Mathematical Example of the Dispersive Effect

Now we will use our newfound physical understanding to help us construct a simple mathematical example showing how the effect arises in our equations.

New coefficients must be added to produce a change in volume under an applied shear stress, and (because of reciprocity) others that produce a change in shear strain under a compressional load (in this case pore pressure). The matrix must also be locally anisotropic.

Mathematical Example (2)

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \\ e_{23} \\ e_{31} \\ e_{23} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & -\beta^{(1)} & & & \\ S_{12} & S_{11} & S_{13} & -\beta^{(1)} & & & \\ S_{13} & S_{13} & S_{33} & -\beta^{(3)} & & & \\ -\beta^{(1)} & -\beta^{(1)} & -\beta^{(3)} & \gamma & & & -\omega \\ & & & & \frac{1}{G_t} & & \\ & & & & & \frac{1}{G_t} & \\ & & & -\omega & & & \frac{1}{G_{dr}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}$$

Mathematical Example (3)

So now we use the standard (Gassmann) trick to eliminate both the liquid increment ζ and the liquid pressure p_f by setting $\zeta = 0$, solving for p_f, which is then given by

$$p_f = -\frac{\beta}{\gamma}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) - \frac{\omega}{\gamma}\sigma_{12},$$

and substituting this back into the previous set of equations.

The result is:

Mathematical Example (4a)

$$
\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{23} \end{pmatrix} =
\begin{pmatrix}
S_{11}^{sat} & S_{12}^{sat} & S_{13}^{sat} & & & \\
S_{12}^{sat} & S_{11}^{sat} & S_{13}^{sat} & & & \\
S_{13}^{sat} & S_{13}^{sat} & S_{33}^{sat} & & & \\
& & & \frac{1}{G_t} & & \\
& & & & \frac{1}{G_t} & \\
& & & & & \frac{1}{G^{sat}}
\end{pmatrix}
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} =
$$

Mathematical Example (4b)

$$
\begin{pmatrix}
S_{11} & S_{12} & S_{13} & & & \\
S_{12} & S_{11} & S_{13} & & & \\
S_{13} & S_{13} & S_{33} & & & \\
 & & & \frac{1}{G_t} & & \\
 & & & & \frac{1}{G_t} & \\
 & & & & & \frac{1}{G_{dr}}
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12}
\end{pmatrix}
$$

Mathematical Example (4c)

$$
-\frac{1}{\gamma}
\begin{pmatrix}
(\beta^{(1)})^2 & (\beta^{(1)})^2 & \beta^{(1)}\beta^{(3)} & & & \beta^{(1)}\omega \\
(\beta^{(1)})^2 & (\beta^{(1)})^2 & \beta^{(1)}\beta^{(3)} & & & \beta^{(1)}\omega \\
\beta^{(1)}\beta^{(3)} & \beta^{(1)}\beta^{(3)} & (\beta^{(3)})^2 & & & \beta^{(3)}\omega \\
 & & & 0 & & \\
 & & & & 0 & \\
\beta^{(1)}\omega & \beta^{(1)}\omega & \beta^{(3)}\omega & & & \omega^2
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12}
\end{pmatrix}.
$$

Mathematical Example (5)

The important result we obtain shows that the shear modulus for the saturated system now contains a term

$$G^{sat} = \frac{G_{dr}}{1 - G_{dr}\omega^2/\gamma},$$

depending on the mechanical properties of the liquid through the coefficients ω and γ.

Note that the saturated shear modulus is always greater than the drained modulus for nonvanishing ω.