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Dispersion in Poroelastic Systems

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ABSTRACT

In a series of laboratory ultrasonic wave propagation experiments on porous glass immersed in a water bath, Plona showed 20 years ago that many of the predictions of the Biot-Gassmann theory of wave propagation in poroelastic media could be observed for these materials. Predictions included the existence of a second (slow) compressional wave, the magnitude of the slow-wave velocity and attenuation, and the resulting enhanced attenuation of the faster compressional wave that corresponds to the usual viscoelastic mode in such media. Chin *et al.*, together with many others by the present time, have shown that the theory explains these data remarkably well.

On the other hand, when the same theory is applied to porous rock, the agreement between theory and experiment is usually not so satisfactory. Slow waves may not be observed at all in rocks, and the velocity and attenuation of the measured compressional waves may not agree very well either quantitatively or even qualitatively in some cases; for example, the dependence of the attenuation on frequency and the corresponding frequency dispersion of the wave velocity is a commonly quoted discrepancy between the theory and the experiments.

This paper addresses one key source of the discrepancy between theory and experiment, *i.e.*, the fact that Gassmann's equations for the bulk and shear moduli predict that the shear modulus is independent of the saturating fluid properties, whereas it is observed that at high enough frequencies the shear modulus can in fact depend on the fluid content in many porous materials. One clue to understanding this behavior comes from effective medium theory, which shows in particular that the shear modulus does in fact depend on the fluid properties in many circumstances. In comparison to values predicted by effective medium theory, Gassmann's equations predict different, smaller values for the bulk and shear moduli of porous media. Sorting through these apparent (but not actual) disagreements among theory and theory, and theory and experiment will be the main thrust of the paper.

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1 Introduction

Velocity dispersion is an inevitable consequence of attenuation in real systems. For example, Aki and Richards[1] show how causality (the requirement that the effect must always follow — never precede — the cause) leads to the Kramers-Kronig relations between the real and imaginary parts of the wavenumber, or equivalently, the velocity of a wave propagating through a dispersive medium. The velocity acquires an inherent frequency dependence in such systems, and this frequency dependence is what we mean by "dispersion" in the present context. We will assume the reader is familiar with this result, and not elaborate here.

The consequences of dispersion are very important in seismology and acoustics, because dispersion makes reconciliation of field data with laboratory data much more difficult than it would be if there were no dispersion. The exploration seismic band is from about 10 – 100 Hz, while earthquake seismology usually considers frequencies from about 10 Hz down. Well-logging tools usually work in the high sonic range, from about 1–20 kHz. On the other hand, laboratory experiments are most often performed in the ultrasonic range from about 100 kHz to 2 MHz. So the gap in frequency between laboratory and field data can be as high as five or six orders of magnitude, but efforts to produce laboratory data below the ultrasonic range have been carried through successfully using resonance bar methods, forced oscillation methods, and some other methods. When they are available, these types of laboratory data are often the most useful ones to us because we can make direct comparisons between field and laboratory systems at the same frequencies. But often we do not have this luxury, so we need to understand both the mechanism (or possibly mechanisms) and the consequences of dispersion in these earth or porous rock systems in order to aid the interpretation (and inversion) of field data.

Probably the most common choice of theory used to try to explain poroelastic data is Biot's theory.[2] In a series of laboratory ultrasonic wave propagation experiments on porous glass immersed in a water bath, Plona[3] showed 20 years ago that many of the predictions of Biot's theory of wave propagation in poroelastic media could be observed for these materials. Predictions included the existence of a second (slow) compressional wave, the magnitude of the slow-wave velocity and attenuation, and the resulting enhanced attenuation of the faster compressional wave that corresponds to the usual viscoelastic mode in such media. Chin *et al.*,[4] Johnson *et al.*,[5] together with many others by the present time, have shown that the theory explains these and similar laboratory data on synthetic materials remarkably well.

On the other hand there are many examples of real earth materials for which Biot's theory does not seem to explain the dispersion very well and it would therefore be most useful to clarify what the physical issues are that limit the use of the theory. Various additions and corrections to Biot's theory have been attempted including treating the porous medium as granular,[6] treating the elastic medium as nonlinear,[7] treating the pore space as a double-porosity system[8] so that high permeability fractures and low permeability but high storage matrix porosity coexist in the theory, and considering the effects of both partial and patchy saturation.[9] Each of these approaches has something important to say about dispersion in poroelasticity systems. But it nevertheless remains difficult to explain some of the data.

The main purpose of this paper is therefore to clarify another of the outstanding questions about dispersion in poroelastic systems, such as those described by Biot's theory.[2] The work to be summarized was motivated in part by ongoing studies of shear velocity in partially saturated porous systems,[9] but partial saturation will not play any role in the discussion here. Our approach will be to reconsider a basic result in the theory, *i.e.*, Gassmann's result[10,11] that the effective shear modulus of the porous, fluid-saturated system is independent of the presence of fluid in the pores.

Section 2 will rederive a basic result in poroelasticity: Gassmann's equations for a fluid-saturated, closed system. Section 3 will present effective medium results for the same system. Section 4 will show that, even though these two approaches are both at least nominally low frequency methods, Gassmann's results differ from the effective medium theory results because they are quasi-static and actually pertinent to a lower frequency band than is allowed for or considered by the effective medium theory approach. Section 5 presents a detailed analysis showing how to reconcile the two results. Section 6 summarizes our conclusions.

2 Gassmann's Equations for Isotropic Porous Media

One of the most fundamental results in poroelasticity concerns the mechanical behavior of an enclosed, undrained, fluid-saturated system. Exact results for the effective bulk and shear moduli of such systems were derived almost 50 years ago by Gassmann.[10] The results are elementary in retrospect, but surely they are not trivial or obvious, as we shall see.

Gassmann's relations are receiving more attention as seismic data are increasingly used for reservoir monitoring. Correct interpretation of underground fluid flow from seismic data requires a quantitative understanding of the relationships among the velocity data and fluid properties in the form of fluid substitution formulas, and these formulas are very commonly based on Gassmann's equations. Nevertheless, confusion persists about the basic assumptions and the derivation of Gassmann's well-known equations in poroelasticity relating dry or drained bulk elastic constants to those for fluid saturated and undrained conditions. It is frequently stated, for example, but quite incorrect to say that Gassmann *assumes* the shear modulus is constant, *i.e.*, mechanically independent of the presence of the saturating fluid. This section of the paper clarifies the situation by repeating a brief derivation[11] of Gassmann's relations that emphasizes the true origin of the constant shear modulus *result*, while also clarifying the role played by the shear modulus in the derivation of the better understood result for the bulk modulus.

2.1 A derivation of Gassmann's equations

We now present a concise, but complete, derivation of Gassmann's famous results. For the sake of simplicity, the analysis of this section is limited to isotropic systems, but it can be generalized with little difficulty to anisotropic systems.[10,12,13] Gassmann's equations[10] relate the bulk and shear moduli of a saturated isotropic porous, monomineralic medium to the bulk and shear moduli of the same medium in the drained case and show furthermore that the shear modulus *must be* mechanically independent of the presence of the fluid. An important implicit assumption is that there is no chemical interaction between porous rock and fluid that affects the moduli. Gassmann's paper is concerned with the quasistatic (very low frequency) analysis of the elastic moduli and that is what we emphasize in this section also. Generalization to higher frequency effects and complications arising in wave propagation due to frequency dispersion will be discussed in the following sections of this paper.

In contrast to simple elasticity with stress tensor σ_{ij} and strain tensor e_{ij}, the presence of a saturating pore fluid in porous media requires the introduction of conjugate variables associated with the fluid. The pressure p_f in the fluid is the new field parameter that can be controlled. Allowing sufficient time (equivalent to a low frequency assumption) for global pressure equilibration will permit us to consider p_f to be a constant throughout the percolating (connected) pore fluid, while restricting the analysis to quasistatic processes. The change ζ in the amount of fluid mass contained in the pores is the new type of strain variable, measuring how much of the original fluid in the pores is squeezed out during the compression of the pore volume, while including the effects of compression or expansion of the pore fluid itself due to changes in p_f. It is most convenient to write the resulting equations in terms of compliances S_{ij} rather than stiffnesses C_{ij}, so for an isotropic porous medium in principal coordinates the basic equation to be considered takes the form:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{12} & -\beta \\ S_{12} & S_{11} & S_{12} & -\beta \\ S_{12} & S_{12} & S_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \tag{1}$$

The constants β and γ appearing in the matrix on the right hand side will be given precise definitions later. For now, they are defined implicitly by (1) as coefficients that can be measured by observing the changes in system strain caused by changes in system stress. It is important to write the equations this way rather than using the inverse relation in terms of the stiffnesses, because the compliances S_{ij} appearing in (1) are simply and directly related to the drained constants λ_{dr} and G_{dr} (the Lamé parameters for the isotropic porous medium in the drained case) in the same way they are related in normal elasticity (the matrix S_{ij} is just the inverse of the matrix C_{ij}), whereas the individual stiffnesses C_{ij}^{sat} (the "*sat*" superscript indicates a constant for the fully liquid saturated case) obtained by inverting the equation in (1) must contain coupling terms through the parameters β and γ that depend on the porous medium and fluid compliances.

Using the standard relations for the isotropic moduli (*e.g.*, Lakes[14]), we find that

$$S_{11} = \frac{1}{E_{dr}} = \frac{\lambda_{dr} + G_{dr}}{G_{dr}(3\lambda_{dr} + 2G_{dr})} = \frac{1}{9K_{dr}} + \frac{1}{3G_{dr}} \tag{2}$$

and

$$S_{12} = -\frac{\nu_{dr}}{E_{dr}} = \frac{1}{9K_{dr}} - \frac{1}{6G_{dr}}, \tag{3}$$

where the drained Young's modulus E_{dr} is defined in terms of the drained bulk modulus K_{dr} and shear modulus G_{dr} by the second equality of (2) and the drained Poisson's ratio is determined by

$$\nu_{dr} = \frac{\lambda_{dr}}{2(\lambda_{dr} + G_{dr})}. \tag{4}$$

The fundamental results of interest are Gassmann's equations, found by considering the saturated (and "undrained," meaning that the liquid is trapped and cannot escape from the volume) case such that

$$\zeta \equiv 0, \tag{5}$$

which — by making use of (1) — implies that the pore pressure must respond to external applied stresses according to

$$p_f = -\frac{\beta}{\gamma}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right). \tag{6}$$

Equation (6) is often called the "pore-pressure buildup" equation (see Skempton[15]). Then, using result (6) to eliminate both ζ and p_f from (1), we obtain

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \end{pmatrix} = \begin{pmatrix} S_{11}^{sat} & S_{12}^{sat} & S_{12}^{sat} \\ S_{12}^{sat} & S_{11}^{sat} & S_{12}^{sat} \\ S_{12}^{sat} & S_{12}^{sat} & S_{11}^{sat} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = \left[\begin{pmatrix} S_{11} & S_{12} & S_{12} \\ S_{12} & S_{11} & S_{12} \\ S_{12} & S_{12} & S_{11} \end{pmatrix} - \frac{\beta^2}{\gamma} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}, \tag{7}$$

where S_{ij}^{sat} is the desired saturated compliance including the effects of the trapped liquid, while S_{ij} is the drained compliance in the absence of the liquid. (We will assume that, when drained, the pores are filled with air, and that $K_{air} \simeq G_{air} \simeq 0$. In fact, K_{air} does not vanish, but it is more than three orders of magnitude smaller than other constants we are considering. This approximation has a negligible effect on the final results, and makes the resulting formulas noticeably simpler — and especially so in the next section on effective medium theory.)

Since for elastic isotropy there are only two independent coefficients (S_{11} and S_{12}), we find that (7) reduces to one expression for the diagonal compliance

$$S_{11}^{sat} = S_{11} - \frac{\beta^2}{\gamma}, \tag{8}$$

and another for the off-diagonal compliance

$$S_{12}^{sat} = S_{12} - \frac{\beta^2}{\gamma}. \tag{9}$$

If K^{sat} and G^{sat} are respectively the undrained bulk and shear moduli, then (2) and (3) together with (8) and (9) imply that

$$\frac{1}{9K^{sat}} + \frac{1}{3G^{sat}} = \frac{1}{9K_{dr}} + \frac{1}{3G_{dr}} - \frac{\beta^2}{\gamma}, \tag{10}$$

and

$$\frac{1}{9K^{sat}} - \frac{1}{6G^{sat}} = \frac{1}{9K_{dr}} - \frac{1}{6G_{dr}} - \frac{\beta^2}{\gamma}. \tag{11}$$

Subtracting (11) from (10) shows immediately that $1/2G^{sat} = 1/2G_{dr}$, or equivalently that

$$G^{sat} = G_{dr}. \tag{12}$$

Thus, the first *result* of Gassmann's thought experiment is that, for purely mechanical effects, the shear modulus for the case with trapped fluid (undrained) is the same as that for the case with no fluid (drained). Then, substituting (12) back into either (10) or (11) gives one form of the result commonly known as Gassmann's equation for the bulk modulus:

$$\frac{1}{K^{sat}} = \frac{1}{K_{dr}} - \frac{9\beta^2}{\gamma}. \tag{13}$$

We want to emphasize that the analysis presented shows clearly that (12) is a definite *result* of this analysis, *not an assumption.* In fact, for isotropic systems, we must have (12) in order for (13) to hold, and furthermore, if (13) holds, then so must (12). Thus, monitoring any changes in shear modulus with changes of liquid content (say through shear velocity measurements) provides a test of both Gassmann's assumptions [*e.g.*, low frequencies, lack of chemical effects on the shear moduli, etc.] and results [*i.e.*, equations (12) and (13)].

2.2 Alternative formulas for K^{sat}

To obtain one of the more common forms of Gassmann's result for the bulk modulus, we now need to define the coefficients β and γ. First note that

$$3\beta \equiv \frac{1}{K_{dr}} - \frac{1}{K_s} \equiv \frac{\alpha}{K_{dr}}, \tag{14}$$

where K_s is the grain modulus of the solid constituent present and α is the Biot-Willis parameter.[16] Furthermore, the parameter γ is related through (6) to Skempton's pore-pressure buildup coefficient[15] B, so that

$$\frac{3\beta}{\gamma} = B. \tag{15}$$

Substituting these results into (13) gives

$$K^{sat} = \frac{K_{dr}}{1 - \alpha B}, \tag{16}$$

which is another form[17] of Gassmann's standard result for the bulk modulus.

One other form of Gassmann's equation for the bulk modulus,[10,11] and one that we will find useful for comparison to the effective medium theory formulas to follow, is

$$K^{sat} = K_{dr} + \frac{\alpha^2}{(\alpha - \phi)/K_s + \phi/K_f}, \tag{17}$$

which highlights how the undrained constant K^{sat} differs from the drained constant K_{dr}. Here ϕ is the porosity and K_f is the fluid bulk modulus. We could alternatively have chosen to replace the first term on the right hand side by $K_{dr} = (1 - \alpha)K_s$. Then, all the geometrical and volume fraction information is contained in α and ϕ, while the pertinent constituent properties are just K_s and K_f.

Next we consider what effective medium theory has to say about the same physical system.

3 Effective Medium Theory

Porous media are inherently inhomogeneous because of the voids (pores) which may contain either air or some type of liquid, such as water or oil. Typical naturally occurring porous media are also random, as the pores are not distributed in any organized (periodic) way throughout the porous body. So it is natural to consider homogenization theory, or equivalently effective medium theory, to estimate physical constants such as the elastic constants of porous media. One implicit and significant difference between effective medium theory and the methods used by Gassmann is that typically we assume that the constituents of the inhomogeneous medium are essentially fixed relative to one another; *i.e.*, when studying elastic constants, we often use a welded-contact assumption: two points in contact remaining in contact throughout a deformation. So implicitly we have a no-slip boundary condition between fluid present in the pores and the solids surrounding (and therefore defining the boundaries of) the pores. Gassmann, on the other hand, assumes that the porous medium has a finite fluid permeability, so pores are connected and fluid is free to move in and out of the pores depending on the state of elastic stress and fluid pore pressure. This difference is important, but — as we shall see — it is not the only source of disagreement between the two approaches.

To highlight the differences in the results, we will first provide a quick derivation of a particular effective medium theory (the CPA, or coherent potential approximation) and make some observations about connections between this theory and rigorous bounds. Then we will study some relevant properties of the so-called[18,19] "canonical functions" which can be used to study and compare both rigorous bounds and the effective medium theory estimates of elastic constants. Then, we will show how Gassmann's results fit into the same framework.

3.1 Derivation of the CPA

Probably the best known of all the effective medium theories in elasticity is the "self-consistent theory." One formulation of these results was presented by Hill[20] and Budiansky[21]. For the special case of spherical inclusions (which is the only case we will consider here), these results are identical to results obtained later by Korringa *et al.*[22] and Berryman[18,19,23] using arguments based on the coherent potential approximation (CPA) from the theory of alloys [see Gubernatis and Krumhansl[24] and references]. But these two "self-consistent" approaches can give very different results when the inclusions have shapes other than spherical. The CPA has the advantage that it is known to provide estimates that always lie between known rigorous bounds,[19,25] such as the Hashin-Shtrikman bounds,[26] whereas the other "self-consistent" formulation is known to violate the bounds in some cases (though not for spheres). The CPA has also been generalized recently for use at higher frequencies by Kaelin[27] and Kaelin and Johnson.[28,29] Since our goal here is to elucidate a point in poroelasticity theory, it will be adequate to concentrate on the noncontroversial case of spherical inclusions — which is therefore what we shall do. Useful reviews of the effective medium topic with emphasis on geophysical (rock physics) applications for further reading are those of Watt *et al.*[30] and Berryman.[31]

We can obtain the CPA formulas quickly by using an argument of Berryman.[18] If we imagine a scattering experiment in which a single sphere of one inclusion material is imbedded in a host matrix, then for a plane compressional wave incident on this sphere, the two pertinent scattering coefficients at infinity are

$$B_0(K_m, K_i, G_m)) = \frac{K_m - K_i}{3K_i + 4G_m} \tag{18}$$

and

$$B_2(G_m, G_i, K_m) = \frac{20G_m(G_i - G_m)/3}{6G_i(K_m + 2G_m) + G_m(9K_m + 8G_m)}, \tag{19}$$

where the moduli for the surrounding matrix material are K_m (bulk) and G_m (shear), and the moduli for the spherical inclusion are K_i and G_i.

The trick used in CPA is to imagine that our composite contains inclusions $i = 1, \dots, n$, where $n \geq 2$, and that the scattering experiment is being performed at such low frequencies (and therefore long wavelengths) that the precise locations of the individual scatterers have no special effect on the results. Then, we can suppose (see Figure 1) that the composite (scattering) medium is imbedded in an adjustable matrix material $m = *$, such that each individual scatterer sees all the other scatterers as composing this matrix. Then, the composite inclusion, when imbedded in the adjustable $*$-matrix, should actually produce no scattering at all at infinity if the single-scattering coefficients satisfy

$$\sum_{i=1}^{n} f_i B_0(K^*, K_i, G^*)|_{*=eff} = 0, \tag{20}$$

and

$$\sum_{i=1}^{n} f_i B_2(G^*, G_i, K^*)|_{*=eff} = 0, \tag{21}$$

where the f_i's are the volume fractions of all the constituents in the composite and therefore $\sum_i f_i = 1$. The particular choices of the adjustable moduli that cause the right hand sides of (20) and (21) to vanish are then defined to be the CPA effective medium constants. Equations (20) and (21) provide a coupled set of equations which uniquely [Note: the only known exceptions to uniqueness occur when some of the constituent moduli vanish — see Berryman[18,23]] determine the effective elastic constants $K^{eff} = K^*$ and $G^{eff} = G^*$. These formulas may be written in many different ways, but the one that we prefer here has the form

$$\frac{1}{K^{eff} + 4G^{eff}/3} = \sum_{i=1}^{n} \frac{f_i}{K_i + 4G^{eff}/3} = \left\langle \frac{1}{K(x) + 4G^{eff}/3} \right\rangle, \tag{22}$$

and

$$\frac{1}{G^{eff} + F^{eff}} = \sum_{i=1}^{n} \frac{f_i}{G_i + F^{eff}} = \left\langle \frac{1}{G(x) + F^{eff}} \right\rangle, \tag{23}$$

where $F \equiv G(9K + 8G)/6(K + 2G)$, with F^{eff} being the same formula with all physical constants replaced by those with "eff" superscripts. The notation $\langle \cdot \rangle$ is introduced as the volume average, and the right hand sides can then easily be shown to be identical to the preceding quantity in each of the two equations.

Note that (22) and (23) are strongly coupled, but the pair of equations can be solved easily by iteration. Also note that the resulting G^{eff} is the same for drained and undrained cases only if F^{eff} is the same, which would require that additionally that K^{eff} be the same for the drained and undrained cases.

3.2 Hashin-Shtrikman-Walpole bounds

Now we will not derive, but merely quote, the results for the Hashin-Shtrikman bounds[26] in elasticity. These results take a very similar form to those for the CPA for spheres. The symbols for the Hashin-Shtrikman upper bounds are K_{HS}^{+} and G_{HS}^{+}, and for the lower bounds the pluses are replaced by minuses. The results are

$$\frac{1}{K_{HS}^{\pm} + 4G_{\pm}/3} = \sum_{i=1}^{n} \frac{f_i}{K_i + 4G_{\pm}/3} = \left\langle \frac{1}{K(x) + 4G_{\pm}/3} \right\rangle, \tag{24}$$

and

$$\frac{1}{G_{HS}^{\pm} + F_{\pm}} = \sum_{i=1}^{n} \frac{f_i}{G_i + F_{\pm}} = \left\langle \frac{1}{G(x) + F_{\pm}} \right\rangle, \tag{25}$$

where $K_{+} = \max_i K_i$, $K_{-} = \min_i K_i$, $G_{+} = \max_i G_i$, $G_{-} = \min_i G_i$, and $F_{\pm}$, where all constants in $F \equiv G(9K+8G)/6(K+2G)$ now take the same subscripts as $F_{\pm}$. If, for a two-component medium, the material properties are well-ordered so that $(G_2 - G_1)(K_2 - K_1) > 0$, then equations (24) and (25) are known as the Hashin-Shtrikman bounds. But — if the constants are not well-ordered, so $(G_2 - G_1)(K_2 - K_1) < 0$ — the formulas presented are still true bounds, known instead as the Walpole bounds.[32–34] Sometimes equations (24) and (25) taken in their entirety are called the Hashin-Shtrikman-Walpole bounds.

3.3 Properties of the canonical functions in elasticity

These results and others of a similar nature using many of the known bounds in elasticity [see Berryman[19]] suggest that a single set of two functions controls the behaviors of both effective medium theories and bounds. We call these expressions the "canonical functions of elasticity" because they occur repeatedly, and they have many nice properties. These function properties include monotonicity as a function of the arguments, which makes them very convenient for comparisons between and among many of the bounds and effective medium results.

We define the canonical function for the bulk modulus as

$$\Lambda(G) \equiv \left\langle \frac{1}{K(x) + 4G/3} \right\rangle^{-1} - \frac{4}{3}G, \tag{26}$$

and the canonical function for the shear modulus as

$$\Gamma(F) \equiv \left\langle \frac{1}{G(x) + F} \right\rangle^{-1} - F. \tag{27}$$

Using these definitions, equations (22) and (23) can be rewritten as

$$K^{eff} = \Lambda(G^{eff}), \tag{28}$$

and

$$G^{eff} = \Gamma(F^{eff}), \tag{29}$$

respectively. Similarly, the Hashin-Shtrikman-Walpole results are written as

$$K_{HS}^{\pm} = \Lambda(G_{\pm}), \tag{30}$$

and

$$G_{HS}^{\pm} = \Gamma(F_{\pm}), \tag{31}$$

It is easy to show that these canonical functions are both monotonic in their arguments. For example,

$$\frac{d\Lambda(G)}{dG} = \frac{4}{3}\left\langle \frac{1}{K(x)+4G/3} \right\rangle^{-2} \left(\left\langle \frac{1}{(K(x)+4G/3)^2} \right\rangle - \left\langle \frac{1}{K(x)+4G/3} \right\rangle^{2} \right) \geq 0. \tag{32}$$

Non-negativity follows easily from the Cauchy-Schwartz inequality[35] $\langle a \rangle^2 \leq \langle a^2 \rangle$. Thus, Λ is a nondecreasing function of its argument. Similarly, Γ is a nondecreasing function of its argument F, which itself has the property that $F = F(K,G)$ is a nondecreasing function of both of its arguments. Furthermore, note that $\Lambda(0) = \langle 1/K(x) \rangle^{-1} = K_R$, $\Gamma(0) = \langle 1/G(x) \rangle^{-1} = G_R$, which are the harmonic means, or Reuss averages[36] of these constants. Also note, $\lim_{G\to\infty} \Lambda(G) = \langle K(x) \rangle = K_V$, $\lim_{F\to\infty} \Gamma(F) = \langle G(x) \rangle = G_V$, which is the mean, or corresponding Voigt average.[37] Thus, the physical range of real arguments for these two functions produces results that lie in the range

$$K_R \leq \Lambda(G) \leq K_V, \tag{33}$$

and

$$G_R \leq \Gamma(F) \leq G_V. \tag{34}$$

It follows that

$$K_{HS}^{-} \leq K^{eff} \leq K_{HS}^{+}, \tag{35}$$

since

$$\Lambda(G_{HS}^{-}) \leq \Lambda(G^{eff}) \leq \Lambda(G_{HS}^{+}), \tag{36}$$

because Λ is monotonic and

$$G_{HS}^{-} \leq G^{eff} \leq G_{HS}^{+}, \tag{37}$$

which we know to be true, independent of the present arguments.

Finally, the canonical functions also have useful monotonicity properties as functions of their constituents' moduli values (not shown in the above argument list). To make these properties explicit in the notation, we will also list the constitutents' moduli K_s, K_f, and G_s in the argument list when it is important to draw attention to them. For example, the drained bulk modulus in effective medium theory from (22) is then expressed as

$$K_{dr}^{eff} = \Lambda(K_s, K_f = 0; G_{dr}^{eff}), \tag{38}$$

and the corresponding result for the saturated modulus is

$$K_{sat}^{eff} = \Lambda(K_s, K_f; G_{sat}^{eff}). \tag{39}$$

Monotonicity of the canonical function in the fluid bulk modulus shows that

$$\Lambda(K_s, K_f = 0; G) \le \Lambda(K_s, K_f; G), \tag{40}$$

assuming that the usual argument G is the same for both expressions. By concatenating inequalties, we therefore obtain useful rigorous relations such as

$$\Lambda(K_s, 0; G_{dr}) \le \Lambda(K_s, K_f; G_{dr}) \le \Lambda(K_s, K_f; G_{sat}), \tag{41}$$

since $0 \le K_f$ and $G_{dr} \le G_{sat}$ and, similarly,

$$\Gamma(K_s, 0; F_{dr}) \le \Gamma(K_s, K_f; F_{dr}) \le \Gamma(K_s, K_f; F_{sat}), \tag{42}$$

since $F_{dr} \le F_{sat}$. We will make use of these properties in the following arguments.

Note that the canonical functions have often played an explicit role in formulating rigorous bounds since the work of Milton,[38] in which he introduced the y-transform concept — closely related to, and in part motivated by, the canonical functions defined here and also earlier by Berryman.[18,19]

3.4 Gassmann's results in terms of canonical functions

Various authors [including Endres and Knight[39] and Kaelin[27]] have noticed that Gassmann's equation for the undrained bulk modulus has a form similar to that of the effective medium theory equation (22), or equivalently to the Hashin-Shtrikman[26] bounds (24). The significance of these observations has remained unclear, however, because — when making these comparisons to the CPA — it has been necessary to sever the relationship between the bulk modulus equation (22) and the shear modulus equation (23) in order to make the similarity apparent. Taking such a step removes the self-consistency condition relating bulk and shear modulus, and replaces the derived result with an *ad hoc* assumption that $K^{eff} \simeq \Lambda(G_{dry}) = K^{sat}$. When similar observations are made relating the Hashin-Shtrikman bulk modulus bounds to the Gassmann result, a similar *ad hoc* step is required, which is $K_{HS}^{\pm} \simeq \Lambda(G_{dry}) = K^{sat}$. The *ad hoc* procedure sidesteps and confuses the real issue, which is the question: Why do the two formulas (either in Gassmann and effective medium theory, or in Gassmann and the Hashin-Shtrikman bounds), although similar in functional form, in fact *disagree*?

To avoid making any unwarranted assumptions, we will first of all show what is true about the Gassmann bulk modulus result and how it is related to the canonical functions. To do this, we will approach the subject from the direction opposite to the one usually taken, and show that effective medium results can be written in a form similar to that of Gassmann's formula. First note that [see Korringa *et al.*[22] for an early application of this formula, and also Pride *et al.*[40]],

$$K_{dr}^{eff} = \frac{(1-\phi)K_s}{1 + a_{dr}\phi}, \tag{43}$$

where $a_{dr} = 3K_s/4G_{dr}^{eff}$. This formula is completely equivalent to the canonical form for the drained case $K_{dr}^{eff} = \Lambda(G_{dr}^{eff})$. In particular, we find that, when there are only two constituents (solid and fluid), then the effective medium theory result can be written in the form

$$K_{sat}^{eff} = (1-\chi)K_s + \frac{\chi^2}{(\chi-\phi)/K_s + \phi/K_f}, \tag{44}$$

where all the microgeometry dependence of the equation appears here in χ, and (coming from the coupling to the shear modulus) is contained in the parameter

$$\chi = \phi\frac{1+a_{sat}}{1+a_{sat}\phi}, \tag{45}$$

where

$$a_{sat} \equiv 3K_s/4G_{sat}^{eff}. \tag{46}$$

Equations (44) and (45) should be compared to

$$K^{sat} = (1-\alpha)K_s + \frac{\alpha^2}{(\alpha-\phi)/K_s + \phi/K_f}, \tag{47}$$

and

$$\alpha = 1 - K_{dr}^{eff}/K_s = \phi\frac{1+a_{dr}}{1+a_{dr}\phi} \tag{48}$$

with a_{dr} given following (43). Thus, it now becomes clear that the only difference between the effective medium result and Gassmann's result for the bulk modulus is determined by which value of the shear modulus is used for evaluating the corresponding parameter a. Thus, it is a definite result that Gassmann's equation for undrained bulk modulus can be written in terms of the canonical function Λ as

$$K^{sat} = \Lambda_{sat}(G_{dr}), \tag{49}$$

where $\Lambda_{sat}(\cdot) \equiv \Lambda(K_s, K_f; \cdot)$ and $\Lambda_{dr}(\cdot) \equiv \Lambda(K_s, 0; \cdot)$. Note the differences among (38), (39), and (49).

By assumption, we also have for both theories (*i.e.*, Gassmann and effective medium theory) that $K_{dr} = \Lambda(K_s, 0; G_{dr})$ and $G_{dr} = \Gamma(K_s, 0; F_{dr})$. Gassmann treats K_{dr} and G_{dr} as purely experimental quantities and there is no inconsistency involved if we choose to treat the K_{dr} and G_{dr} from effective medium theory as our estimates of the drained bulk and shear moduli.

The preceding reasoning shows that it is inappropriate to decouple the effective medium equations (28) and (29), just as it is incorrect to say that Gassmann "assumed" (rather than derived) $G^{sat} = G_{dr}$. [We explicitly and easily derived this result in (12).] It is nevertheless true that (49) is a correct statement of Gassmann's result for the bulk modulus in terms of the canonical function Λ_{sat}. From this statement, (35)-(36), and (41)-(42), we find that

$$K_{dr} \le K^{sat} \le K_{sat}^{eff}, \tag{50}$$

and

$$G_{dr} \leq \Gamma(K_s, K_f; F_{dr}) \leq G_{sat}^{eff}. \tag{51}$$

But, we emphasize that Gassmann's result shows $G^{sat} = G_{dr} \neq \Gamma(K_s, K_f; F_{dr})$ when $K_f \neq 0$, and nothing should be inferred just from the fact that some of these quantities are expressible in terms of the canonical functions.

The canonical functions best serve their purpose when physical approximations (such as CPA) and rigorous mathematical arguments (such as Hashin-Shtrikman bounds) can be applied to derive relationships among the various constants of interest.

4 The Dichotomy

The foregoing analysis shows that we have a theoretically difficult situation to resolve. The dichotomy is this: Gassmann's equations are low frequency and predict that

$$K^{sat} = \Lambda_{sat}(G_{dr}) \qquad \text{and} \qquad G^{sat} = G_{dr}, \tag{52}$$

whereas effective medium theory (which is also for low frequencies) for apparently the same problem predicts that

$$K_{sat}^{eff} = \Lambda_{sat}(G_{sat}^{eff}) \qquad \text{and} \qquad G_{sat}^{eff} = \Gamma_{sat}(F_{sat}^{eff}). \tag{53}$$

Furthermore, because the canonical functions Λ and Γ are monotonic, it is easy to show from the foregoing results that, whenever $K_f > K_{air} \simeq 0$,

$$G^{sat} < G_{sat}^{eff} \tag{54}$$

and, therefore,

$$K^{sat} < K_{sat}^{eff}. \tag{55}$$

How do we explain that these two low frequency theories clearly differ? Even if the numerical difference were not great, the mere existence of the difference (assuming both theories are correct, so it is a real difference) shows that there must be dispersion in such systems. Dispersion also implies attenuation because of Kramers-Kronig relations (see Aki and Richards[1]). So this difference, if true, guarantees that there is more attenuation of sound waves in a poroelastic system than we might expect from other considerations.

The reason for this dispersion is that the Gassmann approach is really quasi-static, and therefore applies at extremely low frequencies, whereas the effective medium theory is clearly not formulated to apply at such low frequencies. The difference arises from how fluid pressure is treated in the two approaches. Gassmann allows the fluid pressure sufficient time to equilibrate throughout the medium, however long it takes – perhaps very long times indeed. The effective medium theory does not preclude the fluid from equilibrating, but does not necessarily allow enough time for equilibration. Time does not play an explicit role in the effective medium theory, only an implicit one in that it must be "long enough" so the frequencies are low and the wavelengths long compared to the microstructure. If liquids occupy isolated pockets scattered throughout the inhomogeneous porous medium, they may have different fluid pressures if they are not permitted sufficient time to equilibrate. In fact, the effective medium theory does not explicitly allow for finite (nonzero) fluid permeability of the porous medium.

These facts suggest that there are some good reasons to think that there could be differences between Gassmann and effective medium theories. But, so far the analysis still leaves the technical question unresolved.

5 Resolution of the Dichotomy

One method of resolving this dichotomy is to use the method of multiple scales for the time-dependent version of this problem. In fact, this has already been done by Burridge and Keller.[41,42] They find the interesting (and perhaps now not surprising) result that there are two possible solutions to the problem. One is essentially that of Biot's theory[2] of wave propagation in poroelastic media. The other is a set of viscoelastic equations. The difference leading to the two quite different results is that, when the scaled viscosity is treated as being of order ϵ^2 (ϵ is the small quantity), they get Biot-Gassmann, whereas when viscosity is treated as order 1, they obtain the viscoelasticity equations instead. These results are consistent with the need for two approaches and two rather different results, even though both are long wavelength, and low frequency results.

We will now take a different approach to show how the effective medium theory result can also arise from Gassmann-style considerations.

5.1 What we need to show

The crux of our problem is to show how the shear modulus can be independent of the fluid properties at quasi-static frequencies, yet become dependent on them at somewhat higher (but still low) frequencies.

Physically, we know what must be happening in the effective medium theory to give rise to the effects discussed. The presence of the liquid results in an increase in the shear modulus, even though the liquid shear modulus is zero. Why is that? The reason is that in an inhomogeneous medium, when we apply stress or strain at the macroscopic scale, that stress or strain gets resolved locally in a complicated way because of the inhomogeneities. It is very easy to see that this must be so in a granular medium, but is clearly true also in most inhomogeneous media. An applied external compression can produce a shear field locally. An applied external pure shear can produce a compression locally. This is the physical source of the effect. If we apply an external shear to a porous medium containing liquid, it matters that the liquid is present and not replaced by air. It matters because the external shear can be resolved into local compression in some regions containing the liquid. In these regions, the liquid can support the compression (but not a shear), and therefore the liquid stores some of the energy applied to the system by the external shearing force. This discussion shows qualitatively (and physically) why the effective medium theory predicts that the shear modulus depends on the bulk modulus of the liquid. (We will show explicitly how it happens in the mathematics later in this section.) On the other hand, if the liquid has enough time (and finite permeability permits it) to move out of the way, it can relax to a state that does not support any of the local compression, and then we have Gassmann's result.

So what we will be attempting to show in the remainder of this section is how these ideas can arise and then be quantified explicitly in the theory.

5.2 Why local isotropy is not sufficient

We know from the arguments given previously that purely isotropic (micro as well as macro) poroelasticity cannot give the effective medium result. No matter what else happens — no matter how we try to make changes in the poroelastic coefficients to see how such effects arise — we will not be able to change the fact that $S_{11} - S_{12} = 1/2G$ [recall (2) and (3)]. Changes in the coupling coefficients that result in an isotropic elastic matrix must satisfy this condition (directly related to system rotational invariance). And this condition guarantees that Gassmann's result for the shear modulus will always hold, *e.g.*, compare (8) and (9). So an isotropic poroelastic medium that is also isotropic everywhere on the *microscale* will not help us resolve our dichotomy.

5.3 Local anisotropy

In contrast, let us now consider one of the simplest cases of local anisotropy, *i.e.*, transverse isotropy (TI). We suppose that this anisotropy could arise from many mechanisms (local layering, or fractures/cracks), and still produce the same formulas we will use in this demonstration. The elastic tensor is not uniquely related to the microstructure. The precise mechanism is also not really important to our present purpose. We are trying merely to establish a link between Gassmann's result and the effective medium results. We have shown this is very difficult (and maybe impossible) to establish for local (microscopic) isotropy. We want to show that it is, however, possible for local anisotropy.

If the inhomogeneous medium is isotropic on the macroscale, it certainly can still be anisotropic on the microscale. This fact has been used extensively in the mathematical homogenization community, where many authors [see, for example, Kohn and Milton[43] and Avellaneda and Milton[44]] have studied laminates to determine realizability results for bounding methods. A typical physical example is an aggregate of randomly oriented crystals (*i.e.*, a polycrystal), where individual crystals or domains are locally anisotropic but the aggregate may be isotropic due to spatial averaging over orientations.

To make a clear connection with rock physics, we will assume that, instead of being locally layered (laminated), the medium has randomly oriented fractures or cracks. The spatial distribution of randomness is such that the overall medium is macroscopically isotropic. We will assume that there exists a scale at which it makes sense to talk about the porous medium's local elastic constants, that there is a single fracture per REV (representative elementary volume) on this scale, and that the elastic compliance tensor can therefore be written locally as that for a transversely isotropic (same as hexagonal symmetry) medium. The axis of symmetry differs from location to location throughout the medium, but locally — if the x_3 direction is the local axis of symmetry — we can write the strain-stress relations in terms of the compliance matrix as

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & -\beta^{(1)} & & & \\ S_{12} & S_{11} & S_{13} & -\beta^{(1)} & & & \\ S_{13} & S_{13} & S_{33} & -\beta^{(3)} & & & \\ -\beta^{(1)} & -\beta^{(1)} & -\beta^{(3)} & \gamma & & & \\ & & & & \frac{1}{G_t} & & \\ & & & & & \frac{1}{G_t} & \\ & & & & & & \frac{1}{G_{dr}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}, \tag{56}$$

analogous to (1).

We still have the condition that $2(S_{11} - S_{12}) = S_{66} = 1/G_{dr}$ for transverse isotropy as we did for the isotropic case. But now the other two shear compliances $(1/G_t)$ are decoupled from the values of the S's in the upper left corner of the matrix. We can use an argument of Schoenberg and Douma[45] [also see Schoenberg and Muir[46] and Dellinger *et al.*[47] for related concepts and techniques] to introduce the effects of the drained fractures/cracks into this matrix. And we assume here that this has already been done. The effects are localized and result in an increase in the compliances $S_{44} = S_{55} = 1/G_t$, and S_{33} (which implies a decrease in associated stiffnesses).

Redoing Gassmann's argument for this case, we have

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} = \begin{pmatrix} S_{11}^{sat} & S_{12}^{sat} & S_{13}^{sat} & & & \\ S_{12}^{sat} & S_{11}^{sat} & S_{13}^{sat} & & & \\ S_{13}^{sat} & S_{13}^{sat} & S_{33}^{sat} & & & \\ & & & \frac{1}{G_t^{sat}} & & \\ & & & & \frac{1}{G_t^{sat}} & \\ & & & & & \frac{1}{G^{sat}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \left[\begin{pmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{12} & S_{11} & S_{13} & & & \\ S_{13} & S_{13} & S_{33} & & & \\ & & & \frac{1}{G_t} & & \\ & & & & \frac{1}{G_t} & \\ & & & & & \frac{1}{G_{dr}} \end{pmatrix} \right.$$

$$\left. - \frac{1}{\gamma} \begin{pmatrix} (\beta^{(1)})^2 & (\beta^{(1)})^2 & \beta^{(1)}\beta^{(3)} & & & \\ (\beta^{(1)})^2 & (\beta^{(1)})^2 & \beta^{(1)}\beta^{(3)} & & & \\ \beta^{(1)}\beta^{(3)} & \beta^{(1)}\beta^{(3)} & (\beta^{(3)})^2 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}. \tag{57}$$

So, although this is more complicated than (1), the result is still basically the same: there is no obvious effect of fluid saturation on the shear modulus. Thus, local anisotropy in the form of transverse isotropy is still not enough to induce the desired response.

Now we must be careful to account for all the effects of the anisotropy on interactions between shear stresses and change of volume. There are some subtle, but well-known, effects contained in (57) that need to be elaborated.

5.3.1 Change of volume under shear

The elastic stiffness and compliance tensors, in the matrix form we are using here, have six eigenvectors. Four of them are universal and independent of the values of the matrix elements. These four are

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{58}$$

When applied to the compliance matrix, all four correspond to states of pure shear, and have eigenvalues: $S_{11} - S_{12}$, S_{44}, $S_{55} = S_{44}$, and S_{66}, respectively.

The remaining two eigenvectors correspond to coupled states of compression and shear. These eigenvectors can be written as

$$\begin{pmatrix} 1 \\ 1 \\ a_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ -2/a_+ \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{59}$$

where a_+ solves

$$\begin{aligned} S_+ &= S_{11} + S_{12} + a_+ S_{13}, \\ S_+ a_+ &= 2S_{13} + a_+ S_{33}, \end{aligned} \tag{60}$$

and S_+ is one of the eigenvalues, which satisfy

$$S_\pm = \frac{1}{2}\left[S_{33} + S_{11} + S_{12} \pm \sqrt{(S_{33} - S_{11} - S_{12})^2 + 8S_{13}^2}\right]. \tag{61}$$

Of these two solutions to (60), a_+ will normally turn out to be the one closest to unity, while the other one ($a_- = -2/a_+$) will be closest to -2. When $a_+ \simeq 1$, then the first eigenvector in (59) is almost a pure compression and the second is almost a pure shear. Otherwise, the eigenvectors are mixed states, that might be called quasi-compressional and quasi-shear states, respectively.

We see now clearly why it is that, if we apply a pure compression to the system, some local shearing must occur. As long as this coupling is contained in the eigenvectors, it is unavoidable. There is no way to construct a compressional state of this system that does not couple to shear. It is possible to construct some pure shear states that do not couple to compression using the first four eigenvectors in (58), but some interaction between compression and shear is nevertheless guaranteed for such anisotropic systems.

Bulk modulus bounds

Effective constants for the overall isotropic system composed from random orientations of the matrix (57) can be determined approximately using Reuss[36] and Voigt[37] estimates, which are also known to be rigorous bounds on the constants.[48,49] So, the best results available for a general polycrystal[44] are the Reuss lower bound and the Voigt upper bound. The Reuss bound on the bulk modulus is

$$\begin{aligned} (K_R)^{-1} &= 2S_{11}^{sat} + 2S_{12}^{sat} + S_{33}^{sat} + 4S_{13}^{sat} \\ &= 2S_{11} + 2S_{12} + S_{33} + 4S_{13} - \left(2\beta^{(1)} + \beta^{(3)}\right)^2/\gamma, \end{aligned} \tag{62}$$

where $\gamma = \left(2\beta^{(1)} + \beta^{(3)}\right)/B$, with B being Skempton's pore-pressure buildup coefficient. The corresponding Voigt bound is

$$\begin{aligned} K_V &= \frac{1}{9}\left(2C_{11}^{sat} + 2C_{12}^{sat} + C_{33}^{sat} + 4C_{13}^{sat}\right) \\ &= \frac{1}{9}\left(2C_{11} + 2C_{12} + C_{33} + 4C_{13}\right) + \frac{a}{9}\left(2\alpha^{(1)} + \alpha^{(3)}\right)^2, \end{aligned} \tag{63}$$

where the α's and β's are related by defining the appropriate column vectors $\bar{\alpha}$ and $\bar{\beta}$ respectively and noting that $\bar{\beta} = \mathbf{S}\bar{\alpha}$ and $\bar{\alpha} = \mathbf{C}\bar{\beta}$, with $\mathbf{S}$ and $\mathbf{C}$ being the drained compliance and stiffness matrices respectively. The constant coefficient a in (63) is determined by $1/a = \gamma - \bar{\alpha}^T\mathbf{S}\bar{\alpha}$.

For hexagonal symmetry (following Nye[50]), we have (the *sat* superscript will be dropped for now)

$$C_{11} + C_{12} = S_{33}/S, \tag{64}$$

$$C_{11} - C_{12} = 1/(S_{11} - S_{12}), \tag{65}$$

$$C_{13} = -S_{13}/S, \tag{66}$$

$$C_{33} = (S_{11} + S_{12})/S, \tag{67}$$

and

$$C_{44} = C_{55} = 1/S_{44}, \tag{68}$$

with

$$S \equiv S_{33}(S_{11} + S_{12}) - 2S_{13}^2. \tag{69}$$

These identities show that [after restoring the *sat* superscripts and using (63)]

$$\begin{aligned} K_V &= \frac{1}{9S^{sat}}\left(S_{11}^{sat} + S_{12}^{sat} + 2S_{33}^{sat} - 4S_{13}^{sat}\right) \\ &= \frac{1}{9S}\left(S_{11} + S_{12} + 2S_{33} - 4S_{13}\right) + \frac{a}{9}\left(2\alpha^{(1)} + \alpha^{(3)}\right)^2. \end{aligned} \tag{70}$$

Also, note that both the Reuss and Voigt bounds on bulk modulus are always larger in the presence of the pore liquid (*i.e.*, $B > 0$) than in its absence. These results are automatic for Reuss and follow for Voigt if $a > 0$. When the TI medium is almost isotropic, we find

$$\frac{1}{a} \simeq \frac{2\alpha^{(1)} + \alpha^{(3)}}{3BK} - \left[\frac{(2\alpha^{(1)} + \alpha^{(3)})^2}{9K} + \frac{(\alpha^{(3)} - \alpha^{(1)})^2}{3G}\right], \tag{71}$$

which reduces to $1/a = \alpha(1 - \alpha B)/BK > 0$ when $\alpha = \alpha^{(1)} = \alpha^{(3)}$. So, with these restrictions, a is positive if $B < 1/\alpha$. For our examples, we will only consider the range $0 \le B \le 1$.

Shear modulus bounds

For the shear modulus, we have the effects of the inhomogeneity showing up in the bounds because the shear can depend on the liquid properties accordingly in the Reuss lower bound

$$\begin{aligned}(G_R)^{-1} &= \frac{1}{15}\left(8S_{11}^{sat} + 4S_{33}^{sat} - 4S_{12}^{sat} - 8S_{13}^{sat} + 6S_{44}^{sat} + 3S_{66}^{sat}\right) \\ &= \frac{1}{15}\left(8S_{11} + 4S_{33} - 4S_{12} - 8S_{13} + 6S_{44} + 3S_{66}\right) - \frac{4}{15\gamma}\left(\beta^{(3)} - \beta^{(1)}\right)^2, \end{aligned} \tag{72}$$

and the Voigt upper bound

$$\begin{aligned} G_V &= \frac{1}{15}\left(2C_{11}^{sat} + C_{33}^{sat} - 2C_{12}^{sat} - C_{13}^{sat} + 6C_{44}^{sat} + 3C_{66}^{sat}\right) \\ &= \frac{1}{15}\left(2C_{11} + C_{33} - 2C_{12} - C_{13} + 6C_{44} + 3C_{66}\right) + \frac{a}{15}\alpha^{(3)}\left(\alpha^{(3)} - \alpha^{(1)}\right) \\ &= \frac{1}{15}\left(\frac{2}{S_{11} - S_{12}} + \frac{6}{S_{44}} + \frac{3}{S_{66}} + \frac{S_{11}^{sat} + S_{12}^{sat} + S_{13}^{sat}}{S^{sat}}\right), \end{aligned} \tag{73}$$

where we have reinstated the *sat* superscript where appropriate for clarity.

Using the Voigt-Reuss-Hill-type estimates (*i.e.*, taking either the average or the geometric mean of the bounds), we find that such estimates for the shear modulus now do depend on the pore liquid properties. Furthermore, the liquid contributions have a definite sign for G_R, showing that the lower bound on shear modulus (and by inference the shear modulus itself) always increases due to the presence of the pore liquid. The sign of liquid corrections to G_V in (73) is not difficult to analyze, and can be seen to depend on the sign of the difference $\alpha^{(3)} - \alpha^{(1)}$. Using the identities relating the column vectors $\bar{\alpha}$ and $\bar{\beta}$, we also find for TI media close to isotropic that

$$\alpha^{(3)}(\alpha^{(3)} - \alpha^{(1)}) \simeq 2G\left[K(2\beta^{(1)} + \beta^{(3)}) + (4G/3)(\beta^{(3)} - \beta^{(1)})\right]\left(\beta^{(3)} - \beta^{(1)}\right). \tag{74}$$

Thus, the condition for increasing G_V is either $\alpha^{(3)} > \alpha^{(1)}$, or, if we assume that $\lambda = K - 2G/3 > 0$, then $\beta^{(3)} > \beta^{(1)}$. It is nevertheless somewhat surprising that there are *any* circumstances (even for choices of the parameters that do not seem very likely) in which the upper bound on G can decrease. We will return to this point in the subsection on constraints.

Numerical examples

To provide one numerical example of these results, we use measured values[51] for a Cretaceous shale as the drained constants. Constants for the drained hexagonal (TI) medium are given in Table 1. Results of the evaluations are shown in Figures 2 and 3. Besides the constants listed in Table 1, we also needed values for the β's. To emphasize the dependence of the shear modulus on the liquid properties, we have chosen to use $\beta^{(1)} = 0.001476$ GPa^{-1} and $\beta^{(3)} = 0.02656$ GPa^{-1}, for this example. These choices are intended to mimic behavior of a sample with cracks oriented normal to the x_3-axis, having its strongest liquid dependence normal to the crack and significantly less dependence parallel to the crack. We use Skempton's coefficient B as a proxy for frequency in the case of the shear modulus (but it does not act as a frequency proxy for the bulk modulus), the lowest values of B corresponding to low frequency, Gassmann-like behavior of the shear modulus, while the highest values of B correspond to behavior expected at high frequencies when liquid saturates the pores. We find as predicted that the bulk modulus is the stronger function of the liquid properties, but that the shear modulus does indeed depend on them also. In this example, the Voigt bounds are seen to be monotonically increasing functions of Skempton's coefficient B for both moduli. The analysis shows that this must be so for the Reuss bounds. However, both the analysis and some examples (not shown) indicate the Voigt upper bound on the shear modulus can decrease slightly as B increases. For such cases, the VRH average will be almost constant, and may not capture the true behavior of the shear modulus. We see that there is a substantial quantitative difference between the changes in the bulk and shear moduli, the bulk modulus changing by as much as 100% in this example, whereas the shear modulus changes by only about 10%. (Also, note that Figure 2 provides graphical confirmation of the optimality[44] of the Voigt and Reuss bounds since they nearly meet here at about $B = 0.7$.)

The effective medium approach being used here and illustrated in Figures 2 and 3 falls short of proving that the shear modulus is dependent on the pore liquid properties. A definitive example would show, for example, that the Voigt upper bound at $B = 0$ lies below the Reuss lower bound at $B = 1$, *i.e.*, $G_V(B = 0) < G_R(B = 1)$. This does not occur in Figure 3, but the results do strongly suggest dependence of the effective G^{sat} on the liquid properties. The next subsection will provide an analytical example where it is easy to see that the shear modulus should increase when pore liquid is present.

A second numerical example is based on a complete set of poroelastic constants for Trafalgar shale from Cheng[52] using data from Aoki *et al.*,[53] and the theoretical formulation of Thompson and Willis.[54] The main elastic constant data are displayed in Table 2. The results are shown in Figures 4 and 5. In addition to the data in Table 2, Cheng quotes generalized Biot-Willis[16] parameters for the transversely isotropic shale having values of $\alpha^{(1)} = 0.733$ and $\alpha^{(3)} = 0.749$. These parameters can be shown to be related to the β's in our formulation by the following expressions: $\beta^{(1)} = \alpha^{(1)}(S_{11} + S_{12}) + \alpha^{(3)}S_{13}$ and $\beta^{(3)} = 2\alpha^{(1)}S_{13} + \alpha^{(3)}S_{33}$, giving values $\beta^{(1)} = 0.01821$ GPa^{-1} and $\beta^{(3)} = 0.02245$ GPa^{-1}. Results obtained for this example are qualitatively similar to those for the first example. The main difference is that the values of the β's used here do not differ as much, and therefore the dependence of the shear modulus on the liquid properties is not as great. We believe that the assumptions of microisotropy and microhomogeneity, which were used by Cheng[52] as a means to reduce the number of equations needed to determine the poroelastic constants from data, may in fact be stronger than necessary and lead directly to the close values obtained here for the β's.

5.3.2 An alternative approach

We have shown that in principle the shear modulus can depend on properties of a liquid in the pores. The results are rather indirect, however, and it might be helpful to see a more explicit way for this behavior to develop in the equations. Furthermore, the corrections to the shear modulus are expected on physical grounds to be positive. While explicit corrections of this type were found for the Reuss lower bound, it would be helpful to see how such corrections arise more intuitively from the mathematics.

Without enumerating all the remaining possible perturbations, we will now jump to another alternative, which makes use of the fact that, in addition to being anisotropic, the compliance matrix can be generalized to include some nonstandard terms. The form of (56) does not permit us to couple simple shear directly into either compression or pore pressure because there are no off-diagonal terms in the lower part of the matrix. Terms that can be added are those that produce a change in volume under an applied shear stress, and others that produce a change in shear strain under a compressional load (in this case pore pressure). We think of this, not as introducing new physics into the problem, but merely as a bookkeeping step to make the analysis simpler in this complex system under study. (The terms introduced could be obtained instead by performing a coordinate transformation to a system not aligned with the principal axes of the compliance matrix.)

There are many possibilities to consider that would achieve the desired result, but the simplest apparently has the form

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & -\beta^{(1)} & & & \\ S_{12} & S_{11} & S_{13} & -\beta^{(1)} & & & \\ S_{13} & S_{13} & S_{33} & -\beta^{(3)} & & & \\ -\beta^{(1)} & -\beta^{(1)} & -\beta^{(3)} & \gamma & & & -\omega \\ & & & & \frac{1}{G_t} & & \\ & & & & & \frac{1}{G_t} & \\ & & & -\omega & & & \frac{1}{G_{dr}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}, \tag{75}$$

The only terms that are new here are those involving ω, which couples σ_{12} to ζ and also couples p_f to e_{12}. Reciprocity requires that both terms are present if either is present. For these purposes, we think of the strains on the left of the equation as being resolved local stains, while the stresses on the right are global stresses.

Repeating Gassmann's argument once more, we have

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} = \begin{pmatrix} S_{11}^{sat} & S_{12}^{sat} & S_{13}^{sat} & & & S_{16}^{sat} \\ S_{12}^{sat} & S_{11}^{sat} & S_{13}^{sat} & & & S_{16}^{sat} \\ S_{13}^{sat} & S_{13}^{sat} & S_{33}^{sat} & & & S_{36}^{sat} \\ & & & \frac{1}{G_t^{sat}} & & \\ & & & & \frac{1}{G_t^{sat}} & \\ S_{16}^{sat} & S_{16}^{sat} & S_{36}^{sat} & & & \frac{1}{G^{sat}} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} \simeq \left[\begin{pmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{12} & S_{11} & S_{13} & & & \\ S_{13} & S_{13} & S_{33} & & & \\ & & & \frac{1}{G_t} & & \\ & & & & \frac{1}{G_t} & \\ & & & & & \frac{1}{G_{dr}} \end{pmatrix} \right.$$

$$\left. - \frac{1}{\gamma} \begin{pmatrix} (\beta^{(1)})^2 & (\beta^{(1)})^2 & \beta^{(1)}\beta^{(3)} & & & \beta^{(1)}\omega \\ (\beta^{(1)})^2 & (\beta^{(1)})^2 & \beta^{(1)}\beta^{(3)} & & & \beta^{(1)}\omega \\ \beta^{(1)}\beta^{(3)} & \beta^{(1)}\beta^{(3)} & (\beta^{(3)})^2 & & & \beta^{(3)}\omega \\ & & & 0 & & \\ & & & & 0 & \\ \beta^{(1)}\omega & \beta^{(1)}\omega & \beta^{(3)}\omega & & & \omega^2 \end{pmatrix} \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix}. \tag{76}$$

Finally, we have a connection of the right type. We see that $1/G^{sat} = 1/G_{dr} - \omega^2/\gamma$, or equivalently that

$$G^{sat} = \frac{G_{dr}}{1 - G_{dr}\omega^2/\gamma}, \tag{77}$$

showing that, since $G_{dr} \geq 0$ and $\gamma \geq 0$, the saturated (undrained) shear modulus always is larger than that of the drained medium regardless of the sign of the new parameter ω.

5.3.3 Constraints on the parameters

We could now study the resulting system by examining its eigenvectors and eigenvalues. Unfortunately, the modified system is more complex than a transversely isotropic system. It is actually monoclinic (see Nye[50]). It is beyond our present needs to study this full system, so we will simplify and neglect the off-diagonal terms that make this system differ from TI. Doing so introduces no error in three of the eigenvalues, but small errors of order $\beta^2\omega^2$ in the remaining three eigenvalues.

Once we have the constants for the saturated system, we can obtain estimates of the effective overall isotropic constants by making use of the Voigt and Reuss bounds as we did in the previous example. This does not produce a formula, but it does give us insight into how the liquid effects can influence the overall isotropic shear modulus of a poroelastic system. Using (62)-(73) again shows that the effective shear modulus depends on the liquid properties.

We also have requirements on the resulting matrix that it be positive semi-definite. This amounts to the physical requirement that the medium be mechanically stable, and the mathematical requirement that the eigenvalues all be non-negative. These requirements are therefore that G_t, G^{sat}, $S_{11} - S_{12}$, and $S_\pm$ [from (61)] must all be nonnegative. This places 4 independent constraints (actually 5 constraints when G^{sat} is decoupled from $S_{11} - S_{12}$ as it is in a monoclinic system) on any models we might want to consider. When $S_{33} \simeq S_{11}$, we have the approximation that $0 \leq S_- \simeq S_{33} - S_{13}^2/S_{12}$, and $0 \leq S_+ \simeq S_{11} + S_{12} + S_{13}^2/S_{12}$, which amount to a shear modulus constraint and a bulk modulus constraint, respectively.

Finally, we should point out that if the off-diagonal terms S_{16}^{sat} and S_{36}^{sat} are retained in the Reuss and Voigt averages, then it not difficult to show that the sign of correction due to fluid effects for G_V can now be guaranteed to be positive if $\omega \geq \beta^{(1)}/12$. Thus, relatively small corrections of the type presented here, which may be present in the real systems but difficult to measure, could be affecting these systems and causing the shear modulus to increase in the presence of the pore liquid.

These various constraints [or their more accurate counterparts for the exact expressions derived in (76)] should be considered when doing forward modeling with these equations to make sure that the stability criteria are always satisfied.

6 Conclusions

We have shown that there are two main issues affecting the possible occurrence of velocity dispersion in poroelastic systems. The first issue concerns the time scales implicitly assumed by Gassmann's derivation and by effective medium theory. Gassmann's theory applies at very low frequencies (long times), and should be thought of as a quasi-static approach. Fluid permeability is required to be finite, pores are connected, and fluid can pass from one pore to another. In contrast, effective medium theory, although formulated at low frequency, is not valid at such low frequencies as envisioned in the quasi-static picture of Gassmann. The practical difference is that Gassmann permits liquid to take as much time as it needs to equilibrate in pore pressure across the whole sample. In contrast, effective medium theory does *not* assume that different pockets of liquid have the same pore pressure values. Permeability in the effective medium picture might be either finite or zero. The time scales of interest may therefore be too fast to achieve the equilibrated pressures needed by Gassmann's arguments.

Although these differences are the most obvious physical ones and these differences are an important source of discrepancy between the two approaches, they are not sufficient to explain the range of dispersion results observed in experiments. If they were sufficient, then we would not be able to explain Plona's ultrasonic data on porous glass so well (Plona,[3] Chin *et al.*,[4] Johnson *et* *al.*[5]), both qualitatively and quantitatively. There must be more at issue.

The second significant issue concerns the fact that Gassmann's arguments (as usually presented) do not treat the porous medium as if it is inhomogeneous, whereas the effective medium theory inherently does so. This difference affects the results because in an inhomogeneous medium, when fields are applied externally, the local impact on the system can be very different from that of the applied field. In particular, if a pure compressional stress is applied externally, this may be resolved into local shear stresses at some points in the medium. Similarly, if a pure shear stress is applied externally, this may be resolved into local compressional stresses at some points in the medium. If the medium is porous, but the pores are empty (air filled), then the effective medium theory and Gassmann are in agreement. But in the same porous medium, if the pores are filled with liquid, then, for example, an external shear stress can be resolved into a local compressional stress that acts on the liquid. Elastic energy can then be stored in the liquid — energy that would not be stored in the pores in the absence of the liquid. Energy also would not be stored if the liquid were not trapped (finite versus zero permeability), so that no compression of the liquid occurred (some liquid leaves to avoid being compressed).

The mathematical expression of these physical arguments has been shown in one example (for transverse isotropy) to follow simply from the fact that, in anisotropic poroelastic media, it is expected that at least two of the eigenvectors of the system will contain coupled compressional and shear behavior.

The result of this effect can be very small in some situations, such as a porous medium with finite permeability but isotropic on the microscale, or it can be quite large for a very similar medium that is anisotropic on the microscale, as would be the case if randomly oriented liquid-filled fractures/cracks were present in the system.

A porous glass system that is uniform on the microscale might very well obey Gassmann's equations all the way up to frequencies at the low end of the megahertz band (before scattering effects become important), while naturally occurring porous systems such as rocks containing microcracks, might have observable frequency dispersion that sets in at frequencies as low as a few kilohertz. Both kinds of systems (as long as both have finite permeability) would, however, still be expected to obey Gassmann's equations at much lower (quasi-static) frequencies, as long as the system has finite permeability.

The main consequence of the foregoing analysis is that differences in shear modulus induced by the presence of liquid in the pores must be explicitly incorporated into the theory at ultrasonic frequencies when attempting to compare theory to laboratory experimental data. Carrying this strategy through will be the subject of future work.

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Table 1. Stiffnesses and compressibilities of a Cretaceous shale, estimated from ultrasonic laboratory measurements assuming hexagonal symmetry.[51]

ij	C_{ij} (GPa)	S_{ij} (GPa^{-1})
11	34.3	0.0370
33	22.7	0.0560
44	5.4	0.1852
66	10.6	0.0943
13	10.7	-0.0126

TABLE 2. Drained stiffnesses and compressibilities of Trafalgar shale, as derived by Cheng[52] using data of Aoki *et al.*[53]

ij	C_{ij} (GPa)	S_{ij} (GPa^{-1})
11	24.1	0.0485
33	21.0	0.0578
44	7.23	0.1383
66	8.66	0.1155
13	7.62	-0.0142
12	6.80	-0.0092

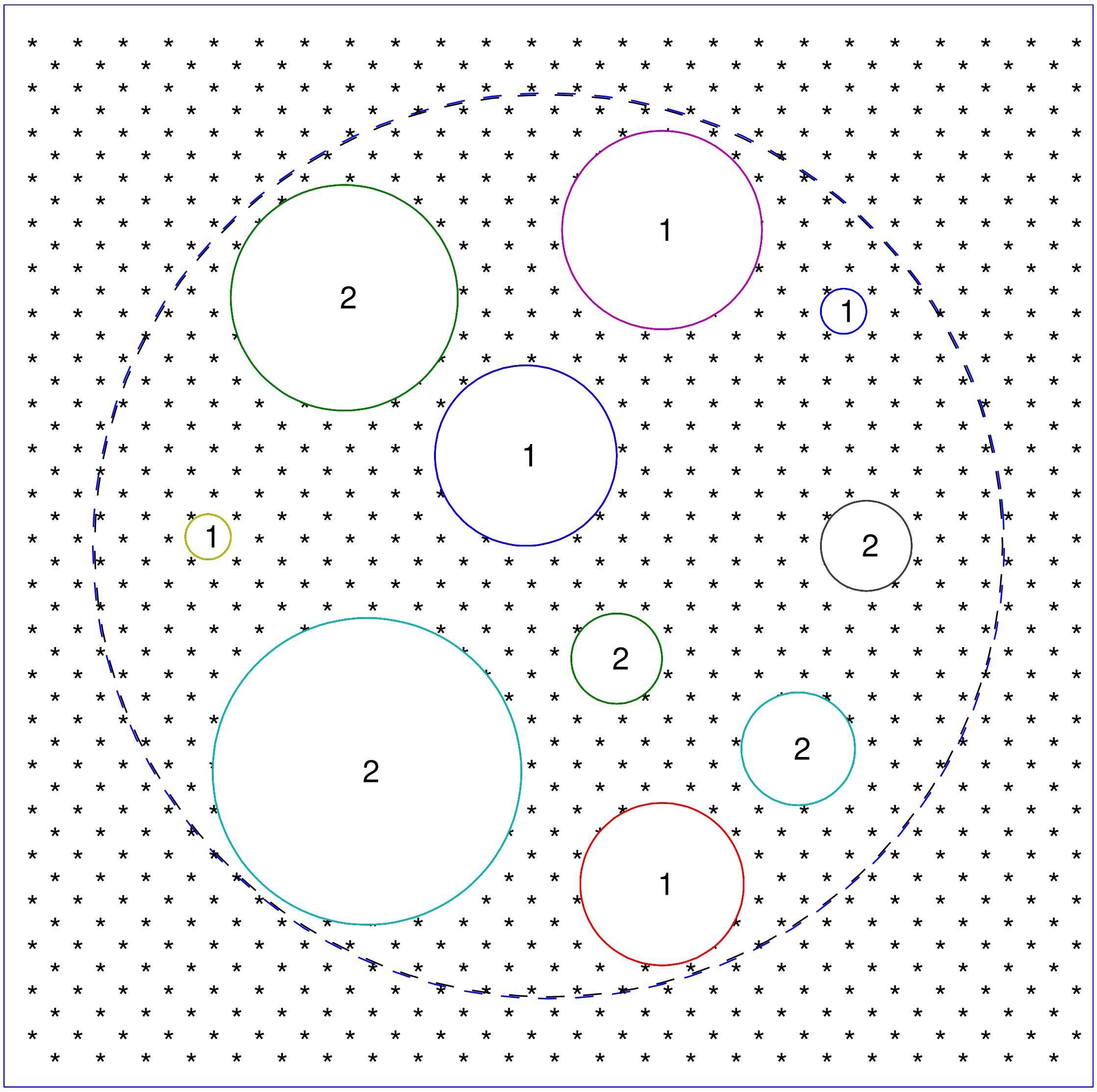

Figure 1: Schematic diagram for the coherent potential approximation (CPA) concept. The true composite is replaced by a large sphere (dashed circle) containing spherical inclusions of type-1 and type-2 constituents in the proper relative proportions. The matrix (type-*) is composed of a material with adjustable elastic constants. When the adjustments have been made so there is no net scattering, the composite and the surrounding matrix presumably have the same effective properties.

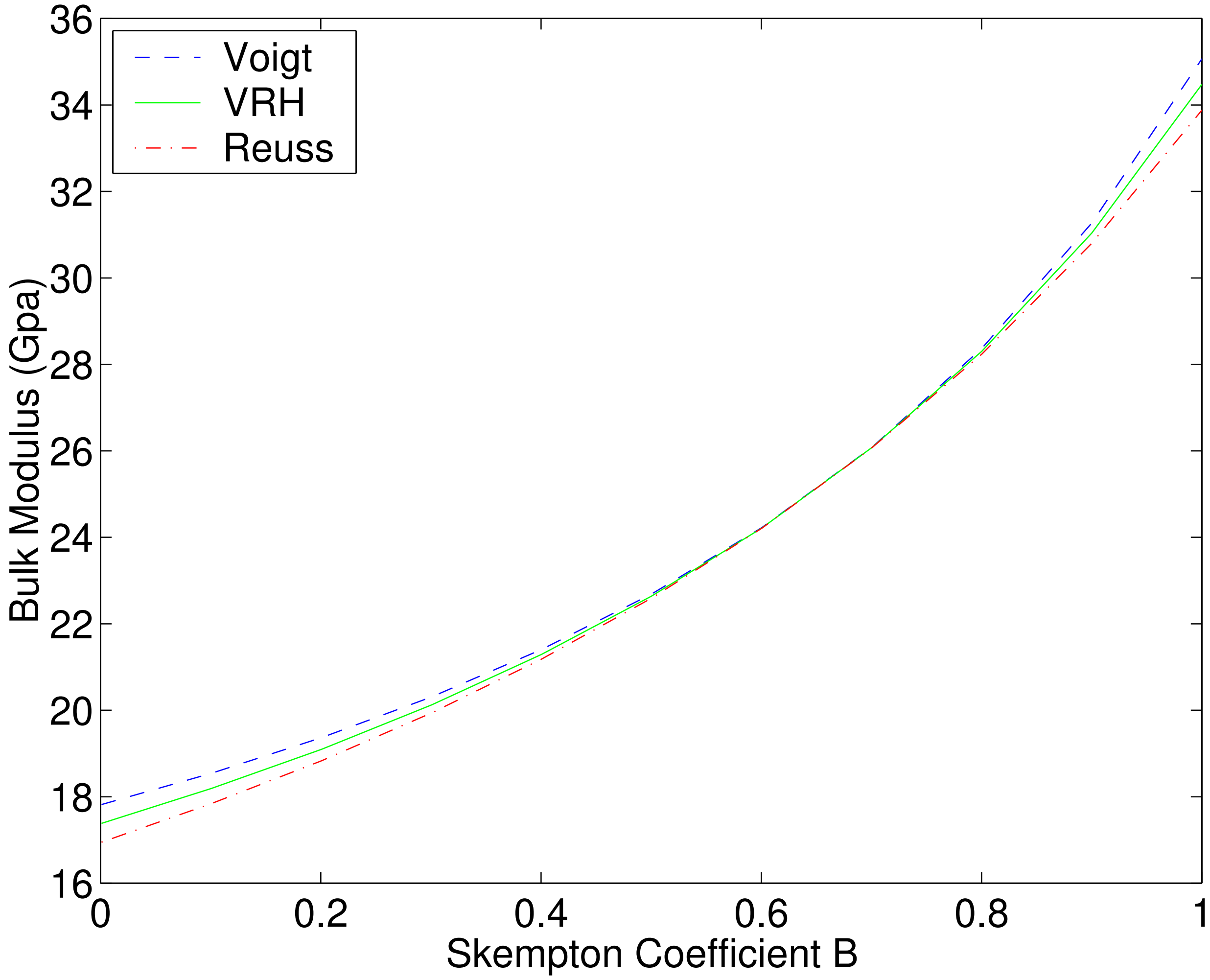

Figure 2: Example of bulk modulus estimates obtained using Reuss and Voigt bounds, and the Voigt-Reuss-Hill average. Constants for the drained hexagonal (TI) medium are given in Table 1. Note that this figure provides graphical confirmation of the optimality[44] of the Voigt and Reuss bounds since they nearly meet here at about $B = 0.7$.

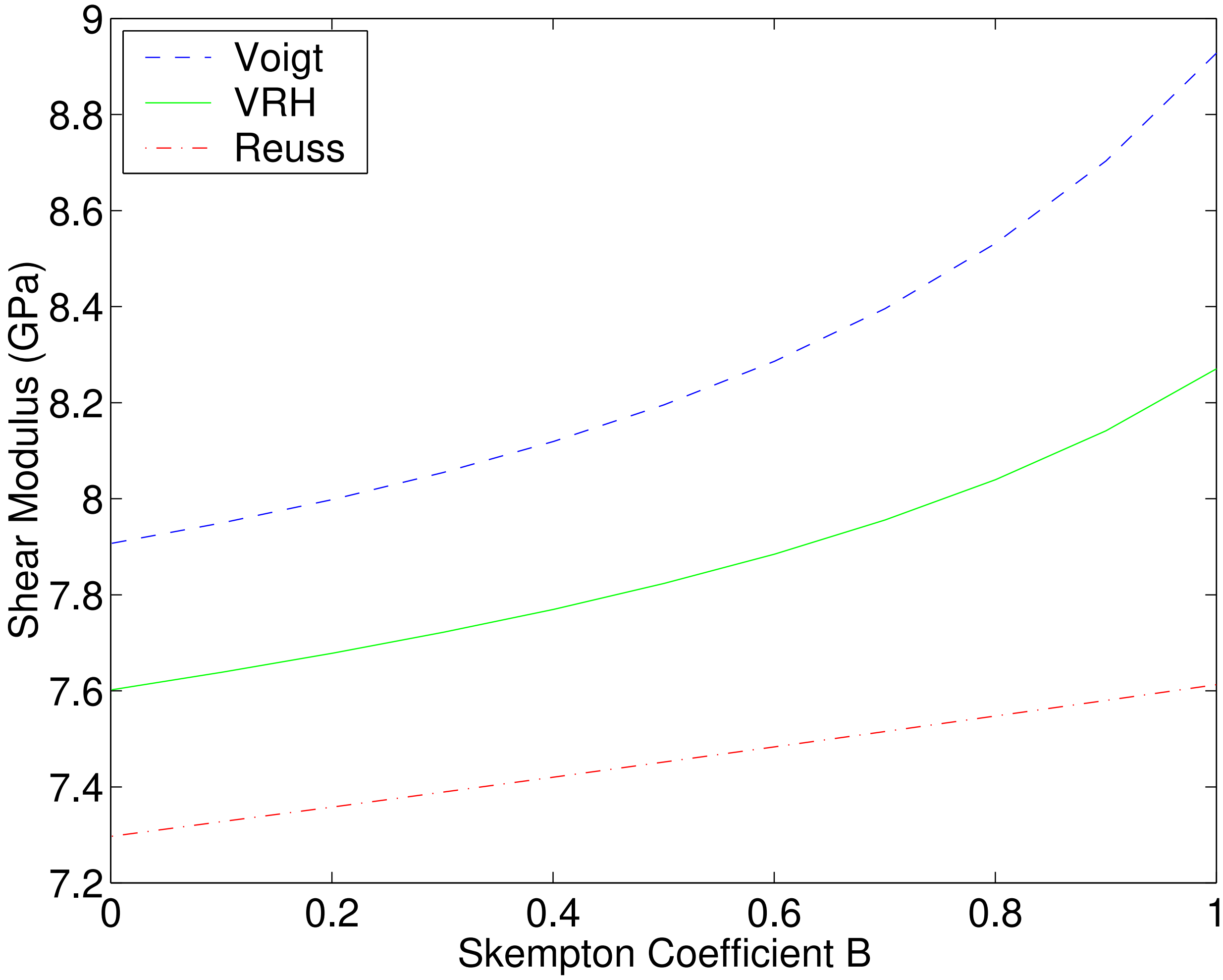

Figure 3: Example of shear modulus estimates obtained using Reuss and Voigt bounds, and the Voigt-Reuss-Hill average. Constants for the drained hexagonal (TI) medium are given in Table 1. Other poroelastic constants are in the text.

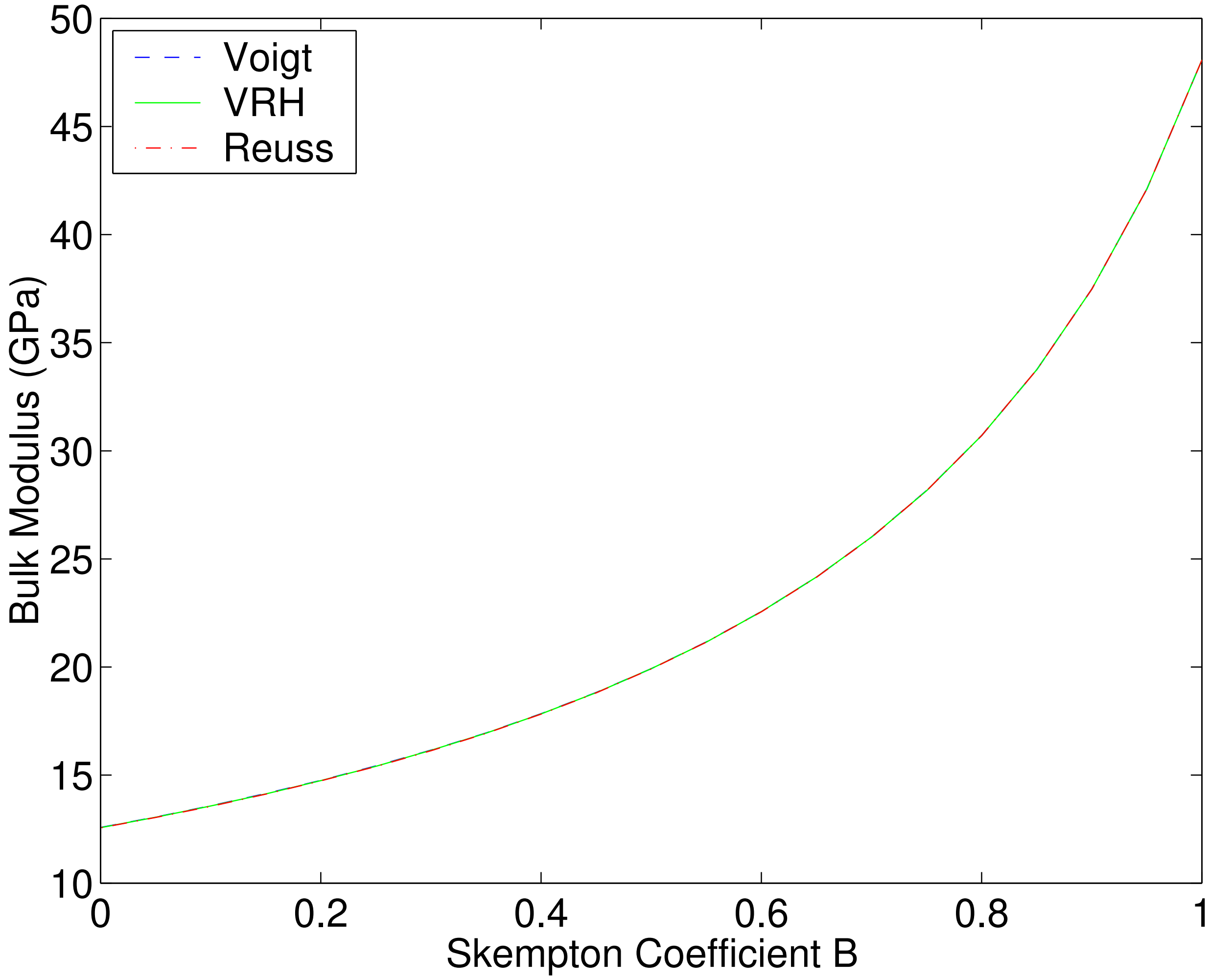

Figure 4: Example of bulk modulus estimates obtained using Reuss and Voigt bounds, and the Voigt-Reuss-Hill average. Constants for the drained Trafalgar shale are given in Table 2. This figure provides further graphical confirmation of the optimality[44] of the Voigt and Reuss bounds since they are very close over the entire range plotted here.

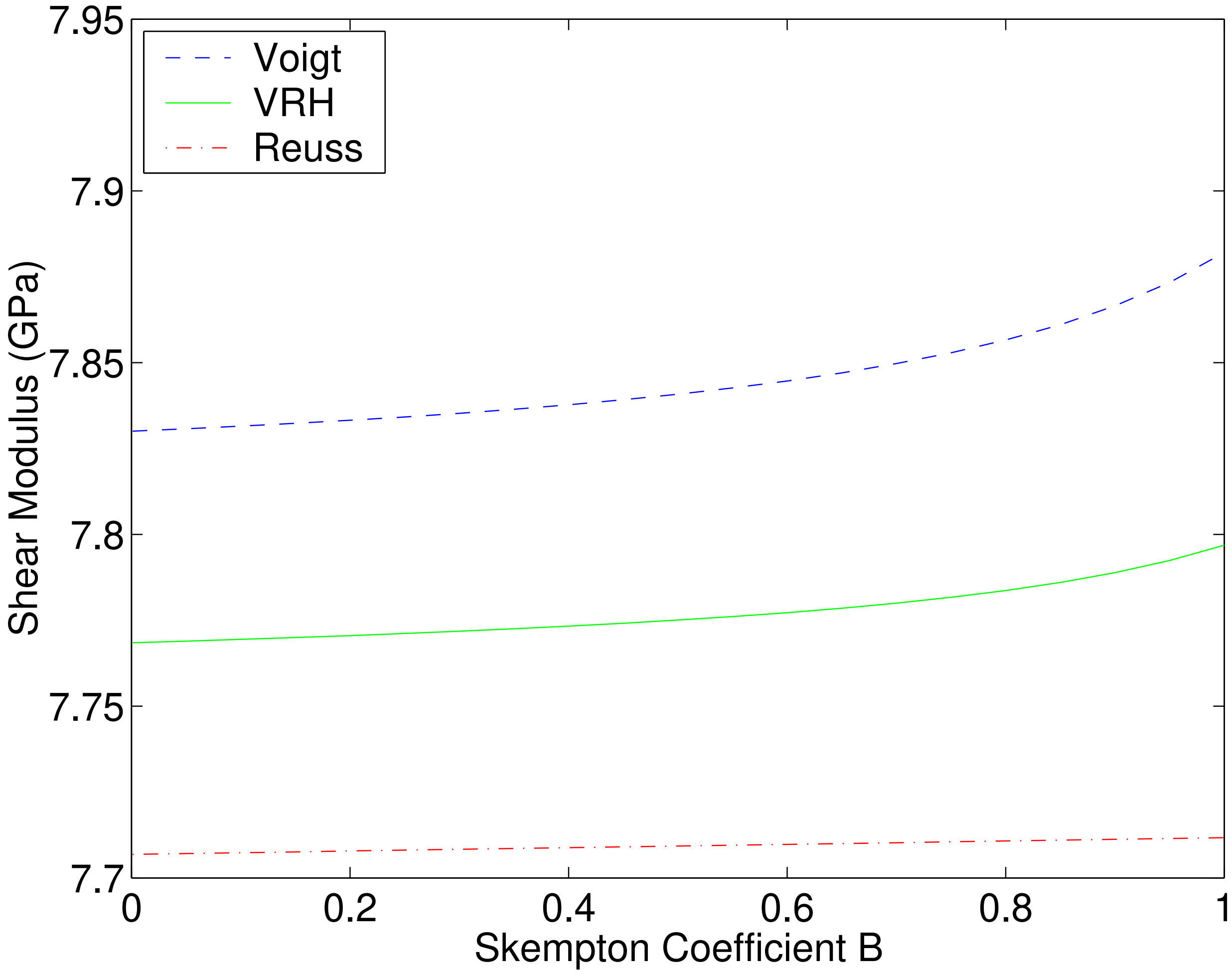

Figure 5: Example of shear modulus estimates obtained using Reuss and Voigt bounds, and the Voigt-Reuss-Hill average. Constants for the drained Trafalgar shale are given in Table 2. Other poroelastic constants are in the text.