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Single-scattering approximations for coefficients in Biot's equations of poroelasticity

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Abstract

Three single-scattering approximations for coefficients in Biot's equations of poroelasticity are considered: the average T-matrix approximation (ATA), the coherent potential approximation (CPA), and the differential effective medium (DEM). The scattering coefficients used here are exact results obtained previously for scattering from a spherical inclusion of one Biot material imbedded in another otherwise homogeneous Biot material. The CPA has been shown previously to guarantee that, if the coefficients for the scattering materials satisfy Gassmann's equation, then the effective coefficients for the composite medium satisfy Brown and Korringa's generalization of Gassmann's equation. A collection of similar results is obtained here showing that the coefficients derived from ATA, CPA, or DEM all satisfy the required conditions for consistency. It is also shown that Gassmann's equation will result from any of these single-scattering approximations if the collection of scatterers includes only spheres of fluid and of a single type of elastic solid.
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1 Introduction

The equations for elastic wave propagation through fluid-saturated porous media were first derived by *Biot (1956)*. The main limitation to the use of these equations of poroelasticity for studying elastic waves in rocks is that relationships between the material properties of the constituents of the rock and coefficients appearing in the equations are still not well understood. The fundamental result of *Gassmann (1951)* shows how the coefficients depend on the compressibility of the saturating pore fluid. But Gassmann's result is actually of rather limited applicability, since its derivation assumes that the frame of the porous medium is composed of only one type of solid constituent. This situation virtually never applies to real materials like rocks, but does have limited use in explaining the behavior of artificial materials like sintered-glass-bead packs. *Brown and Korringa (1975)* have generalized Gassmann's result to show how the coefficients must depend on the fluid compressibility in general, but their result contains two new compressibilities that are unknown and have unknown dependence on the constituent's properties. Thus, the difficulty has been moved but not removed.

One reasonable way to attack this problem is to try to generalize methods that have been developed for estimating the material constants of elastic composites. The author made such an attempt some years ago *(Berryman, 1986)* using a method we call the coherent potential approximation (CPA). The present effort generalizes the earlier ideas so we now study three different approximations for the coefficients. First, we review the history of these methods in the context of random elastic composites.

Foldy (1945) developed a procedure for studying multiple scattering effects in inhomogeneous, isotropic, elastic media, and showed how the effective index of a medium could be related to the averaged forward scattering amplitudes. The theory is valid for strong scattering, but limited to low concentrations of scatterers. *Lax (1951)* generalized Foldy's method in a variety of ways, and introduced self-consistency so the approach was no longer limited to low concentrations. The average T-matrix approximation (ATA) was introduced for multiple scattering in quantum-mechanical systems by *Watson (1957)* and for electrical conduction in disordered alloys by *Korringa (1958)*. The coherent potential approximation (CPA) was introduced into solid state physics by *Soven (1967)* and *Taylor (1967)*. Based on Lax's multiple-scattering theory, the CPA provides a self-consistent means of computing the quantum mechanical densities of states for electrons and phonons in random alloys. *Velický, Kirkpatrick, and Ehrenreich (1968)* showed that the CPA neglects statistical correlations due to short range order and to multiple-scattering effects. However, in applications of interest, short range order is normally eliminated by hypothesis, while effects of multiple scattering, though always present, are effectively minimized by the self-consistent approach. The CPA is a single-scattering approximation, since it neglects these multiple-scattering contributions. Perturbation theory has shown that the CPA is exact for strong scattering with low concentrations of inclusions, or for weak scattering at arbitrary concentrations of inclusions. As an interpolation scheme for estimating effective properties of alloys, the CPA is very successful. A review of the extensive progress made on applications of the CPA to a variety of problems was presented by *Elliott, Krumhansl, and Leath (1974)*. *Gubernatis and Krumhansl (1975)* showed how to apply the CPA to estimate the material properties of randomly polycrystalline rock.

For classical percolation and conduction problems, an effective medium theory equivalent to the CPA was developed by *Bruggeman (1935)* and *Landauer (1951)*. A review of this approach was presented by *Kirkpatrick (1973)*. *Bruggeman (1935)* and *Roscoe (1952)* also developed another approach to the estimation problem which is now called the differential effective medium (DEM) method. In this approach, the approximation is found by successively computing the change in the effective constants after infinitesimal amounts of the inclusion phase are added to a homogeneous material whose effective constants are the same as those computed for the composite up to the current volume fraction. Whereas the CPA (for spherical inclusions) always gives formulas symmetric in the constituents, the DEM is necessarily asymmetric because the medium used to start the imbedding process (called the host) is always continuous *(Yonezawa and Cohen, 1983)*. This fundamental difference between the two approximations has advantages in some circumstances *(Sen, Scala, and Cohen, 1981; Sheng and Callegari, 1984; Sheng, 1990; Sheng, 1991)*. *Cleary, Chen, and Lee (1980)* review the earlier work on elastic DEM theory.

Berryman (1980; 1982) showed that the CPA for isotropic elastic composites produces estimates of the elastic constants that are always consistent with known rigorous bounds, including both the *Hashin-Shtrikman (1962; 1963)* and more restrictive bounds. *Milton (1985)* subsequently showed the CPA is actually a realizable model, and therefore it is guaranteed that the estimates of constants obtained this way always satisfy the rigorous bounds. *McLaughlin (1977)* showed that the DEM satisfies the Hashin-Shtrikman bounds. *Milton (1984)*, *Norris (1985)*, *Norris, Sheng, and Callegari (1985)*, and *Avellaneda (1987)* have shown that the DEM is also a realizable model.

The calculations presented here use single-scattering results for spherical inhomogeneities in a fluid-saturated porous medium *(Berryman, 1985)* to construct three different approximations for coefficients in Biot's equations of poroelasticity *(Biot, 1956)* describing elastic wave propagation through fluid-saturated porous media. Certain facts due to *Gassmann (1951)* and to *Brown and Korringa (1975)* are known about the general dependence of these coefficients on the properties of the saturating pore fluid. These relations may be used as stringent tests of the approximations. Until recently, neither rigorous results (such as formulas or bounds) nor approximation hierarchies were available for these coefficients. *Berryman and Milton (1991)* have obtained exact results for these coefficients when the composite has only two porous constituents. The results of all approximation schemes derived here have been compared to these exact formulas and are found to agree; they also give correct values when exact results are known for all the parameters.

Section 2 reviews single-scattering approximations for the bulk and shear moduli *(Berryman, 1980)*. Thought experiments used to justify these approximations are described and relevant details of the approximations are presented. It is also demonstrated that all three of these approximations are the same when the shear modulus is constant throughout the composite, and shown that they agree with the exact result for the bulk modulus derived for this limit by *Hill (1963)*. Section 3 introduces Biot's equations of poroelasticity and shows how the saturating fluid bulk modulus enters the coefficients. Gassmann's equation and the equations of Brown and Korringa are presented. Section 4 presents the exact results for the scattering coefficients of spherical scatterers imbedded in Biot material. Section 5 presents sufficient conditions for the various approximations to agree with either Gassmann's equation or Brown and Korringa's equations.

The main results of the paper are three single-scattering approximations for the coefficients in Biot's equations derived in Section 6. We find that all three of the approximations considered give estimates of coefficients that are in agreement with the general results of *Brown and Korringa (1975)* when the scatterers are spheres of Biot material satisfying *Gassmann's (1951)* equation. These results generalize and extend results published earlier by *Berryman (1986)* for the CPA. Section 7 shows when it is possible to derive estimates of coefficients consistent with Gassmann's equation starting with spherical scatterers that are either purely elastic or purely fluid. Section 8 concentrates on analytical comparisons among various approximations and values of the constituents' moduli. Rigorous bounds on results of these approximations are obtained. In addition, a rigorous bound on the actual effective constants is obtained and shown to provide a useful check on the consistency of experimental data. An analytical solution to the equations of DEM is found in a special case, and compared to the corresponding analytical solution for CPA in Section 9. Section 10 presents numerical comparisons among the approximations. Various examples are considered including clayey sandstone. Section 11 summarizes the conclusions of the paper. A brief mathematical appendix shows how the equations obtained using the differential effective medium may generally be analyzed to provide useful inequalities.

2 Approximations for K^* and μ^*

The three single-scattering approximations we will study in this paper are: the average T-matrix approximation (ATA), the coherent potential approximation (CPA), and the differential effedctive medium (DEM). To provide a simple review of the nature of these approximations and to compare and contrast these methods, we will start by deriving results for elastic scattering from spherical inclusions of radius a. The host medium has bulk modulus K and shear modulus μ. The spherical inclusion has bulk modulus K' and shear modulus μ'.

If an incident compressional wave has the form

$$\mathbf{u} = \hat{z}(A_0/ik)\exp i(kz - \omega t), \tag{1}$$

where k is the wavenumber, ω is the angular frequency, and A_0 is the amplitude of the dilatation ($\nabla \cdot \mathbf{u}$), then the radial component of the scattered wave field is

$$u_r = (ik)^{-1}\exp i(kr - \omega t)\left[B_0 - B_1\cos\theta - B_2(3\cos 2\theta + 1)/4\right]. \tag{2}$$

These scattering coefficients have been computed by *Ying and Truell (1956)* and by *Yamakawa (1962)*. The first scattering coefficient is

$$B_0 = -\frac{k^3 a^3 A_0}{3} \frac{K' - K}{K' + \frac{4}{3}\mu}. \tag{3}$$

The third coefficient is

$$B_2 = \frac{k^3 a^3 A_0}{3} \frac{10\mu}{3(K + 2\mu)} \frac{\mu' - \mu}{\mu' + F}, \tag{4}$$

where

$$F = (\mu/6)(9K + 8\mu)/(K + 2\mu). \tag{5}$$

The coefficient B_1 depends on material densities. Based on this coefficient, the methods used here always show that the effective density is just the average density [Berryman, 1980].

We will use the convention that effective constants are distinguished by the $*$ superscript (*e.g.*, K^* and μ^*). The results obtained in this section are valid for composites constructed either from solid grains or from porous (drained) grains.

2.1 Average T-matrix approximation

The average T-matrix approximation is justified by the following thought experiment (see Fig. 1): If the host medium is type-1, then the scattering coefficient for a spherical inclusion of radius a of the composite composed of type-1 and type-2 with volume fractions $v^{(1)}$ and $v^{(2)} = 1 - v^{(1)}$ is porportional to

$$B_0 \propto a^3 \frac{K^* - K^{(1)}}{K^* + \frac{4}{3}\mu^{(1)}}. \tag{6}$$

If the composite is actually composed of n spherical inclusions of type-2 of radii a_j for $j = 1, \ldots, n$ imbedded within the sphere of radius a, then (including only the single-scattering effects) the scattering coefficient must also satisfy

$$B_0 \propto \sum_{j=1}^{n} a_j^3 \frac{K^{(2)} - K^{(1)}}{K^{(2)} + \frac{4}{3}\mu^{(1)}}, \tag{7}$$

which is correct to $O(k^3)$. Higher order $O(k^5)$ terms due to the offset of the small scatterers from the origin may be properly neglected in this approximation. Equating (6) and (7) and noting that the volume fraction $v^{(2)}$ is given by

$$v^{(2)} = \frac{\sum_{j=1}^{n} a_j^3}{a^3}, \tag{8}$$

the resulting expression for the average T-matrix approximation for the bulk modulus K^*_{ATA} is

$$\frac{K^*_{ATA} - K^{(1)}}{K^*_{ATA} + \frac{4}{3}\mu^{(1)}} = v^{(2)} \frac{K^{(2)} - K^{(1)}}{K^{(2)} + \frac{4}{3}\mu^{(1)}}. \tag{9}$$

It is not hard to see that (9) can be generalized for multiple types of inclusions to

$$\frac{K^*_{ATA} - K^{(1)}}{K^*_{ATA} + \frac{4}{3}\mu^{(1)}} = \left\langle \frac{K(\mathbf{x}) - K^{(1)}}{K(\mathbf{x}) + \frac{4}{3}\mu^{(1)}} \right\rangle, \tag{10}$$

since the volume average $\langle\cdot\rangle$ over the spatial coordinate $\mathbf{x}$ produces the same weighting as in (9), while there is no contribution to (10) from regions where $K(\mathbf{x}) = K^{(1)}$. Since the right hand side of (10) is just the average of the scattering coefficient, we see now why this approximation is called the "average T-matrix approximation." Equation (10) can be rearranged into the convenient form

$$\frac{1}{K^*_{ATA} + \frac{4}{3}\mu^{(1)}} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}\mu^{(1)}} \right\rangle. \tag{11}$$

A similar computation based on B_2 gives the ATA for the shear modulus

$$\frac{1}{\mu^*_{ATA} + F^{(1)}} = \left\langle \frac{1}{\mu(\mathbf{x}) + F^{(1)}} \right\rangle, \tag{12}$$

where F is defined in (5). The formulas (11) and (12) are not coupled and therefore give explicit equations for the effective constants.

Note that the ATA is not one approximation, but many — one for each constituent of the composite. If the composite has multiple constituents, there is a distinct ATA obtained by treating each component in turn as the host. Furthermore, this approximation in the form (10) or (11) can be generalized again by replacing the host medium by any material — not necessarily one of the constituent materials. To make such a generalization useful, we suppose that the host medium (distinguished by a $\dagger$) has properties satisfying $K_{min} \le K^\dagger \le K_{max}$ and $\mu_{min} \le \mu^\dagger \le \mu_{max}$. Then, the generalized ATA takes the form

$$\frac{1}{K^*_{ATA} + \frac{4}{3}\mu^\dagger} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}\mu^\dagger} \right\rangle. \tag{13}$$

together with a corresponding generalization of (12). It will become clear that the CPA is just a self-consistent generalized ATA.

An example of the use of the ATA for elastic constants was presented by *Kuster and Toksöz (1974)*. It should also be noted that, when the shear modulus of the host $\mu^{(1)}$ is the smallest (largest), the ATA produces the same value of the bulk modulus as the Hashin-Shtrikman upper (lower) bound *(Hashin and Shtrikman, 1962; 1963)*.

2.2 Coherent potential approximation

The coherent potential approximation is justified by a slightly different thought experiment (see Fig. 2): If the host medium is the composite itself (*i.e.*, type-*), then the scattering coefficient for scattering from a spherical inclusion of type-i material is just

$$B_0 \propto a_i^3 \frac{K^{(i)} - K^*}{K^{(i)} + \frac{4}{3}\mu^*}. \tag{14}$$

The value of the composite bulk modulus may be found by treating K^* as a tunable quantity. We average the single-scattering contributions at infinity and adjust K^* until the sum vanishes:

$$\sum_{i=1}^{m} v^{(i)} \frac{K^{(i)} - K^*_{CPA}}{K^{(i)} + \frac{4}{3}\mu^*_{CPA}} = 0, \tag{15}$$

assuming m constituents whose volume fractions satisfy $\sum_{i=1}^{m} v^{(i)} = 1$. This equation can also be rearranged and written in terms of volume averages

$$\frac{1}{K^*_{CPA} + \frac{4}{3}\mu^*_{CPA}} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}\mu^*_{CPA}} \right\rangle. \tag{16}$$

Notice that this formula also depends on knowledge of the effective value of the shear modulus μ^*_{CPA}. This value is determined self-consistently using the analogous computation based on B_2 as described by Berryman.[1] The result is

$$\frac{1}{\mu^*_{CPA} + F^*_{CPA}} = \left\langle \frac{1}{\mu(\mathbf{x}) + F^*_{CPA}} \right\rangle, \tag{17}$$

where F is defined in (5). Note that (16) and (17) are coupled and therefore provide only implicit formulas for the effective constants. Normally these equations are solved by iteration.

The CPA produces a single formula in which all components are treated equally. Such a symmetric formula is appropriate when there is no one constituent acting as host to all the others. Other approximations of this type may be obtained by using the single scattering coefficients for non-spherical (often ellipsoidal) scatterers. The self-consistent approach of *Hill (1965)* and *Budiansky (1965)* for K^* and μ^* with spherical inclusions produces formulas identical to (16) and (17), but for other shapes of inclusions their approach is known to produce results different from the CPA. Since *Milton (1985)* has shown that the CPA is realizable (*i.e.*, in principle a model can be found that has the same value of the effective constants as that computed from the CPA), in many circumstances it is the CPA that is the preferred method of generating estimates.

2.3 Differential effective medium

The differential effective medium is constructed using another *gedanken* experiment (see Fig. 3): If there are only two constituents whose volume fractions are $x = v^{(1)}$ and $y = v^{(2)} = 1 - x$, then suppose we know the value of the effective bulk modulus $K^*_{DEM}(y)$ at one value of y. Treating $K^*_{DEM}(y)$ as the host medium and $K^*_{DEM}(y + dy)$ as the effective constant after a small proportion $dy/(1 - y)$ of the host has been replaced by spherical inclusions of type-2, we find

$$\frac{K^*_{DEM}(y + dy) - K^*_{DEM}(y)}{K^*_{DEM}(y + dy) + \frac{4}{3}\mu^*_{DEM}(y)} = \frac{dy}{(1 - y)} \frac{K^{(2)} - K^*_{DEM}(y)}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)}. \tag{18}$$

Since the host contains the volume fraction x of type-1 and y of type-2, on average a fraction $dy/(1 - y)$ of the host must be replaced by type-2 in order to change the overall fraction of type-2 to $y + dy$. Taking the limit $dy \to 0$ gives the result

$$(1 - y)\frac{d}{dy}\left[K^*_{DEM}(y)\right] = \frac{K^{(2)} - K^*_{DEM}(y)}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)} \left[K^*_{DEM}(y) + \frac{4}{3}\mu^*_{DEM}(y)\right], \tag{19}$$

where the initial host is pure type-1 so $K^*_{DEM}(0) = K^{(1)}$. The corresponding formula for the shear modulus is

$$(1-y)\frac{d}{dy}\left[\mu^*_{DEM}(y)\right] = \frac{\mu^{(2)} - \mu^*_{DEM}(y)}{\mu^{(2)} + F^*_{DEM}(y)}\left[\mu^*_{DEM}(y) + F^*_{DEM}(y)\right], \tag{20}$$

where F is given by (5). Note that (19) and (20) are coupled and must therefore be integrated simultaneously.

Like the ATA, the DEM formulas are not symmetrical in the constituents. Equations (19) and (20) have been derived assuming the host is of type-1 and the inclusion of type-2. A second distinct DEM is found by interchanging the roles of type-1 and type-2. The imbedding process guarantees that the host material remains connected in ths composite.

If the composite of interest contains more than two constituents (as is likely to be the case for most rocks), it is more difficult to include the additional components in the DEM than in ATA or CPA. The essential difference is that the result for the DEM depends on the path taken, *i.e.*, the order in which the constituents are added to the composite. *Norris (1985)* presented one method of generalizing (19) to multiple constituents. Another recent and relevant example was presented in *Sheng (1990)* and *Sheng (1991)*. Sheng's approach to modeling fluid-saturated rocks is first to compute the DEM for a fluid/cement mixture with the fluid as the host and the cement as the inclusion. Then, he computes the DEM for fluid/cement/grain with the fluid/cement composite as host and the grain as the inclusion. The point of this exercise is to construct a model where the fluid forms a connected phase and the frame is composed of cemented solid particles. We do something similar in Section 10.3, where we model a clayey sandstone. The main difference in philosophy is that we use formulas that are proven to have the correct dependence on the fluid bulk modulus, so we may effectively eliminate the fluid from further consideration and concentrate our efforts on the more difficult problem of evaluating the various moduli of the composite porous frame.

2.4 Comparison to Hill's exact formula

Hill (1963) has derived an exact formula for the effective bulk modulus of a composite when the shear modulus of all constituents is the same $\mu^{(1)} = \ldots = \mu^{(m)} = \mu^*$. His result is

$$\frac{1}{K^* + \frac{4}{3}\mu^*} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}\mu^*} \right\rangle. \tag{21}$$

It is easy to see (by inspection) that the ATA (11)-(12) and the CPA (16)-(17) reduce correctly to (21) when the μs are all equal.

When the shear moduli are constant, (20) shows that $\mu^*(y) = \mu^*$ is constant for the DEM. Thus, (19) may be rewritten in this limit as

$$(1-y)\frac{d}{dy}\left[\frac{1}{K^*_{DEM}(y) + \frac{4}{3}\mu^*}\right] = \frac{1}{K^{(2)} + \frac{4}{3}\mu^*} - \frac{1}{K^*_{DEM}(y) + \frac{4}{3}\mu^*}. \tag{22}$$

To check the agreement of the DEM (22) with Hill's result, take the y derivative of (21) in the case of only two constituents. Then, we find

$$\frac{d}{dy}\left[\frac{1}{K^*(y) + \frac{4}{3}\mu^*}\right] = \frac{1}{K^{(2)} + \frac{4}{3}\mu^*} - \frac{1}{K^{(1)} + \frac{4}{3}\mu^*}, \tag{23}$$

whereas substituting (21) into the right hand side of (22) (*i.e.*, $K^* \to K^*_{DEM}$) produces exactly the same expression as (23). *Norris (1985)* has also established the agreement of DEM with Hill's result. (Also see Appendix A of the present paper for another proof.) So we conclude that all three approximations become exact when the shear modulus of the composite is constant throughout.

An alternative formulation of the estimation problem can be based directly on Hill's formula (21). The goal of this new procedure is to arrive at an effective shear modulus μ^* for the composite by some means, and then use (21) as if the constituents are imbedded in a (composite) medium with homogeneous shear modulus $\mu(\mathbf{x}) = \mu^*$. In its simplest form, this approach is just another way of deriving the generalized ATA. Adding a self-consistency condition, it reproduces the CPA. An approximation similar to the DEM could also be obtained this way, but results would differ slightly since simultaneous integration for $K^*(y)$ and $\mu^*(y)$ is not required; in fact, treating (21) as a formula for $K^*(y)$, only a single integration is performed to find $\mu^*(y)$. This procedure produces a hybrid approximation, closer to the CPA than the usual DEM.

This discussion highlights the major difference between DEM and CPA. The DEM takes the fluctuations in the shear modulus into account approximately when calculating the estimate K^*, while ATA and CPA ignore effects due to fluctuations.

3 Equations of Poroelasticity

Consider a porous medium whose connected pore space is saturated with a single-phase viscous fluid. The fraction of the total volume occupied by the fluid is the porosity ϕ, which is assumed to be uniform on some appropriate length scale. The bulk modulus and density of the fluid are K_f and ρ_f, respectively. The bulk and shear moduli of the drained porous frame are K and μ. For simplicity, we assume the frame is composed of a single constituent whose bulk and shear moduli and density are K_m, μ_m, and ρ_m. The frame moduli may be measured on drained samples, or they may be estimated using one of a variety of methods from the theory of composites.

For long wavelength disturbances ($\lambda >> h$ where h is a typical pore size) propagating through such a porous medium, we define average values of the local displacements in the solid and also in the saturating fluid. The average displacement vector in the solid frame is $\mathbf{u}$, while that in the pore fluid is $\mathbf{u}_f$. A more useful way of quantifying the fluid displacement is to introduce the average displacement of the fluid relative to the frame which is $\mathbf{w} = \phi(\mathbf{u}_f - \mathbf{u})$. For small strains, the frame dilatation is

$$e = \nabla \cdot \mathbf{u}. \tag{24}$$

Similarly, the average fluid dilatation is

$$e_f = \nabla \cdot \mathbf{u}_f, \tag{25}$$

which includes fluid flow terms as well as dilatation. The increment of fluid content is defined by

$$\zeta = -\nabla \cdot \mathbf{w} = \phi(e - e_f). \tag{26}$$

With these definitions, *Biot (1956)* introduces a quadratic strain-energy functional of the independent variables e and ζ for an isotropic, linear porous medium

$$2E = He^2 - 2Ce\zeta + M\zeta^2 - 4\mu I_2, \tag{27}$$

where I_2 is the second strain invariant *(Berryman and Thigpen 1985)*. Elementary bounds on coefficients in the equations of poroelasticity are presented by *Thigpen and Berryman (1985)*. Mechanical stability requires non-negativity of E, which implies that $H \geq 0$, $M \geq 0$, $HM - C^2 \geq 0$, and $\mu \geq 0$. Two coupled equations of motion for small disturbances in the fluid-saturated medium may be derived easily from this functional.

With time dependence of the form $\exp(-i\omega t)$, Biot's equations of poroelasticity are, using the notation of *Biot (1962)*,

$$\mu\nabla^2\mathbf{u} + (H-\mu)\nabla e - C\nabla\zeta + \omega^2(\rho\mathbf{u} + \rho_f\mathbf{w}) = 0, \tag{28}$$

$$C\nabla e - M\nabla\zeta + \omega^2(\rho_f\mathbf{u} + q\mathbf{w}) = 0, \tag{29}$$

where

$$\rho = \phi\rho_f + (1-\phi)\rho_m \tag{30}$$

and

$$q(\omega) = \rho_f[\alpha/\phi + iF(\xi)\eta/\kappa\omega]. \tag{31}$$

The tortuosity $\alpha \geq 1$ is a pure number related to the frame inertia which has been measured *(Brown, 1980; Johnson et al., 1982)* and can also be estimated theoretically *(Berryman, 1983)*. The kinematic viscosity of the saturating fluid is η; the permeability of the porous frame is κ; the dynamic viscosity factor is given (for our choice of sign for the frequency dependence) by

$$F(\xi) = \frac{1}{4}\left\{\xi T(\xi)/\left[1 + 2T(\xi)/i\xi\right]\right\}, \tag{32}$$

where

$$T(\xi) = \frac{ber'(\xi) - ibei'(\xi)}{ber(\xi) - ibei(\xi)} \tag{33}$$

and

$$\xi = (\omega h^2/\eta)^{\frac{1}{2}}. \tag{34}$$

The functions $ber(\xi)$ and $bei(\xi)$ are the real and imaginary parts of the Kelvin function. The dynamic parameter h is a characteristic length generally associated with the steady-flow hydraulic radius of the pores, or with a typical pore size. With some modifications to all terms involving properties of the saturating fluid, these equations have also been shown to apply to partially saturated porous media as well *(Berryman, Thigpen, and Chin, 1988)*.

The coupled equations (28) and (29) give rise to three distinct modes of wave propagation: two compressional waves (fast and slow) with wavenumbers k_+ and k_- and a single shear wave speed having wavenumber k_s.

The coefficients appearing in Biot's equations of poroelasticity must be known before quantitative predictions can be made with the theory. *Brown and Korringa (1975)* have shown that these coefficients are given for general isotropic porous media by

$$H = K + \sigma C + \frac{4}{3}\mu, \tag{35}$$

$$C = \sigma / \left[\frac{\sigma}{K_s} + \phi \left(\frac{1}{K_f} - \frac{1}{K_\phi} \right) \right], \tag{36}$$

and

$$M = C/\sigma, \tag{37}$$

where

$$\sigma = 1 - K/K_s. \tag{38}$$

The three bulk moduli characteristic of the porous frame are defined by Brown and Korringa through the expressions:

$$\frac{1}{K} = -\frac{1}{V}\left(\frac{\partial V}{\partial p_d}\right)_{p_f}, \tag{39}$$

$$\frac{1}{K_s} = -\frac{1}{V}\left(\frac{\partial V}{\partial p_f}\right)_{p_d}, \tag{40}$$

and

$$\frac{1}{K_\phi} = -\frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial p_f}\right)_{p_d}, \tag{41}$$

where V is the total sample volume, $V_\phi = \phi V$ is the pore volume, p is the external pressure, p_f is the pore pressure, and $p_d = p - p_f$ is the differential pressure. *Brown and Korringa (1975)* state that, although these three bulk moduli have simple physical interpretations, this "does not necessarily help in knowing their values."

It is implicitly assumed that K, K_s, and K_ϕ are properties of the solid frame alone, and therefore independent of the pore fluid modulus K_f. This basic assumption of poroelasticity is not as restrictive as it might at first appear. The pore fluid modulus K_f is associated with connected (primary) porosity, while any isolated or unconnected (secondary) porosity is treated as part of the solid frame. Thus, isolated fluid inclusions may also be treated by the methods developed here.

The constant K is just the bulk modulus of the drained porous frame that we introduced earlier. However, the values of the two remaining constants K_s and K_ϕ are generally not known unless the porous frame is homogeneous on the microscopic scale. For this special circumstance [which is also the only one considered by *Gassmann (1951)*] with a single type of elastic solid composing the frame, these two moduli are both equal to the bulk modulus K_m of the single granular constituent

$$K_s = K_\phi = K_m. \tag{42}$$

Thus, Gassmann's equation is equivalent to

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{\sigma - \phi}{K_m}, \quad \sigma = 1 - K/K_m, \tag{43}$$

while Brown and Korringa's more general result is equivalent to

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{\sigma}{K_s} - \frac{\phi}{K_\phi}, \quad \sigma = 1 - K/K_s. \tag{44}$$

Gassmann's result has also been derived within the context of Biot's theory of poroelasticity by *Biot and Willis (1957)* and *Geertsma (1957)*, and from a micromechanical theory based on classical elasticity by *Zimmerman, Somerton, and King (1986)*. Results essentially equivalent to those of Brown and Korringa were also obtained later by *Rice and Cleary (1976)* and by *Palciauskas and Domenico (1989)*. The more general constants of Brown and Korringa, K_s and K_ϕ, must somehow be related to the material properties of the multiple solid constituents of the porous frame. Finding such relations using the single-scattering approximations is the main focus of the remainder of this paper.

4 Scattering Coefficients

Let the spherical inhomogeneity have radius a. The spherical region is internally homogeneous, but otherwise its properties are arbitrary. Thus, the bulk and shear moduli, grain bulk modulus, density, porosity, and permeability of the solids included may all be different from those of the host. Similarly, the bulk modulus, density, and viscosity of the fluids included may also differ from those of the host fluid.

Suppose now that a plane fast compressional wave is generated at a freee surface far from the inclusion. If the incident fast compressional wave with wavenumber k_+ has the form

$$\mathbf{u} = \hat{z}(A_0/ik_+)\exp i(k_+z - \omega t), \tag{45}$$

then, due to mode conversion at the sphere interface, the radial component of the scattered compressional wave contains both fast and slow parts in the far field. The general expression for this radial component is

$$\begin{aligned} u_r = (ik_+)^{-1}\exp i(k_+r - \omega t) \times [B_0^{(+)} - B_1^{(+)}\cos\theta \\ -B_2^{(+)}(3\cos 2\theta + 1)/4] \\ -(ik_-)^{-1}\exp i(k_-r - \omega t) \times [B_0^{(-)} - B_1^{(-)}\cos\theta \\ -B_2^{(-)}(3\cos 2\theta + 1)/4]. \end{aligned} \tag{46}$$

The wavenumber for the slow compressional wave is k_-. The coefficients for the scattered fast wave are $B_j^{(+)}$ for $j = 0, 1, 2$; for the scattered slow wave, they are $B_j^{(-)}$. For the present application, only the first coefficient in each set is needed and these two coefficients are known exactly.

At low frequencies, the coefficients of interest are given by

$$B_0^{(-)} = \frac{k_-^3 a^3 C A_0}{3HM'(K' + \frac{4}{3}\mu)} \left[C(K' + \frac{4}{3}\mu + \sigma' C') - C'(K + \frac{4}{3}\mu + \sigma C) \right], \tag{47}$$

and

$$B_0^{(+)} = -\frac{k_+^3 a^3 A_0}{3} \frac{[K' - K + (\sigma' - \sigma)C]}{K' + \frac{4}{3}\mu} + (k_+/k_-)^3 B_0^{(-)}. \tag{48}$$

It is easy to check that, if the fluid bulk modulus vanishes $K_f \to 0$, then $C \to 0$ so $B_0^{(-)} \to 0$ and (48) reduces to (3). Thus, in the absence of a fluid, the equations reduce to those for elastic wave scattering from a spherical inclusion.

5 Consistency Conditions

The remainder of this paper will involve detailed calculations of the single-scattering approximations and comparisons of the results with Gassmann's equation and Brown and Korringa's equation [our equations (43) and (44), respectively]. It will simplify the presentation of these results somewhat if we first determine some general criteria to show what it means for an approximation to be consistent with either *Gassmann (1951)* or *Brown and Korringa (1975)*.

The first fact is that the constant σ^* must be independent of the fluid properties unless the fluid is trapped in a way that makes it effectively part of the porous frame. Furthermore, if there is only one type of elastic solid grain in the composite porous medium, then Gassmann's derivation is valid and (42) should hold. This requirement implies that, if $K_m^{(1)} = K_m^{(2)} = K_m$, then $K_s^* = K_m$. If this requirement is not satisfied, then of course (42) would be violated and the approximation would be inconsistent with Gassmann's equation.

The next fact to note concerns the effective bulk modulus of a composite fluid. It is well-known that the effective bulk modulus K_f^* of a fluid composed of two fluids with bulk moduli $K_f^{(1)}$ and $K_f^{(2)}$ and volume fractions $v^{(1)}$ and $v^{(2)} = 1 - v^{(1)}$, respectively, is given by the harmonic mean

$$\frac{1}{K_f^*} = \frac{v^{(1)}}{K_f^{(1)}} + \frac{v^{(2)}}{K_f^{(2)}}, \tag{49}$$

which is sometimes called Wood's formula *(Wood 1957)*. Equation (49) may be generalized for an arbitrary number of fluids to

$$\frac{1}{K_f^*} = \left\langle \frac{1}{K_f(\mathbf{x})} \right\rangle, \tag{50}$$

where the bracket $< \cdot >$ is a volume average. From (50) and the form of M given in both (43) and (44), it is clear that a *sufficient condition* for consistency is that the effective constant M^* should satisfy

$$\frac{1}{M^*} = \left\langle \frac{1}{M(\mathbf{x})} \right\rangle, \tag{51}$$

while the effective constant σ^* is independent of the fluid properties.

For the differential effective medium for a composite fluid, the defining equation is derived from the single scattering coefficient

$$B_0^{(+)} \propto \frac{K_f' - K_f}{K_f'}, \tag{52}$$

where K_f is the bulk modulus of the host and K_f' is the bulk modulus of the spherical scatterer. If there are only two fluids, we define $x = v^{(1)}$ and $y = v^{(2)}$ where $x + y = 1$. Then, if the initial host medium is pure type-1 (*i.e.*, $K_f^*(0) = K_f^{(1)}$), the differential effective medium approximation is

$$\frac{K_f^*(y+dy) - K_f^*(y)}{K_f^*(y+dy)} = \frac{dy}{(1-y)} \frac{K_f^{(2)} - K_f^*(y)}{K_f^{(2)}}. \tag{53}$$

Taking the limit $dy \to 0$ in (53), we obtain

$$(1-y)\frac{d}{dy}\left[\frac{1}{K_f^*(y)}\right] = \frac{1}{K_f^{(2)}} - \frac{1}{K_f^*(y)}. \tag{54}$$

It is not difficult to check (see Appendix A) that Wood's formula (49) satisfies (54). Again from the form of M in (43) and (44), it follows that a *sufficient condition* for consistency of the differential effective medium for Biot's equations is

$$(1-y)\frac{d}{dy}\left[\frac{1}{M^*(y)}\right] = \frac{1}{M^{(2)}} - \frac{1}{M^*(y)}, \tag{55}$$

while the effective constant σ^* must again be independent of the fluid constituents.

We stress that the conditions (51) and (55) are sufficient but not necessary. In fact, the actual expressions for the effective constant M^* are more complex than either of these expressions, but we will find that they share their main features.

6 Single-scattering Approximations

Now we develop the single-scattering formulas for σ and M based on the scattering coefficients (48) and (47). The methods used are completely analogous to those used in deriving the expressions for K^* and μ^*, so we will not repeat the arguments here. In this section, we assume that the individual scatterers are porous fluid-saturated materials having coefficients that satisfy Gassmann's equation. Then, we check to see if the results for a composite porous material are consistent with Brown and Korringa's result generally and with Gassmann's result specifically when only one type of solid grain is involved.

When computing the scattering formula for σ, in each case we assume that the terms in (48) proportional to $(K' - K)$ have already been averaged to find the effective constant for the drained porous frame. Then, these terms are guaranteed to cancel from the equations derived in this section. This assumption is used repeatedly in the analysis for σ^*.

6.1 Average T-matrix approximation

For two fluid-saturated porous components, the ATA for σ^* is

$$\frac{\sigma^*_{ATA} - \sigma^{(1)}}{K^*_{ATA} + \frac{4}{3}\mu^{(1)}} = v^{(2)} \frac{\sigma^{(2)} - \sigma^{(1)}}{K^{(2)} + \frac{4}{3}\mu^{(1)}}. \tag{56}$$

Generalizing to multiple components and using (11) to simplify the result, we find

$$\frac{\sigma^*_{ATA}}{K^*_{ATA} + \frac{4}{3}\mu^{(1)}} = \left\langle \frac{\sigma(\mathbf{x})}{K(\mathbf{x}) + \frac{4}{3}\mu^{(1)}} \right\rangle. \tag{57}$$

Equation (57) clearly satisfies the requirement that σ^*_{ATA} is independent of the fluid properties.

For computations, (57) is the most useful form of the equation. However, to check the other consistency conditions, it is helpful to substitute $\sigma^* = 1 - K^*/K^*_s$ and $\sigma^{(i)} = 1 - K^{(i)}/K^{(i)}_m$. Then,

$$\frac{1}{K^*_s} = \left\langle \frac{1}{K_m(\mathbf{x})} \cdot \frac{1 + \frac{4}{3}\mu^{(1)}/K^*_{ATA}}{1 + \frac{4}{3}\mu^{(1)}/K(\mathbf{x})} \right\rangle. \tag{58}$$

Using (11) again, it is not difficult to see that (58) implies that $K^*_s = K_m$ if $K(\mathbf{x}) = K_m$ is constant throughout. This result is required for compatibility with Gassmann's equation.

For two components, it follows from (47) and (56) that

$$\frac{1}{M^*_{ATA}} + \frac{(\sigma^*)^2}{K^*_{ATA} + \frac{4}{3}\mu^{(1)}} = v^{(2)} \left[\frac{1}{M^{(2)}} + \frac{(\sigma^{(2)})^2}{K^{(2)} + \frac{4}{3}\mu^{(1)}} \right]. \tag{59}$$

Generalizing to multiple constituents and using (57), we find that

$$\frac{1}{M^*_{ATA}} = \left\langle \frac{1}{M(\mathbf{x})} + \frac{[\sigma(\mathbf{x}) - \sigma^*_{ATA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^{(1)}} \right\rangle. \tag{60}$$

Comparing (60) to (51), we see that the only difference between the two is a term in (60) that depends only on the solid properties. Thus, (60) is indeed consistent with the results of *Brown and Korringa (1975)*.

We can now use (60) to obtain an estimate of K^*_ϕ. Substituting (44) into (60), we find that

$$\frac{\phi^*}{K^*_\phi} = \left\langle \frac{\phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle + \left[\frac{\sigma^*_{ATA}}{K^*_s} - \left\langle \frac{\sigma(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle - \left\langle \frac{[\sigma(\mathbf{x}) - \sigma^*_{ATA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^*_{ATA}} \right\rangle \right], \tag{61}$$

where $1/K^*_s = (1 - \sigma^*_{ATA})/K^*_{ATA}$ and $\phi^* = \langle \phi(\mathbf{x}) \rangle$. To check the consistency of this formula with Gassmann's equation, consider the limit $K^{(1)}_m = K^{(2)}_m = K_m$ in which we have already shown that $K^*_s = K_m$. Then, using the identity (in this limit) that

$$\left\langle \frac{[\sigma(\mathbf{x}) - \sigma^*_{ATA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^{(1)}} \right\rangle = \langle K(\mathbf{x}) - K^*_{ATA} \rangle / K^2_m = \langle \sigma^*_{ATA} - \sigma(\mathbf{x}) \rangle / K_m, \tag{62}$$

we find that the bracketed expression in (61) vanishes, so $K^*_\phi = K_m$ as required for agreement with Gassmann. Also note that (62) incidentally provides a simple proof that $K^*_{ATA} \le \langle K(\mathbf{x}) \rangle$ in general and that $\sigma^*_{ATA} \ge \langle \sigma(\mathbf{x}) \rangle$ in this limit, since the left hand side of (62) is clearly non-negative.

6.2 Coherent potential approximation

The main results of this subsection have appeared before in *Berryman (1986)*.

For two fluid-saturated porous components, the CPA for σ^* is

$$v^{(1)}\frac{\sigma^{(1)} - \sigma^*_{CPA}}{K^{(1)} + \frac{4}{3}\mu^*_{CPA}} + v^{(2)}\frac{\sigma^{(2)} - \sigma^*_{CPA}}{K^{(2)} + \frac{4}{3}\mu^*_{CPA}} = 0. \tag{63}$$

Generalizing to multiple components and using (16) to simplify the result, we find

$$\frac{\sigma^*_{CPA}}{K^*_{CPA} + \frac{4}{3}\mu^*_{CPA}} = \left\langle \frac{\sigma(\mathbf{x})}{K(\mathbf{x}) + \frac{4}{3}\mu^*_{CPA}} \right\rangle. \tag{64}$$

Equation (64) clearly satisfies the requirement that σ^*_{CPA} is independent of the fluid properties.

For computations, (64) is the most useful form of the equation. However, to check the other consistency conditions, it is helpful to substitute $\sigma^* = 1 - K^*/K^*_s$ and $\sigma^{(i)} = 1 - K^{(i)}/K^{(i)}_m$. Then,

$$\frac{1}{K^*_s} = \left\langle \frac{1}{K_m(\mathbf{x})} \cdot \frac{1 + \frac{4}{3}\mu^*_{CPA}/K^*_{CPA}}{1 + \frac{4}{3}\mu^*_{CPA}/K(\mathbf{x})} \right\rangle. \tag{65}$$

Using (16) again, it is not difficult to see that (65) implies that $K^*_s = K_m$ if $K(\mathbf{x}) = K_m$ is constant throughout. This result is required for compatibility with Gassmann's equation.

The CPA for M^* is most conveniently written for the case of multiple components as

$$\frac{1}{M^*_{CPA}} = \left\langle \frac{1}{M(\mathbf{x})} + \frac{[\sigma(\mathbf{x}) - \sigma^*_{CPA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^*_{CPA}} \right\rangle. \tag{66}$$

Comparing (66) with (51), we see again that the fluid dependent terms are identical and that is all we require for the approximation to be consistent with the result of *Brown and Korringa (1975)*. Note that the ATA (60) and the CPA (66) are identical when the shear modulus is constant.

Equation (66) can now be used to obtain an estimate of K^*_ϕ. Substituting (44) into (66), we find that

$$\frac{\phi^*}{K^*_\phi} = \left\langle \frac{\phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle + \left[\frac{\sigma^*_{CPA}}{K^*_s} - \left\langle \frac{\sigma(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle - \left\langle \frac{[\sigma(\mathbf{x}) - \sigma^*_{CPA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^*_{CPA}} \right\rangle \right], \tag{67}$$

where $1/K^*_s = (1 - \sigma^*_{CPA})/K^*_{CPA}$ and $\phi^* = \langle \phi(\mathbf{x}) \rangle$. To check the consistency of this formula with Gassmann's equation, consider the limit $K^{(1)}_m = K^{(2)}_m = K_m$ in which we have already shown that $K^*_s = K_m$. Then, using the identity (in this limit) that

$$\left\langle \frac{[\sigma(\mathbf{x}) - \sigma^*_{CPA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^*} \right\rangle = \langle K(\mathbf{x}) - K^*_{CPA} \rangle / K^2_m = \langle \sigma^*_{CPA} - \sigma(\mathbf{x}) \rangle / K_m, \tag{68}$$

we find that the bracketed expression in (67) vanishes, so $K^*_\phi = K_m$ as required for agreement with Gassmann. Equation (68) also shows that $K^*_{CPA} \leq \langle K(\mathbf{x}) \rangle$ in general, and that $\sigma^*_{CPA} \geq \langle \sigma(\mathbf{x}) \rangle$ in this limit.

6.3 Differential effective medium

When there are only two components in the composite material, the defining DEM equation for σ is

$$\frac{\sigma^*_{DEM}(y+dy)-\sigma^*_{DEM}(y)}{K^*_{DEM}(y+dy)+\frac{4}{3}\mu^*_{DEM}(y)} = \frac{dy}{(1-y)}\frac{\sigma^{(2)}-\sigma^*_{DEM}(y)}{K^{(2)}+\frac{4}{3}\mu^*_{DEM}(y)}, \tag{69}$$

from which it follows that

$$(1-y)\frac{d}{dy}\left[\sigma^*_{DEM}(y)\right] = \frac{\sigma^{(2)}-\sigma^*_{DEM}(y)}{K^{(2)}+\frac{4}{3}\mu^*_{DEM}(y)}\left[K^*_{DEM}(y)+\frac{4}{3}\mu^*_{DEM}(y)\right], \tag{70}$$

with $\sigma^*_{DEM}(0)=\sigma^{(1)}$. Comparing (70) and (19), we find that

$$\frac{d}{dy}\left[\sigma^*_{DEM}(y)\right]/\frac{d}{dy}\left[K^*_{DEM}(y)\right] = \frac{\sigma^{(2)}-\sigma^*_{DEM}(y)}{K^{(2)}-K^*_{DEM}(y)}, \tag{71}$$

which can be integrated to yield

$$\frac{\sigma^*_{DEM}(y)-\sigma^{(2)}}{\sigma^{(1)}-\sigma^{(2)}} = \frac{K^*_{DEM}(y)-K^{(2)}}{K^{(1)}-K^{(2)}}. \tag{72}$$

Equation (72) is the main result for $\sigma^*_{DEM}(y)$.

To compare the DEM result with that of ATA, note that (56) and (9) show that

$$\frac{\sigma^*_{ATA}-\sigma^{(1)}}{\sigma^{(2)}-\sigma^{(1)}} = v^{(2)}\frac{K^*_{ATA}+\frac{4}{3}\mu^{(1)}}{K^{(2)}+\frac{4}{3}\mu^{(1)}} = \frac{K^*_{ATA}-K^{(1)}}{K^{(2)}-K^{(1)}}, \tag{73}$$

from which we can easily derive

$$\frac{\sigma^*_{ATA}-\sigma^{(2)}}{\sigma^{(1)}-\sigma^{(2)}} = \frac{K^*_{ATA}-K^{(2)}}{K^{(1)}-K^{(2)}}. \tag{74}$$

Equations (74) and (72) are very similar. Furthermore, recall that, if the shear modulus is constant so $\mu^{(1)}=\mu^{(2)}=\mu^*$, then $K^*_{ATA}=K^*_{DEM}$ as we have shown previously. So it also follows from (74) and (72) that $\sigma^*_{ATA}=\sigma^*_{DEM}$ when the shear modulus is constant.

Similarly, to compare the DEM result with that of CPA, note that (63) and (15) show that

$$\frac{\sigma^{(1)}-\sigma^*_{CPA}}{\sigma^{(2)}-\sigma^*_{CPA}} = -\frac{v^{(2)}}{v^{(1)}}\frac{K^{(1)}+\frac{4}{3}\mu^*_{CPA}}{K^{(2)}+\frac{4}{3}\mu^*_{CPA}} = \frac{K^{(1)}-K^*_{CPA}}{K^{(2)}-K^*_{CPA}}, \tag{75}$$

from which it is easy to show that

$$\frac{\sigma^*_{CPA}-\sigma^{(2)}}{\sigma^{(1)}-\sigma^{(2)}} = \frac{K^*_{CPA}-K^{(2)}}{K^{(1)}-K^{(2)}}. \tag{76}$$

Equation (76) should be compared to (74) and (72). Clearly, if the shear modulus is constant, all three approximations give the same results for σ^* as well as for K^*. Furthermore, it has been shown elsewhere by *Berryman and Milton (1991)* that the result

$$\frac{\sigma^*-\sigma^{(2)}}{\sigma^{(1)}-\sigma^{(2)}} = \frac{K^*-K^{(2)}}{K^{(1)}-K^{(2)}} \tag{77}$$

is exact for two component composite porous media. Thus, all three approximations produce expressions for K_s^* or for σ^* in agreement with exact results for generalized Gassmann's equations.

To check further for consistency with Gassmann's equation, now substitute $\sigma^* = 1 - K^*/K_s^*$ and $\sigma^{(i)} = 1 - K^{(i)}/K_m^{(i)}$ into (72). We find that

$$\frac{K_{DEM}^*(y)/K_s^*(y) - K^{(2)}/K_m^{(2)}}{K^{(1)}/K_m^{(1)} - K^{(2)}/K_m^{(2)}} = \frac{K_{DEM}^*(y) - K^{(2)}}{K^{(1)} - K^{(2)}}. \tag{78}$$

Solving for $K_s^*(y)$, we find

$$K_s^*(y) = \frac{K^{(1)} - K^{(2)}}{(K^{(1)}/K_m^{(1)} - K^{(2)}/K_m^{(2)}) + \frac{K^{(1)}K^{(2)}}{K_{DEM}^*(y)}\left(1/K_m^{(2)} - 1/K_m^{(1)}\right)}. \tag{79}$$

Another useful form of the result [see *Berryman and Milton (1991)*] is

$$\frac{1/K_s^*(y) - 1/K_m^{(2)}}{1/K_m^{(1)} - 1/K_m^{(2)}} = \frac{1/K_{DEM}^*(y) - 1/K^{(2)}}{1/K^{(1)} - 1/K^{(2)}}. \tag{80}$$

It follows from (79) or (80) that, if $K_m^{(1)} = K_m^{(2)} = K_m$, then $K_s^* = K_m$ — consistent with Gassmann's equation. Expressions corresponding to (79) and (80) may also be derived for the ATA and the CPA.

We now compute the DEM for the coefficient M^*. From (47), we find that

$$\begin{aligned}\frac{1}{M_{DEM}^*(y+dy)} - \frac{1}{M_{DEM}^*(y)}&\frac{\sigma_{DEM}^*(y+dy)}{\sigma_{DEM}^*(y)}\frac{K_{DEM}^*(y) + \frac{4}{3}\mu_{DEM}^*(y)}{K_{DEM}^*(y+dy) + \frac{4}{3}\mu_{DEM}^*(y)} \\ &+\frac{\sigma_{DEM}^*(y+dy)(\sigma_{DEM}^*(y+dy) - \sigma_{DEM}^*(y))}{K_{DEM}^*(y+dy) + \frac{4}{3}\mu_{DEM}^*(y)} \\ = \frac{dy}{(1-y)}\Big[\frac{1}{M^{(2)}} - &\frac{1}{M_{DEM}^*(y)}\frac{\sigma^{(2)}}{\sigma_{DEM}^*(y)}\frac{K_{DEM}^*(y) + \frac{4}{3}\mu_{DEM}^*(y)}{K^{(2)} + \frac{4}{3}\mu_{DEM}^*(y)} \\ &+\frac{\sigma^{(2)}(\sigma^{(2)} - \sigma_{DEM}^*(y))}{K^{(2)} + \frac{4}{3}\mu_{DEM}^*(y)}\Big].\end{aligned} \tag{81}$$

To simplify (81), we need the following identities

$$\begin{aligned}&\frac{\sigma_{DEM}^*(y+dy)}{\sigma_{DEM}^*(y)}\frac{K_{DEM}^*(y) + \frac{4}{3}\mu_{DEM}^*(y)}{K_{DEM}^*(y+dy) + \frac{4}{3}\mu_{DEM}^*(y)} - 1 \\ &\underset{dy\to 0}{\to} dy\left[\frac{d\sigma_{DEM}^*(y)/dy}{\sigma_{DEM}^*(y)} - \frac{dK_{DEM}^*(y)/dy}{K_{DEM}^*(y) + \frac{4}{3}\mu_{DEM}^*(y)}\right],\end{aligned} \tag{82}$$

and, using the results (19) and (70),

$$(1-y)\left[\frac{d\sigma_{DEM}^*(y)/dy}{\sigma_{DEM}^*(y)} - \frac{dK_{DEM}^*(y)/dy}{K_{DEM}^*(y) + \frac{4}{3}\mu_{DEM}^*(y)}\right]$$

$$= \left[\frac{\sigma^{(2)}}{\sigma^*_{DEM}(y)} - 1\right] \frac{K^*_{DEM}(y) + \frac{4}{3}\mu^*_{DEM}(y)}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)} - \frac{K^{(2)} - K^*_{DEM}(y)}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)}$$

$$= \left[\frac{\sigma^{(2)}}{\sigma^*_{DEM}(y)} \frac{K^*_{DEM}(y) + \frac{4}{3}\mu^*_{DEM}(y)}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)} - 1\right]. \tag{83}$$

So, when we take the limit $dy \to 0$ in (81), terms of the form (83) cancel from both sides of the equation and the result is

$$(1-y)\frac{d}{dy}\left[\frac{1}{M^*_{DEM}(y)}\right] = \frac{1}{M^{(2)}} - \frac{1}{M^*_{DEM}(y)} + \frac{[\sigma^{(2)} - \sigma^*_{DEM}(y)]^2}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)}. \tag{84}$$

Comparing (84) with (55), we find that all terms depending on fluid properties are consistent, while the remaining term in (84) depends only on solid properties. This result is required for the DEM to be consistent with the results of *Brown and Korringa (1975)*.

Considering the case with constant shear modulus, we find that (60) and (66) (being the same in this limit) are both solutions of (84). Thus, we find again that all three approximations are the same for constant shear modulus composites.

Using (72) we can further simplify (84) by noting that

$$\frac{[\sigma^{(2)} - \sigma^*_{DEM}(y)]^2}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)} = (\sigma^{(2)} - \sigma^*_{DEM}(y))\left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right)\left[1 - \frac{K^*_{DEM}(y) + \frac{4}{3}\mu^*_{DEM}(y)}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)}\right]$$

$$= \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right)\left[\sigma^{(2)} - \sigma^*_{DEM}(y) - (1-y)\frac{d}{dy}\left(\sigma^*_{DEM}(y)\right)\right], \tag{85}$$

where (70) was used in the last step of (85). Equation (84) can now be rearranged into the form

$$(1-y)\frac{d}{dy}\left[\frac{1}{M^*_{DEM}(y)} + \sigma^*_{DEM}(y)\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right] + \frac{1}{M^*_{DEM}(y)} + \sigma^*_{DEM}(y)\left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right)$$

$$= \frac{1}{M^{(2)}} + \sigma^{(2)}\left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right), \tag{86}$$

which is in the form shown to be integrable in Appendix A. Thus, we find the general result for the DEM that

$$\frac{1}{M^*_{DEM}(y)} = \left\langle \frac{1}{M(\mathbf{x})} \right\rangle + \langle \sigma(\mathbf{x}) - \sigma^*_{DEM}(y) \rangle \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right). \tag{87}$$

To compare (87) to the corresponding result for CPA when only two constituents are present, note that

$$\left\langle \frac{[\sigma(\mathbf{x}) - \sigma^*_{CPA}]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^*_{CPA}} \right\rangle = \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right)^2 \langle K(\mathbf{x}) - K^*_{CPA} \rangle = \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right) \langle \sigma(\mathbf{x}) - \sigma^*_{CPA} \rangle \tag{88}$$

Substituting (88) into (66), we obtain

$$\frac{1}{M^*_{CPA}} = \left\langle \frac{1}{M(\mathbf{x})} \right\rangle + \langle \sigma(\mathbf{x}) - \sigma^*_{CPA} \rangle \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}} \right), \tag{89}$$

which is of the same form as (87). A similar result holds for the ATA.

We can use (87) to find the DEM estimate of K^*_ϕ. Eliminating the fluid terms from (87), we find

$$\frac{\phi^*}{K^*_\phi(y)} = \left\langle \frac{\phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle + \left[\frac{\sigma^*_{DEM}(y)}{K^*_s(y)} - \left\langle \frac{\sigma(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle - \langle \sigma(\mathbf{x}) - \sigma^*_{DEM}(y) \rangle \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}} \right) \right], \tag{90}$$

where $\phi^* \equiv \langle \phi(\mathbf{x}) \rangle$. To check the agreement of (90) with Gassmann's equation, consider $K_m^{(1)} = K_m^{(2)} = K_m$. We have shown before that, in this limit, $K^*_s(y) = K_m$. Substituting these values into (90) and simplifying, we find that the expression in brackets in (90) vanishes so the unique solution is $K^*_\phi(y) = K_m$. Thus, the DEM does satisfy Gassmann's equation in this limit.

Finally, it has been shown by *Berryman and Milton (1991)* that the result

$$\frac{\phi^*}{K^*_\phi} = \frac{\sigma^*}{K^*_s} - \left\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle - \langle \sigma(\mathbf{x}) - \sigma^* \rangle \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}} \right) \tag{91}$$

is exact for two component composite media. The agreement in form between (90) and (91) is clear. Furthermore, the results (61) and (67) for ATA and CPA respectively can be transformed into the same form. Thus, we find that all three approximations produce formulas for K^*_ϕ of the right form to agree with all known exact results.

7 Gassmann's Equation

In the previous section, we assumed that the scatterers had Biot coefficients that were themselves consistent with Gassmann's equation. Then, we derived formulas that were consistent with Brown and Korringa's general result or with Gassmann's result in special cases. Here we will assume that the scatterers are spheres of fluid or spheres of a single type of elastic solid and check to see if Gassmann's equation is the result.

For a fluid inclusion, $K' = \mu' = 0$, $\sigma' = 1$, and

$$\frac{1}{M_f} = \frac{1}{K_f}. \tag{92}$$

For a solid inclusion, $K' = K_m$, $\mu' = \mu_m$, $\sigma' = 0$, and

$$\frac{1}{M_s} = 0, \tag{93}$$

since the porosity of the solid vanishes. The volume fraction of fluid is ϕ and that of the solid is $1 - \phi$.

7.1 Generalized ATA

We will now show that the generalized ATA for a mixture of fluid and solid spheres satisfies Gassmann's equation. In this section, we will not use any subscript to distinguish the starred quantities. Also, it is required in what follows that the host shear modulus $\mu^\dagger > 0$. Figure 4 illustrates the concept of the generalized ATA considered here.

Starting with the effective porous frame bulk modulus for the generalized ATA, we find that

$$\frac{1}{K^* + \frac{4}{3}\mu^\dagger} = \frac{\phi}{\frac{4}{3}\mu^\dagger} + \frac{1-\phi}{K_m + \frac{4}{3}\mu^\dagger}, \tag{94}$$

which can be rearranged to show that

$$K^* = (1-\phi)K_m \frac{K^* + \frac{4}{3}\mu^\dagger}{K_m + \frac{4}{3}\mu^\dagger}. \tag{95}$$

Similarly, the effective value of σ^* is given by

$$\frac{\sigma^*}{K^* + \frac{4}{3}\mu^\dagger} = \frac{\phi}{\frac{4}{3}\mu^\dagger}, \tag{96}$$

so

$$\sigma^* = \left(\frac{\phi}{\frac{4}{3}\mu^\dagger}\right)(K^* + \frac{4}{3}\mu^\dagger) = 1 - (1-\phi)\frac{K^* + \frac{4}{3}\mu^\dagger}{K_m + \frac{4}{3}\mu^\dagger} = 1 - K^*/K_m, \tag{97}$$

where (94) was used in the second step of (97), and (95) was used in the last step. Thus, (97) shows that $K_s^* = K_m$, as required by Gassmann's equation.

The effective value of M^* is given in this approximation by

$$\frac{1}{M^*} = \left\langle \frac{1}{M(\mathbf{x})} + \frac{[\sigma(\mathbf{x}) - \sigma^*]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^\dagger} \right\rangle. \tag{98}$$

First, note that

$$\left\langle \frac{1}{M(\mathbf{x})} \right\rangle = \frac{\phi}{K_f} \tag{99}$$

in this limit. Then, note that, since $\sigma(\mathbf{x}) = 0, 1$ for the fluid/solid composite, we have $\sigma(\mathbf{x}) = \sigma(\mathbf{x})^2$, so

$$\left\langle \frac{[\sigma(\mathbf{x}) - \sigma^*]^2}{K(\mathbf{x}) + \frac{4}{3}\mu^\dagger} \right\rangle = \left\langle \frac{\sigma(\mathbf{x}) - (\sigma^*)^2}{K(\mathbf{x}) + \frac{4}{3}\mu^\dagger} \right\rangle = \frac{\sigma^*(1-\sigma^*)}{K^* + \frac{4}{3}\mu^\dagger}. \tag{100}$$

Finally, we have

$$\frac{\sigma^*(1-\sigma^*)}{K^* + \frac{4}{3}\mu^\dagger} = \frac{\sigma^*}{K_m}\frac{K^*}{K^* + \frac{4}{3}\mu^\dagger} = \frac{\sigma^*}{K_m}\left(1 - \frac{\frac{4}{3}\mu^\dagger}{K^* + \frac{4}{3}\mu^\dagger}\right) = \frac{\sigma^*}{K_m}\left(1 - \frac{\phi}{\sigma^*}\right) = \frac{\sigma^* - \phi}{K_m}, \tag{101}$$

so

$$\frac{1}{M^*} = \frac{\phi}{K_f} + \frac{\sigma^* - \phi}{K_m}, \tag{102}$$

which is equivalent to Gassmann's equation.

Thus, we have shown that Gassmann's equation does result from any of the generalized ATA approximations including the CPA as long as $\mu^\dagger > 0$. If $\mu^\dagger = 0$, this approximation fails because much of the formal mathematics used here fails. This problem is significant for some of these approximations since, for example, the CPA is known to produce $\mu^* = 0$ for solid fractions $(1-\phi) \leq 0.4$. For the ATA, the significance of the problem is a little more subtle and has to do with the choice of the host material. Clearly, we cannot choose the fluid to be the host for the ATA since the fluid has $\mu^\dagger = 0$. However, if we choose the solid to be the host, then $\mu^\dagger = \mu_m$ which is satisfactory as far as the formal mathematical requirements, but this choice introduces another problem. If the solid is used as the host in the ATA, then the thought process used to derive the ATA clearly implies that the composite must have a connected (percolating) solid frame for any value of the solid volume fraction. The fluid may or may not be connected in this model; if not, then elasticity is sufficient and Biot's equations need not be introduced. Also significant is the fact that the scattering picture is one of Biot material imbedded in a solid host. In contrast, since the CPA uses the effective medium as the host, the imbedding medium for CPA is the fluid-saturated porous medium.

7.2 Differential effective medium

For the DEM, it will be instructive to derive the results twice: once with the fluid as the initial host and once with solid as host.

Starting with fluid and adding solid, we have $M^*(0) = M^{(1)} = K_f$, $\sigma^*(0) = \sigma^{(1)} = 1$, $K^*(0) = K^{(1)} = 0$, $\mu^*(0) = \mu^{(1)} = 0$, $1/M^{(2)} = 0$, $\sigma^{(2)} = 0$, $K^{(2)} = K_m$, and $\phi = 1 - y$. From (19) and (70), it follows that

$$\frac{d}{dy}\log\sigma^*(y) = \frac{d}{dy}\log\left[K_m - K^*(y)\right], \tag{103}$$

which can be integrated to yield

$$\sigma^*(y) = 1 - K^*(y)/K_m, \tag{104}$$

as required. The equation for $M^*(y)$ is (84), which becomes

$$(1-y)\frac{d}{dy}\left[\frac{1}{M^*(y)}\right] + \frac{1}{M^*(y)} = \frac{[\sigma^*(y)]^2}{K_m + \frac{4}{3}\mu^*(y)}. \tag{105}$$

To satisfy Gassmann's equation, we must have

$$\frac{1}{M^*_G(y)} = \frac{1-y}{K_f} + \frac{\sigma^*(y) - (1-y)}{K_m}. \tag{106}$$

Substituting (106) into the left hand side of (105), we find

$$(1-y)\frac{d}{dy}\left[\frac{1}{M^*_G(y)}\right] + \frac{1}{M^*_G(y)} = \frac{\sigma^*(y)}{K_m}\left[1 - \frac{K^*(y) + \frac{4}{3}\mu^*(y)}{K_m + \frac{4}{3}\mu^*(y)}\right] = \frac{\sigma^*(y)}{K_m}\frac{K_m - K^*(y)}{K_m + \frac{4}{3}\mu^*(y)}, \tag{107}$$

which reduces correctly to (105). Thus, $M^*(y) = M_G^*(y)$, so the DEM does give Gassmann's equation. The fact that $\mu^*(0) = 0$ plays no role in this result; furthermore, having $\mu^*(y) = 0$ for any value of $y > 0$ would also not affect the result. This conclusion is significant because, if we assume $K^*(0) = \mu^*(0) = 0$, then a solution is $K^*(y) = \mu^*(y) = 0$ for all $y < 1$. So in this case the DEM produces the correct moduli for a suspension (a fluid containing solid particles not touching each other), which also satisfies Gassmann's equation.

Now starting with the solid as host and adding the fluid, we have $1/M^*(0) = 1/M^{(1)} = 0$, $\sigma^*(0) = \sigma^{(1)} = 0$, $K^*(0) = K^{(1)} = K_m$, $\mu^*(0) = \mu^{(1)} = \mu_m$, $M^{(2)} = K_f$, $\sigma^{(2)} = 1$, $K^{(2)} = 0$, and $\phi = y$. Substituting these values into (19) and (70) gives the result

$$\frac{d}{dy}\log\left[1 - \sigma^*(y)\right] = \frac{d}{dy}\log K^*(y), \tag{108}$$

which can be integrated to yield

$$1 - \sigma^*(y) = K^*(y)/K_m, \tag{109}$$

again as required. Substituting into (84) gives the result

$$(1-y)\frac{d}{dy}\left[\frac{1}{M^*(y)}\right] + \frac{1}{M^*(y)} = \frac{1}{K_f} + \frac{\left[1-\sigma^*(y)\right]^2}{\frac{4}{3}\mu^*(y)} = \frac{1}{K_f} + \frac{\left[K^*(y)\right]^2}{K_m^2\frac{4}{3}\mu^*(y)}. \tag{110}$$

If $M^*(y)$ is to satisfy Gassmann's equation in this case, it must be of the form

$$\frac{1}{M_G^*(y)} = \frac{y}{K_f} + \frac{\sigma^*(y) - y}{K_m}. \tag{111}$$

Substituting (111) into the left hand side of (110) gives

$$\begin{aligned}(1-y)\frac{d}{dy}\left[\frac{1}{M_G^*(y)}\right] + \frac{1}{M_G^*(y)} &= \frac{1}{K_f} - \frac{1}{K_m} + \frac{1}{K_m}\left[(1-y)\frac{d}{dy}\sigma^*(y) + \sigma^*(y)\right] \\ &= \frac{1}{K_f} + \frac{\left[K^*(y)\right]^2}{K_m^2\frac{4}{3}\mu^*(y)},\end{aligned} \tag{112}$$

showing again that $M^*(y) = M_G^*(y)$, so the DEM satisfies Gassmann's equation. In contrast to the previous derivation, this one would clearly break down if $\mu^*(y) = 0$ for any value of $y < 1$.

8 Inequalities

In this section, we will develop some relationships that show the range of possible variation of the coefficients in Biot's equations when they are computed using one of the three approximations (ATA, CPA, or DEM) that we have been studying. One motivation for finding such bounds is to provide a more transparent method of checking the results of the previous sections. Another motivation is that knowledge of limits and the range of possible values is very helpful for checking the results of numerical algorithms such as those to be developed in the next section. We stress that the results in this section are bounds not only on the *estimates of the effective constants*, but also bounds on the constants themselves when only two constituents are present [because then the formulas used are exact *(Berryman and Milton, 1991)*]. To the author's knowledge, no work has been published previously providing nontrivial bounds on the coefficients in Biot's equations. However, the theory of composites has been used to find rigorous bounds on the elastic constants for the frame and the known results for porous media have been enumerated earlier by *Berryman and Milton (1988)*.

First, recall the known inequalities satisfied by the effective bulk and shear moduli in all three approximations. The inequalities for the generalized ATA and CPA follow from the monotonicity of the functional

$$\mathcal{K}(\mu) = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}\mu} \right\rangle^{-1} - \frac{4}{3}\mu, \tag{113}$$

as has been shown previously by *Berryman (1982)*. The resulting set of inequalities for K^* computed using either ATA or CPA is

$$\min K(\mathbf{x}) \leq \langle 1/K(\mathbf{x})\rangle^{-1} \leq K^* \leq \langle K(\mathbf{x})\rangle \leq \max K(\mathbf{x}). \tag{114}$$

To see that K^*_{DEM} also satisfies (114), note the following rearrangements of (19)

$$(1-y)\frac{d}{dy}\left[K^*_{DEM}(y)\right] + K^*_{DEM}(y) = K^{(2)} - \frac{\left[K^{(2)} - K^*_{DEM}(y)\right]^2}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)} \leq K^{(2)} \tag{115}$$

and

$$(1-y)\frac{d}{dy}\left[\frac{1}{K^*_{DEM}(y)}\right] + \frac{1}{K^*_{DEM}(y)} = \frac{1}{K^{(2)}} - \left[K^{(2)}/K^*_{DEM}(y) - 1\right]^2 \times \left[\frac{1}{K^{(2)}} - \frac{1}{K^{(2)} + \frac{4}{3}\mu^*_{DEM}(y)}\right] \leq \frac{1}{K^{(2)}}. \tag{116}$$

The inequalities on the right of (115) and (116) follow from inspection and the fact that $\mu^*_{DEM}(y) \geq 0$. The analysis in Appendix A shows that integrating (115) gives $K^*_{DEM}(y) \leq \langle K(\mathbf{x})\rangle$, while integrating (116) gives $1/K^*_{DEM}(y) \leq \langle 1/K(\mathbf{x})\rangle$. Thus, we have shown that the set of inequalities given in (114) is valid for all three approximations considered. Analogous arguments for the shear modulus may be used to show that

$$\min \mu(\mathbf{x}) \leq \langle 1/\mu(\mathbf{x})\rangle^{-1} \leq \mu^* \leq \langle \mu(\mathbf{x})\rangle \leq \max \mu(\mathbf{x}). \tag{117}$$

The monotonicity of $\mathcal{K}(\mu)$ may also be used to show that

$$K^-_{ATA} \leq K^*_{CPA} \leq K^+_{ATA}, \tag{118}$$

where K^-_{ATA} is the value of K^*_{ATA} for $\mu^\dagger = \min \mu(\mathbf{x})$, while K^+_{ATA} is the value of K^*_{ATA} for $\mu^\dagger = \max \mu(\mathbf{x})$. Similarly, we find that

$$K^-_{ATA} \leq K^*_{DEM}(y) \leq K^+_{ATA} \tag{119}$$

follows from the fact that

$$(1-y)\frac{d}{dy}\left[\frac{1}{K^*_{DEM}(y) + \frac{4}{3}\mu^\dagger}\right] + \frac{1}{K^*_{DEM}(y) + \frac{4}{3}\mu^\dagger} = \frac{1}{K^{(2)} + \frac{4}{3}\mu^\dagger}$$

$$- \left[\frac{K^{(2)}+\frac{4}{3}\mu^\dagger}{K^*_{DEM}(y)+\frac{4}{3}\mu^\dagger} - 1\right]^2 \left[\frac{1}{K^{(2)}+\frac{4}{3}\mu^\dagger} - \frac{1}{K^*_{DEM}(y)+\frac{4}{3}\mu^\dagger}\right] \leq \frac{1}{K^{(2)}+\frac{4}{3}\mu^\dagger}, \tag{120}$$

since the right hand side of (120) is less than or equal to $1/(K^{(2)}+\frac{4}{3}\mu^\dagger)$ when $\mu^*_{DEM}(y) \geq \mu^\dagger$, while it is greater than or equal to $1/(K^{(2)}+\frac{4}{3}\mu^\dagger)$ when $\mu^*_{DEM}(y) \leq \mu^\dagger$. The result (119) then follows from the two extreme cases of (117). Since K^+_{ATA} and K^-_{ATA} have values identical to the upper and lower bounds of Hashin and Shtrikman, (118) and (119) provide proofs that the estimates obtained from CPA and DEM lie between these rigorous bounds. A similar proof can be given for the shear modulus estimates. See *Berryman (1982)* and *McLaughlin (1977)* for earlier proofs.

Recall that the absolute bounds on σ given by $0 \leq \phi(\mathbf{x}) \leq \sigma(\mathbf{x}) \leq 1$ follow from its definition, the non-negativity of the bulk moduli, and the Voigt bound *(Watt et al. 1976)* on the frame modulus $K(\mathbf{x}) \leq (1-\phi(\mathbf{x}))K_m(\mathbf{x})$. When only two constituents are present in the composite, (72), (74), and (76) have the same form as the general result (77), shown to be exact in this limit. Inequalities for σ^* then follow from (114) and (77). First, it follows easily that

$$\min \sigma(\mathbf{x}) \leq \sigma^* \leq \max \sigma(\mathbf{x}). \tag{121}$$

Then, assuming without loss of generality that $K^{(1)} > K^{(2)}$, we also have

$$\frac{\langle K^{-1}(\mathbf{x})\rangle^{-1} - K^{(2)}}{K^{(1)}-K^{(2)}} \leq \frac{\sigma^* - \sigma^{(2)}}{\sigma^{(1)}-\sigma^{(2)}} \leq \frac{\langle K(\mathbf{x})\rangle - K^{(2)}}{K^{(1)}-K^{(2)}}. \tag{122}$$

Furthermore, we can show that

$$\frac{\sigma^* - \langle\sigma\rangle}{\sigma^{(1)}-\sigma^{(2)}} = \frac{K^* - \langle K\rangle}{K^{(1)}-K^{(2)}}. \tag{123}$$

If the grain modulus is constant $K_m^{(1)} = K_m^{(2)} = K_m$, then (123) implies $\sigma^* \geq \langle\sigma\rangle$, which also follows in this simple case from $K^* \leq \langle K\rangle$.

For the ATA and CPA estimates of K_s^*, we find bounds by treating the shear modulus in the formulas (58) and (65) as a parameter. Then, letting $\mu \to 0$ and $\mu \to \infty$, we find the limits of $1/K_s^*$ are $\langle 1/K_m(\mathbf{x})\rangle$ and $\langle K(\mathbf{x})/K_m(\mathbf{x})\rangle/\langle K(\mathbf{x})\rangle$, respectively. Which of these limits is the upper and which the lower bound depends on the relative values of the constituent constants. For two constituents if $(K^{(2)}-K^{(1)})(K_m^{(1)}-K_m^{(2)}) \geq 0$, then

$$\left\langle\frac{1}{K_m(\mathbf{x})}\right\rangle \leq \frac{1}{K_s^*} \leq \frac{\langle K(\mathbf{x})/K_m(\mathbf{x})\rangle}{\langle K(\mathbf{x})\rangle}. \tag{124}$$

The inequalities in (124) are reversed if $(K^{(2)}-K^{(1)})(K_m^{(1)}-K_m^{(2)}) < 0$. Also, note that it is easy to show from (124) and (114) that $\langle\sigma(\mathbf{x})\rangle \leq \sigma^*$, whereas this inequality is also reversed if the either of the relative values is reversed. A bound on K_s^* for the DEM follows from (79) when only two constituents are present and $K^{(2)} < K^{(1)}$. Then, if $K_m^{(2)} \leq K_m^{(1)}$, we find

$$K_m^{(2)} \leq K_s^* \leq K_m^{(1)}. \tag{125}$$

If the inequality on the grain moduli is reversed, then the inequalities in (125) are also reversed. The inequalities in (125) are valid for all three approximations. All of these bounds on K_s^* show that $K_s^* = K_m$ if the grain modulus $K(\mathbf{x}) = K_m$ is constant. Furthermore, a rigorous bound on K_s^* that follows from (80) and (114) is

$$\min K_m(\mathbf{x}) \leq K_s^* \leq \max K_m(\mathbf{x}). \tag{126}$$

which is valid for two constituent porous composites.

Using the fact that $\sigma(\mathbf{x}) \geq \phi(\mathbf{x})$, we find for the ATA and for the CPA that

$$\frac{\sigma^*}{K_s^*} - \frac{\phi^*}{K_\phi^*} \geq \left\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle \geq 0. \tag{127}$$

Equation (127) follows easily by rearranging (61) and (67). Similarly, (90) may be rearranged to give

$$\begin{aligned}(1-y)\frac{d}{dy}\left(\frac{\sigma_{DEM}^*}{K_s^*} - \frac{\phi^*}{K_\phi^*}\right) + \left(\frac{\sigma_{DEM}^*}{K_s^*} - \frac{\phi^*}{K_\phi^*}\right) &= \frac{\sigma^{(2)} - \phi^{(2)}}{K_m^{(2)}} \\ &+ \frac{(\sigma^{(2)} - \sigma_{DEM}^*)^2}{K^{(2)} + \frac{4}{3}\mu_{DEM}^*} \geq \frac{\sigma^{(2)} - \phi^{(2)}}{K_m^{(2)}}.\end{aligned} \tag{128}$$

The two constituent version of (127) for DEM follows from (128) using the analysis of Appendix A. Equation (127) does not provide an optimal (tight) bound on K_ϕ^*. To obtain a better bound, we solve (61) and (67) for K_ϕ^*, so that

$$K_\phi^* = \frac{\phi^*}{\sigma^*/K_s^* - \langle[\sigma(\mathbf{x}) - \phi(\mathbf{x})]/K_m(\mathbf{x})\rangle - \langle[\sigma(\mathbf{x}) - \sigma^*]^2/[K(\mathbf{x}) + \frac{4}{3}\mu^*]\rangle}. \tag{129}$$

Then, the tightest possible bounds on K_ϕ^* are obtained from (129) by substituting the optimal bounds on K_s^* and σ^* found previously. The resulting inequalities are not illuminating however, so we will not present them here. Furthermore, for special values of the constituents' parameters, it is possible for K_ϕ^* to take virtually any real value $-\infty < K_\phi^* < +\infty$. This point will be clarified in the remaining paragraphs of this section and in more detail in the next section.

We will conclude this section by presenting a rigorous bound on the actual effective constants (not just a bound on the estimates). Recalling that the coefficients H and M are coefficients in the quadratic form for the internal energy (27), we then have in general that both of these coefficients must be non-negative in all circumstances, otherwise the fluid-saturated porous material would collapse due to an inherent mechanical instability. Using (36) and (37) together with the implicit assumption that K_s^* and K_ϕ^* are independent of the pore fluid modulus K_f, the inequality $M \geq 0$ from mechanical stability is sufficient to show, in general, that

$$\frac{\sigma^*}{K_s^*} - \frac{\phi^*}{K_\phi^*} \geq 0. \tag{130}$$

This follows because, if (130) were not true for some porous material, then with a nearly incompressible fluid in the pores C and M would become negative. Inequality (130) guarantees that the porous frame is mechanically stable against collapse, regardless of the properties of the saturating fluid.

Now there are two ways in which (130) can be satisfied. First, it will be trivially satisfied if $K_s^* > 0$ and $K_\phi^* < 0$. From the definition (41), we see that K_ϕ^* could be negative if an increase in the fluid pressure p_f at constant differential pressure p_d resulted in an increase in the pore volume. Although negativity of K_ϕ^* may seem odd, it does not appear to be physically impossible. Second (and we expect the more common situation), (130) will be satisfied if $K_s^* > 0$, $K_\phi^* > 0$, and

$$K_\phi^* \geq \phi^* K_s^*/\sigma^* \geq \phi^* K_s^*. \tag{131}$$

Of course, Gassmann's equation automatically satisfies (131), since in that case $K_\phi^* = K_s^* \geq \phi^* K_s^*$ is satisfied trivially.

Inequality (131) provides an elementary check on the consistency of experimental data. For example, if we consider the data presented for Berea sandstone by *Green and Wang (1986)*, we find that in one case ($p_d = 2.0$ MPa) the data do satisfy the inequality $K_\phi^* \geq \phi^* K_s^*$, while in the other two cases ($p_d = 0$ and $p_d = 0.9$ MPa) the data violate the inequality. This lack of consistency may be due to the fact that Green and Wang used values of the compressibilities $1/K^*$ and $1/K_s^*$ taken from the literature, while their own measurements on different samples were used to estimate $1/K_\phi^*$. The consistency relation (131) is a sensitive check on the input parameters to Brown and Korringa's equations; *Green and Wang (1986)* noted some small ($< 5\%$) discrepancies between the calculated and measured values of the pore pressure buildup coefficient, but were unaware of the demonstrable inconsistencies in the data they used.

Monotonicity properties of the formulas for K_s^* and K_ϕ^* at fixed values of volume fraction and as a function of K^* are discussed in Section 10. These monotonicity properties could also be used to generate inequalities for the constants.

9 Special Analytical Results

In this section, we will show that, in some special cases, the DEM formulas as well as the CPA and ATA can be solved exactly. These results are not expected to have much direct practical importance, but they do provide very convenient checks on the numerical methods used to solve the equations.

Starting again with the solid as host and adding the fluid, we have $1/M^*(0) = 1/M^{(1)} = 0$, $\sigma^*(0) = \sigma^{(1)} = 0$, $K^*(0) = K^{(1)} = K_m$, $\mu^*(0) = \mu^{(1)} = \mu_m$, $M^{(2)} = K_f$, $\sigma^{(2)} = 1$, $K^{(2)} = 0$, and $\phi = y$. Substituting these values into (19) and (20), we find

$$(1-y)\frac{d}{dy}\left[\frac{1}{K^*(y)}\right] = \frac{1}{K^*(y)} + \frac{1}{\frac{4}{3}\mu^*(y)}, \tag{132}$$

and

$$(1-y)\frac{d}{dy}\left[\frac{1}{\mu^*(y)}\right] = \frac{1}{\mu^*(y)} + \frac{1}{F^*(y)}. \tag{133}$$

The resulting elastic constants satisfy a power law relation if and only if $K_m \equiv \frac{4}{3}\mu_m = \frac{4}{3}F_m$ (which is approximately true for glass beads), in which case

$$K^*(y)/K_m = \mu^*(y)/\mu_m = (1-y)^2 = (1-\phi)^2. \tag{134}$$

Another exact result is obtained from (19) and (20) when $K^{(1)} = \frac{4}{3}\mu^{(1)}$ and $K^{(2)} = \frac{4}{3}\mu^{(2)}$ (*i.e.*, when Poisson's ratio takes the constant value of 1/5). Then, $K^*_{DEM}(y)/K^{(2)} = \mu^*_{DEM}(y)/\mu^{(2)} \equiv f(y)$, where the function $f(y)$ satisfies the equation

$$(1-y)\frac{d}{dy}f(y) = \frac{2f(y)\,[1-f(y)]}{1+f(y)}. \tag{135}$$

The existence of two distinct solutions to the nonlinear ordinary differential equation (135) follows easily from the observation that $f(y)$ and $1/f(y)$ both satisfy the same equation. Equation (135) can be integrated analytically to yield

$$f_{\pm}(y) = 1 + \psi(y)/2 \pm \left[\psi(y) + \psi(y)^2/4\right]^{\frac{1}{2}}, \tag{136}$$

where

$$\psi(y) = \frac{[1-f(0)]^2}{f(0)}(1-y)^2, \tag{137}$$

with $f(0) = K^{(1)}/K^{(2)} > 0$. Only one of these solutions satisfies the initial condition; thus, the minus sign in (136) holds when $f(0) < 1$ and the plus sign when $f(0) > 1$. Furthermore, note that $f_+ \cdot f_- = 1$.

The CPA can also be solved exactly when the constituents satisfy $K^{(1)} = \frac{4}{3}\mu^{(1)}$ and $K^{(2)} = \frac{4}{3}\mu^{(2)}$. Then, the equations for the bulk and shear moduli are proportional since $K^* = \frac{4}{3}\mu^* = \frac{4}{3}F^*$. Solving

$$\frac{1}{2K^*} = \left\langle \frac{1}{K(\mathbf{x}) + K^*} \right\rangle \tag{138}$$

for K^*, we find that

$$K^*(y) = \theta(y) + \left[\theta^2(y) + K^{(1)}K^{(2)}\right]^{\frac{1}{2}}, \tag{139}$$

where

$$\theta(y) = \tfrac{1}{2}(K^{(1)} - K^{(2)})(1-2y). \tag{140}$$

At $y = \frac{1}{2}$, the CPA gives the geometric mean $K^*(\frac{1}{2}) = (K^{(1)}K^{(2)})^{\frac{1}{2}}$. To compare this result with that of DEM, we can prove (using the general fact for positive f that $[1+f(0)] \geq 2f^{\frac{1}{2}}(0)$ and carrying through some tedious algebra) that

$$f_-(\tfrac{1}{2}) \leq f^{\frac{1}{2}}(0). \tag{141}$$

Since $K^* = f^{\frac{1}{2}}(0)K^{(2)}$ is the geometric mean, (141) shows that the DEM gives a lower value of the effective constant than the CPA at $y = \frac{1}{2}$ when $f(0) < 1$. The fact that $f_+(y) = 1/f_-(y)$ together with (141) then shows that the DEM gives a higher value of the effective constant than CPA at $y = \frac{1}{2}$ when $f(0) > 1$, since

$$f_+(\tfrac{1}{2}) \geq f^{-\frac{1}{2}}(0). \tag{142}$$

The exact results (134) and (136), although very special and therefore unlikely to apply in real problems, are very useful checks on our numerical integration scheme. We have chosen to use a Runge-Kutta integration scheme *(Hildebrand, 1956)*, to improve the accuracy and thereby avoid the necessity of using very small steps in y. Likewise, the exact formula (139) serves as a check on the convergence of our iteration scheme for the CPA.

10 Numerical Comparisons

Detailed discussion of numerical methods for ATA and CPA were presented by *Berryman (1980)*. A brief discussion of methods for DEM will be followed by numerical comparisons among the various approaches.

Since the formulas (77) and (91) are known to be exact for two component composite porous media, the only differences that will occur in the computations of the approximations aarise from the differences in the computed values of the frame bulk modulus K^*. It follows from (77) that

$$\frac{d\sigma^*}{dK^*} = \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\right), \tag{143}$$

or from (80) that

$$\frac{d(1/K_s^*)}{d(1/K^*)} = \frac{1/K_m^{(1)} - 1/K_m^{(2)}}{1/K^{(1)} - 1/K^{(2)}} \equiv -\gamma, \tag{144}$$

so σ^* is a monotonic function of K^* and $1/K_s^*$ is a monotonic function of $1/K^*$. Whether these are monotonically increasing or decreasing depends on the signs of the derivatives in (143) and (144). Similarly, it follows from (91) that

$$\phi^* \frac{d(1/K_\phi^*)}{d(1/K^*)} = -\gamma(1+\gamma). \tag{145}$$

Also notice that the results (143) and (144) are independent of the volume fractions of the constituents, while (145) is dependent on both the volume fractions and the porosities of the constituents.

For fixed values of the volume fractions, we see that σ^* is a linear function of K^*, while K_s^* is hyperbolic in K^*. The behavior of K_ϕ^* implied by (145) is generally hyperbolic in K^*, but it is also complicated by the fact that in some (rare) cases K_ϕ^* can be negative. If (again for fixed values of the volume fractions) there exists a value of K^* for which the right hand side of (91) vanishes, then as K^* varies near this value from slightly smaller to slightly larger values K_ϕ^* will jump from $-\infty$ to $+\infty$ (or vice versa) and then continue to vary monotonically as K^* continues to increase. Clearly, either a positively or negatively infinite value of K_ϕ^* has the same physical significance: the pore space becomes incompressible for this special value of K^*. Except for this unusual case, the monotonicity properties of the moduli are quite simple. Depending on the signs of the slopes in (144) and (145), we expect predictable relationships between estimates of K_s^* and K_ϕ^* for different estimates of K^*.

We will now consider some particular examples.

10.1 Analytical model

Our first example will be the artificial model introduced in Section 9 that can be solved analytically. The defining requirement for this model is that the materials constituting the porous composite have a constant Poisson's ratio of 1/5. This condition is equivalent to requiring $K^{(a)} = \frac{4}{3}\mu^{(a)}$, $K^{(b)} = \frac{4}{3}\mu^{(b)}$, etc. Table 1 lists four such artificial materials. Units in Table 1 are arbitrary, but could be taken to be 10 GPa; then, material a behaves like a porous glass.

The main purpose of this exercise is to check the numerical accuracy of our codes since the exact formulas can be directly compared to the results of the integration and iteration performed for the DEM and the CPA, respectively. We find that the two calculations of the DEM agree in all cases to four significant figures when we compute the DEM in steps $\Delta f = 0.01$ of the increment in volume fraction.

Tables 2, 3, and 4 give details of three sample calculations, each succeeding case with more extreme contrast between the constituents' parameters than the one that preceded it. In each Table, the columns labeled DEM^- and DEM^+ refer to DEM calculations starting with the low modulus material or the high modulus material as host, respectively. The CPA is symmetric in the constituents so only one result is obtained, assuming as we must here that the inclusions are spherical. We see in general that the values of K^* satisfy

$$K^*_{DEM^-} \leq K^*_{CPA} \leq K^*_{DEM^+}, \tag{146}$$

as we expect from the preceding analysis. But, note that (146) has only been rigorously proven for $f = 0.5$ at present. Nevertheless, (146) holds for this analytical model without exception empirically. Given (146) the preceding analysis (144) in this section shows that

$$K_s^*(DEM^-) \leq K_s^*(CPA) \leq K_s^*(DEM^+), \tag{147}$$

as is again observed to hold in the examples. No such simple relation holds or can be expected to hold for the bulk parameter K_ϕ^*. Indeed the observed variations for this parameter are neither monotonic nor bounded by the extreme values of K_m. As the contrast increases, it is also observed that negative values of K_ϕ^* occur. As explained previously, this behavior is actually required by the mechanical and thermodynamical constraint (130).

10.2 Mixture of two sands

Next we consider a mixture of two types of sandstone whose properties are listed in Table 5. We chose the type-A sand to be the same as one considered by *Korringa and Thompson (1977)*, and type-B is a modified version of their type-B. *Korringa and Thompson (1977)* used essentially the same method as the CPA we use here to arrive at their frame constants. Korringa and Thompson also used more general CPA formulas based on nonspherical pore shapes to generate other examples; this approach is presumably similar to that discussed in more detail by Berryman (1980). We had to modify the type-B sand because Korringa and Thompson treat two sands with the same grain bulk modulus K_m. But we know that the results for K_s^* and K_ϕ^* will then be trivial for all values of the mixing fractions, since Gassmann's equations requires that $K_s^* = K_\phi^* = K_m$ in this situation.

Table 6 shows the results of the calculations. We see that with only moderate contrasts in the material properties all the moduli behave in a very regular, monotonic manner as a function of the volume fraction. In addition to the relations (146) and (147) which are again observed to hold for these synthetic data, now we also observe that

$$K_\phi^*(DEM^-) \le K_\phi^*(CPA) \le K_\phi^*(DEM^+) \tag{148}$$

and furthermore that

$$K_s^* \le K_\phi^* \tag{149}$$

within each class of approximation. The results of the previous subsection show that (148) and (147) are not general, but nevertheless we expect them to hold whenever the properties of the constituents are not too dissimilar.

10.3 Clayey sandstone

The presence of clay in sandstones has been shown to have a significant effect on the compressional and shear wave speeds of such rocks *(Han, Nur, and Morgan, 1986)*, and to have a strong effect on the slow wave propagation and fast wave attenuation as well *(Klimentos and McCann, 1988; 1990)*. The present results together with the exact results of *Berryman and Milton (1991)* may be used to derive formulas for clayey sandstone, depending only on known quantities and the overall bulk modulus K^* of the composite. This calculation is a two step process.

First, consider a porous clay composed of a single type of solid grain, but with a substantial amount of void space in the interstices among the mineral grains making up the clay and also with occasional large voids. We will call this the type-a composite. Such a material may be treated by taking the porous clay as the first constituent (type-1) and the large voids as the second constituent so that $\phi^{(2)} = \sigma^{(2)} = 1$ and $K^{(2)} = 0$. Then, $[f^{(1)} + f^{(2)}]\phi^{(a)} = f^{(1)}\phi^{(1)} + f^{(2)}$ and it is not difficult to show that (77) and (91) reduce to $K_s^{(a)} = K_\phi^{(a)} = K_m^{(1)}$, also in agreement with Gassmann. We could have anticipated this result since, by assumption, there is only one type of solid grain in such a clay with large voids.

Second, consider a clayey sandstone in which sand grains are imbedded in a matrix of clay including some large pores (see Figure 5). Thus, we have a mixture of types a and b where we computed the coefficients for type-a in the preceding paragraph and the coefficients for the solid grains (type-b) are given by $\phi^{(b)} = \sigma^{(b)} = 0$ and $K^{(b)} = K_m^{(b)}$. The overall results for the clayey sandstone as determined by (77) and (91) are then given by

$$\sigma^* = \sigma^{(a)} \frac{K^* - K_m^{(b)}}{K^{(a)} - K_m^{(b)}} \tag{150}$$

and

$$\frac{\phi^*}{K_\phi^*} = \frac{\sigma^*}{K_s^*} - f^{(a)} \frac{\sigma^{(a)} - \phi^{(a)}}{K_m^{(a)}} - (f^{(a)}\sigma^{(a)} - \sigma^*)(\frac{\sigma^{(a)}}{K^{(a)} - K_m^{(b)}}). \tag{151}$$

In these formulas, $K_m^{(a)} = K_m^{(1)}$, the volume fractions satisfy $f^{(1)} + f^{(2)} = f^{(a)} = 1 - f^{(b)}$, and the porosities $\phi^* = f^{(a)}\phi^{(a)} = f^{(1)}\phi^{(1)} + f^{(2)}$.

To obtain some insight into the behavior of (150) and (151), suppose that the clay is composed of nearly incompressible grains so that $K_m^{(a)} \to \infty$ and $\sigma^{(a)} \to 1$. Then, if the bulk modulus of the clay/void mixture is much smaller than the bulk modulus of the sand grains, we have $K^{(a)} \ll K_m^{(b)}$ and

$$K_s^* \simeq K_m^{(b)} \tag{152}$$

follows from (150). Similarly, letting $K_m^{(a)} \to \infty$ and $K^{(a)} \to 0$ in (151), we have

$$K_\phi^* \simeq \phi^{(a)} K_m^{(b)}. \tag{153}$$

There is an implicit assumption in the derivations of (152) and (153) that $K^{(a)} \ll K^* \simeq K_m^{(b)}$; if instead $K^* \simeq K^{(a)}$, then $K_s^* \to \infty$ and either $K_\phi^* \to \infty$ or K_ϕ^* takes negative values in order to satisfy the physical requirement (130).

We also note that both of these results (152) and (153) are essentially independent of both K^* and $K^{(a)}$ as long as the assumption that $K^{(a)} \ll K_m^{(b)} \simeq K^*$ holds. Both coefficients are strong functions of K^* if $K^* \to K^{(a)}$.

Table 7 shows the properties of a clay and a sand (Kayenta). Both of these sets of values correspond to examples discussed by *McTigue (1986)* and *Palciauskas and Domenico (1989).*

Table 8 shows the results of our calculations. The sand is assumed to occupy the fixed fraction $f^{(b)} = 0.8$, while the large void fraction $f^{(2)} = 1 - f^{(1)}$ varies with the clay fraction. For this example, the clay fraction varies from 0.0 to 0.2. First, the effective constant $K^{(a)}$ is computed for the clay/void composite. Then, K^* is computed for the sand/clay/void composite. The results substantially agree with the formulas (152) and (153). For example, notice that for this calculation $0.4 \le \phi^{(a)} \le 1$, whereas $0.08 \le \phi^*/\sigma^* \le 0.2/0.34 = 0.59$. Thus, the requirement from (130) that $K_\phi^* \ge \phi^* K_s^*/\sigma^*$ reduces to $\phi^{(a)} \ge \phi^*/\sigma^*$, which it can be see from the values in Table 8 is easily satisfied for all volume fractions.

11 Conclusions

All three of the approximations considered (ATA, CPA, DEM) give estimates of the coefficients that are in agreement with the general results of *Brown and Korringa (1975)* when the scatterers are spheres of Biot material satisfying *Gassmann's (1951)* equation. All three approximations are identical when the shear modulus is constant throughout the porous composite and agree with Hill's exact formula *(Hill, 1963)* for the effective bulk modulus K^*. It is reasonable to conjecture in general that the formulas obtained for the coefficients in Biot's equations are exact in the limit of uniform shear modulus. As further evidence in favor of this conjecture, it is known that the formulas are also exact when the composite porous material contains only two porous constituents with the same shear modulus *(Berryman and Milton, 1991).*

Having proven that all three of these approximations have the proper dependence on the fluid bulk modulus, we were able to eliminate the fluid from further consideration and concentrate on the more difficult problem of evaluating the various moduli of the composite porous frame.

The ATA is the easiest approximation to compute since the formulas are explicit. The other two approximations require either iteration or integration for their solution. However, ATA is usually an extreme approximation; with only two constituents, ATA provides either an upper or lower bound on the actual values of the effective constants. Because of the inherent path dependence of the integration scheme, the DEM is difficult to generalize for multiple constituents [but see *Norris (1985)* for a discussion], while the ATA and CPA are easily generalized as shown here.

When only two constituents are present in the porous composite and these constituents fit perfectly to fill the volume, exact formulas are now known for all the moduli in terms of the effective bulk modulus of the porous frame *(Berryman and Milton, 1991)*. All three approximations found in the present paper agree in form with these exact results, but they may nevertheless produce differing values because the estimates of the frame bulk modulus K^* generally differ in the various approximations. The ATA and CPA are also more versatile than the exact formulas since they provide approximate formulas for problems with two or more types of spherically shaped constituents, whereas the exact results are valid only for composites having two constituents of arbitrary shape.

The results presented in this paper are based on scattering coefficients for spherical inclusions, thus limiting their applicability. Scattering coefficients for non-spherical inclusions of Biot material are not known at present, so the generalization to composites containing other inclusion shapes must await the calculation of these coefficients.

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A Analysis of DEM Equations

A recurring equation in analysis of DEM formulas is

$$(1-y)\frac{d}{dy}[\chi(y)] + \chi(y) = h(y), \tag{154}$$

for $0 \le y \le 1$ with the initial condition $\chi(0) = G$. If the right hand side is constant [say $h(y) = H$], then the solution is

$$\chi_H(y) = (1-y)G + yH. \tag{155}$$

If the right hand side is positive $h(y) \ge 0$, then (154) may be analyzed by considering

$$(1-y)\frac{d}{dy}[\chi(y)] + \chi(y) \ge 0, \tag{156}$$

which can be integrated to yield

$$\chi(y) \ge (1-y)\chi(0) = (1-y)G. \tag{157}$$

Now, if the right hand side of (154) is less than some constant so $h(y) \le H$, it is straightforward to see that

$$(1-y)\frac{d}{dy}[\chi_H(y) - \chi(y)] + [\chi_H(y) - \chi(y)] = H - h(y) \ge 0. \tag{158}$$

Noting that $\chi_H(0) - \chi(0) = 0$ and using the result (157), we finally have

$$\chi(y) \le \chi_H(y). \tag{159}$$

This inequality is used repeatedly in the analysis of DEM equations. In particular, it shows that $\langle 1/K(\mathbf{x})\rangle^{-1} \le K^*_{DEM}(y) \le \langle K(\mathbf{x})\rangle$.

Table 1. Four materials with Poisson's ratio equal to 1/5. Units are arbitrary.

material	ϕ	K	K_m	μ	μ_m
a	0.35	1.00	4.00	0.7500	3.00
b	0.35	0.20	0.80	0.1500	0.60
c	0.35	0.10	0.40	0.0750	0.30
d	0.35	0.01	0.04	0.0075	0.03

Table 2. Computations for a composite of types a and b from Table 1. Units are arbitrary.

fraction	DEM^-			CPA			DEM^+		
$f^{(a)} = 1 - f^{(b)}$	K^*	K^*_s	K^*_ϕ	K^*	K^*_s	K^*_ϕ	K^*	K^*_s	K^*_ϕ
0.0	0.20	0.80	0.80	0.20	0.80	0.80	0.20	0.80	0.80
0.2	0.26	1.06	1.21	0.27	1.07	1.25	0.29	1.14	1.48
0.5	0.42	1.68	2.39	0.45	1.79	2.93	0.48	1.90	3.73
0.8	0.70	2.80	3.99	0.75	2.99	4.94	0.76	3.03	5.16
1.0	1.00	4.00	4.00	1.00	4.00	4.00	1.00	4.00	4.00

Table 3. Computations for a composite of types a and c from Table 1. Units are arbitrary.

fraction	DEM^-			CPA			DEM^+		
$f^{(a)} = 1 - f^{(c)}$	K^*	K^*_s	K^*_ϕ	K^*	K^*_s	K^*_ϕ	K^*	K^*_s	K^*_ϕ
0.0	0.10	0.40	0.40	0.10	0.40	0.40	0.10	0.40	0.40
0.2	0.14	0.57	0.70	0.15	0.58	0.75	0.18	0.70	1.41
0.5	0.27	1.06	2.26	0.32	1.26	8.15	0.38	1.50	-6.83
0.8	0.57	2.28	7.17	0.69	2.74	-53.02	0.70	2.82	-25.46
1.0	1.00	4.00	4.00	1.00	4.00	4.00	1.00	4.00	4.00

Table 4. Computations for a composite of types a and d from Table 1. Units are arbitrary.

fraction	DEM^-			CPA			DEM^+		
$f^{(a)} = 1 - f^{(d)}$	K^*	K^*_s	K^*_ϕ	K^*	K^*_s	K^*_ϕ	K^*	K^*_s	K^*_ϕ
0.0	0.01	0.04	0.04	0.01	0.04	0.04	0.01	0.04	0.04
0.2	0.02	0.06	0.09	0.02	0.07	0.10	0.06	0.23	-0.15
0.5	0.04	0.15	-4.01	0.10	0.40	-0.11	0.26	1.06	-0.11
0.8	0.17	0.70	-0.35	0.61	2.44	-0.20	0.65	2.59	-0.22
1.0	1.00	4.00	4.00	1.00	4.00	4.00	1.00	4.00	4.00

Table 5. Material constants for two representative sandstones.

material	ϕ	K_{CPA}(GPa)	K_m(GPa)	μ_{CPA}(GPa)	μ_m(GPa)
sand A	0.3	17.76	40.0	15.62	40.0
sand B	0.3	11.44	30.0	8.07	20.0

Table 6. Computations for a sandstone mixture of types A and B from Table 5. Units for bulk moduli are GPa.

fraction	DEM^-			CPA			DEM^+		
$f^{(A)}$	K^*	K_s^*	K_ϕ^*	K^*	K_s^*	K_ϕ^*	K^*	K_s^*	K_ϕ^*
0.0	11.44	30.00	30.00	11.44	30.00	30.00	11.44	30.00	30.00
0.2	12.48	31.86	32.27	12.48	31.86	32.30	12.51	31.91	32.41
0.5	14.23	34.80	35.56	14.26	34.84	35.68	14.29	34.89	35.80
0.8	16.26	37.89	38.46	16.29	37.93	38.58	16.30	37.95	38.61
1.0	17.76	40.00	40.00	17.76	40.00	40.00	17.76	40.00	40.00

Table 7. Material constants for a clay and a sandstone (Kayenta).

material	ϕ	K(GPa)	K_m(GPa)	μ(GPa)	μ_m(GPa)
clay	0.4	0.0625	50.0	0.001	0.002
sand	0.0	37.88	37.88	29.0	29.0

Table 8. Computations for a clayey sandstone composed of the materials in Table 7. The sand is assumed to occupy the fixed fraction $f^{(b)} = 0.8 = 1 - f^{(a)}$, while the large void fraction $f^{(2)} = f^{(a)} - f^{(1)}$. Units of bulk moduli are GPa.

clay	*porosity*	DEM^-			CPA			DEM^+		
$f^{(1)}$	ϕ^*	K^*	K_s^*	K_ϕ^*	K^*	K_s^*	K_ϕ^*	K^*	K_s^*	K_ϕ^*
0.00	0.200	0.000	37.88	37.88	22.84	37.88	37.88	24.95	37.88	37.88
0.02	0.188	0.001	38.25	39.28	22.84	37.88	37.30	24.95	37.88	37.30
0.04	0.176	0.002	38.29	38.97	22.84	37.88	36.67	24.95	37.88	36.67
0.06	0.164	0.005	38.33	38.62	22.84	37.88	35.97	24.95	37.88	35.97
0.08	0.152	0.008	38.38	38.25	22.84	37.88	35.19	24.95	37.88	35.19
0.10	0.140	0.012	38.44	37.92	22.84	37.88	34.32	24.95	37.88	34.32
0.12	0.128	0.019	38.52	37.66	22.84	37.88	33.33	24.95	37.88	33.34
0.14	0.116	0.028	38.63	37.58	22.84	37.88	32.22	24.95	37.88	32.23
0.16	0.104	0.043	38.79	37.92	22.84	37.88	30.95	24.95	37.88	30.96
0.18	0.092	0.077	39.04	39.42	22.84	37.88	29.49	24.95	37.88	29.50
0.20	0.080	0.352	39.57	45.69	22.89	37.89	27.84	24.99	37.89	27.83

Table 9. Same as Table 8 except that the sand is assumed to occupy the fixed fraction $f^{(b)} = 0.6 = 1 - f^{(a)}$. Units of bulk moduli are GPa.

clay	*porosity*	DEM^-			CPA			DEM^+		
$f^{(1)}$	ϕ^*	K^*	K_s^*	K_ϕ^*	K^*	K_s^*	K_ϕ^*	K^*	K_s^*	K_ϕ^*
0.00	0.400	0.0000	37.88	37.88	7.67	37.88	37.88	13.74	37.88	37.88
0.04	0.376	0.0002	39.40	41.49	7.67	37.88	37.30	13.74	37.88	37.30
0.08	0.352	0.0006	39.54	41.45	7.67	37.88	36.67	13.74	37.88	36.67
0.12	0.328	0.0012	39.67	41.37	7.67	37.88	35.97	13.75	37.88	35.97
0.16	0.304	0.0021	39.82	41.33	7.67	37.88	35.19	13.75	37.88	35.19
0.20	0.280	0.0034	39.98	41.36	7.67	37.88	34.32	13.75	37.88	34.32
0.24	0.256	0.0054	40.19	41.54	7.67	37.88	33.34	13.75	37.88	33.34
0.28	0.232	0.0086	40.44	41.96	7.67	37.88	32.23	13.75	37.88	32.23
0.32	0.208	0.0143	40.76	42.83	7.68	37.88	30.96	13.75	37.88	30.96
0.36	0.184	0.0287	41.18	44.60	7.69	37.89	29.51	13.75	37.88	29.50
0.40	0.160	0.1622	41.77	48.48	7.79	37.94	27.97	13.81	37.91	27.86

Figure Captions

Fig. 1. The average T-matrix approximation (ATA) treats one of the constituents (type-1 here) as the host and sets the single-scattering contributions from the inclusions equal to the scattering from a sphere of the effective composite.

Fig. 2. The coherent potential approximation (CPA) treats the composite itself (type-*) as the host and sets the single-scattering contributions from all the inclusions equal to zero.

Fig. 3. The differential effective medium (DEM) approach treats the last computed effective constant $K(y)$ as the host and sets the single-scattering contributions for infinitesimal concentrations of inclusions equal to the scattering from a sphere of the next effective constant $K(y + dy)$. After taking the limit $dy \to 0$, a differential equation for the effective constants is obtained.

Fig. 4. The generalized ATA uses an arbitrary host (type-†). For comparison with Gassmann's equation, the derivation is based on having pure solid (s) and pure fluid (f) inclusions.

Fig. 5. A clayey sandstone contains solid sand grains, porous clay adsorbed on the grain surfaces, and large pores which in some cases (in 3-D) provide a connected pathway for fluid transport.