Coherent Potential Approximation for Elasticity

Low Frequency Scattering Coefficients

$$B_0(K_m, K_i, G_m) = \frac{K_m - K_i}{3K_i + 4G_m}$$

$$B_2(G_m, G_i, K_m) = \frac{20G_m(G_i - G_m)/3}{6G_i(K_m + 2G_m) + G_m(9K_m + 8G_m)}$$

Host medium uses subscript m, inclusions i.

Inclusion is a spherical scatterer imbedded in the host for the present argument.

Derivation of the CPA Formulas

The composite (scattering) medium is imbedded in an adjustable matrix material $m = *$, such that each individual scatterer sees all the other scatterers as composing this matrix. Then, the composite inclusion, when imbedded in the $*$-matrix, should actually produce no scattering at all at infinity if the single-scattering coefficients satisfy

$$\sum_{i=1}^{n} f_i B_0(K^*, K_i, G^*) = 0,$$

$$\sum_{i=1}^{n} f_i B_2(G^*, G_i, K^*) = 0.$$

CPA Formulas

The final results we obtain using the CPA are

$$\frac{1}{K^{eff}+4G^{eff}/3} = \sum_{i=1}^{n} \frac{f_i}{K_i+4G^{eff}/3} = \left\langle \frac{1}{K(x)+4G^{eff}/3} \right\rangle,$$

and

$$\frac{1}{G^{eff}+F^{eff}} = \sum_{i=1}^{n} \frac{f_i}{G_i+F^{eff}} = \left\langle \frac{1}{G(x)+F^{eff}} \right\rangle,$$

where $F \equiv G(9K + 8G)/6(K + 2G)$. Note that these equations are coupled and must be solved iteratively. The notation $\langle \cdot \rangle$ is just the volume average.