|^2\, d^3x \rightarrow dl_i^*[s],$$

$$P_i \rightarrow T_i.$$

The significance of this correspondence lies in its impact on algorithmic structure of inversion codes: Programming from the top down, these two problems look essentially identical. Of course, from the bottom up, they are very different because the routines required to compute the trial potential fields and the trial ray paths are completely different. Nevertheless, the top down equivalence of these two problems suggests that algorithms that have been found to be successful for one of the problems will also work for the other as well. Berryman[1] has developed a method of stabilizing iterative traveltime tomography algorithms with least-square error as objective function by using the minimum number of Fermat feasibility violations as a figure of merit to help choose an underrelaxation parameter to modify the step size of the model correction.

Since it has also been shown[10] that in general least-squares methods produce models in the infeasible region (unless an exact solution of the inversion problem has been obtained), a criterion for underrelaxation based on minimum feasibility violation number is expected to be applicable to any inversion scheme that has this same variational structure.

For electrical impedance tomography, this idea has been tested extensively using both synthetic data (with and without added noise) and real laboratory data with conducting and insulating anomalies in a small tank filled with brine. The tests have shown that the several different inversion algorithms for electrical impedance tomography can all be stabilized in the presence of noise by using these feasibility constraints. For example, when the inverse problem is formulated so the primary data are the measured powers dissipated across the injection electrodes, a least-squares fitting procedure based on comparing the predicted and measured powers may be used; then, the feasibility constraints determine the size of the underrelaxation parameter (modifying the size of the model update) required to stabilize the resulting iterative algorithm as in the traveltime problem.[1] Alternatively, when the powers cannot be measured very accurately and it is therefore undesirable to have the powers play the role of primary data, the constraints may be used to stabilize any iterative reconstruction method while the voltage differences across other measurement electrodes are the primary data. General arguments based on the convexity of the Dirichlet feasibility region and the fact that the correction step tends to be tangent to the feasibility surface show that inaccuracies in the power measurements are relatively unimportant in this context. Even moderately large inaccuracies (5 to 10%) in the power measurements are negligible in this approach because the feasibility constraints are only used to determine the underrelaxation parameter (*i.e.*, a multiplier for the correction step) not the direction of the correction step itself.

Another remarkable fact is this: For the electrical impedance tomography problem, there are two different sets of feasibility constraints. One set is for Dirichlet's principle as presented here. The other is for its dual (Thomson's principle)

$$P_i \leq \int |\vec{J}_i|^2/\sigma \, d^3x, \tag{7}$$

where $\vec{J}_i(\vec{x})$ is a trial current distribution for the i-th current injection pair that satisfies the continuity equation $\nabla \cdot \vec{J}_i = 0$. The current distribution $\vec{J}_i$ in (7) and the gradient of the potential ϕ_i in (2) are generally unrelated except that when the minimum of both variational functionals is attained, then $\vec{J}_i^* = -\sigma\nabla\phi_i^*[\sigma](\vec{x})$. The existence of the dual variational principle is a general result whenever the pertinent variational principles are true minimum principles. (By contrast, Fermat's principle is not in this class, since it is only a stationary principle; the analysis given previously for Fermat's principle remains valid however since the first arrival traveltime data are truly minima.)

The existence of dual variational principles for this inversion problem suggests that the quality of a reconstructed conductivity distribution can be estimated. To see why this is so, discretize (3) and (7) according to

$$P_i \leq \sum_{j=1}^{n} \sigma_j \int_{\Omega_j} |\nabla\phi_i^{(trial)}|^2 \, d^3x \tag{8}$$

and

$$P_i \leq \sum_{j=1}^{n} \frac{1}{\sigma_j} \int_{\Omega_j} |\vec{J}_i|^2 \, d^3x, \tag{9}$$

where we now model the conductivity distribution in the body by dividing it into n cells labelled by j of volume Ω_j and constant conductivity σ_j. The limiting equalities in (8) determine a set of hyperplanes in the vector space for the conductivity vector $\bar{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, while (9) determines another set of hyperplanes in the resistivity vector space $\bar{\rho} = (\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_n})$.

To provide a simplified example, suppose there are only two cells in the reconstruction problem; then,

$$P_i \leq \sigma_1 A_i + \sigma_2 B_i \qquad (Dirichlet) \tag{10}$$

and

$$P_i \leq \frac{C_i}{\sigma_1} + \frac{D_i}{\sigma_2}. \qquad (Thomson) \tag{11}$$

The constants A_i, B_i, C_i, and D_i are all positive, dependent on the injection pair i and on the trial field or current distribution. Including the positivity and finiteness constraints on conductivity, the dual feasibility conditions for σ_2 as a function of σ_1 become

$$\sigma_2 \geq \begin{cases} \dfrac{P_i - \sigma_1 A_i}{B_i} & \text{for } \sigma_1 \leq \dfrac{P_i}{A_i} \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

and

$$\sigma_2 \leq \begin{cases} \dfrac{\sigma_1 D_i}{\sigma_1 P_i - C_i} & \text{for } \sigma_1 \geq \dfrac{C_i}{P_i} \\ \infty & \text{otherwise.} \end{cases} \tag{13}$$

The complete set of inequalities implied by (12) and (13) is as large as the number of injection pairs m $(1 \leq i \leq m)$ times the number of trial fields and currents considered.

Figure 2 shows the kind of picture that emerges from this type of analysis when the constraints for a single power measurement are considered. Then, the two variational principles have essentially disjoint infeasibility regions, while the dual feasibility region is sandwiched between. Thus, a reconstructed conductivity $\sigma(\vec{x})$ may be compared against the two feasibility boundaries and error bars on the accuracy of a reconstruction may therefore be obtained. For example, if all cell conductivities save one are held fixed, then the feasibility boundaries determine a definite range of values for the one allowed to vary. To our knowledge, this approach is the first example of a nonlinear inversion problem for which error bars of this type may be obtained for the reconstructed model. Furthermore, it is clear that, if an exact solution to the inversion problem exists, this solution must lie on both the Dirichlet and the Thomson feasibility boundaries. Thus, any points of intersection of these two boundaries might play a special role in other formulations of the reconstruction problem.

The picture becomes considerably more complex when more than one power measurement is used and these powers are subject to inaccuracies and therefore may provide an inconsistent set of data. Then, for each power measurement, we have a dual feasibility region similar to the one pictured in Fig. 2. In the absence of measurement errors, a solution of the inversion problem must lie in the intersection of all the resulting dual feasibility regions. However, the inconsistencies introduced by the measurment errors may result in an empty intersection set. In this situation (which is actually common in practice), we introduce the concept of the combined (or dual) feasibility violation number; then, the optimum solution of this inversion problem must lie in a region of model space where the minimum total number of feasibility violations occurs (considering all constraints from both variational principles). This approach is the one that has been used in developing our reconstruction algorithms and it has been found to provide the desired stabilizing mechanism needed for robust nonlinear inversion in the presence of noise.

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FIGURE CAPTIONS

Fig. 1. Data for electrical impedance tomography is generally gathered by injecting a current I between two electrodes while measuring the voltage differences V between other pairs of electrodes.

Fig. 2. Illustrating the dual feasibility region for discrete conductivity (σ_1, σ_2) between the Dirichlet and Thomson regions of infeasibility for a single power measurement.