Example: Seismic Traveltime Tomography

Various examples of the methods to be discussed in this talk could be provided, but to be keep the analysis both simple and concrete we will limit most of our discussion to physical problems based on seismic traveltime tomography. Other choices such ocean acoustic tomography or electrical impedance tomography , among many others, could have been made, but an understanding of seismic traveltime tomography is sufficient to introduce all the main ideas and yet complex enough to be a good introduction to the main difficulties with these methods as well. We will introduce the problem of seismic traveltime tomography now.

A typical problem in seismic traveltime tomography is to infer the (isotropic) compressional-wave velocity distribution of an inhomogeneous medium, given a set of observed first-arrival traveltimes between sources and receivers of known location within the medium. This problem is common for borehole-to-borehole seismic tomography in oil field applications. We might also consider the problem of inverting for wave slowness when the absolute traveltimes are not known, as is normally the case in earthquake seismology but will not discuss this further today.

Slowness Models

We can consider three kinds of slowness (reciprocal of velocity) models. If we permit the slowness to be a general function of position, $s(\mathbf{x})$, then this is a continuum model, which are explicitly excluding from consideration here. We can instead make one of two more restrictive assumptions that (i) the model comprises homogeneous blocks, or cells, with s_j denoting the slowness value of the jth cell, or (ii) the model is composed of a grid with values of slowness assigned at the grid points with some interpolation scheme to assign the values between grid points.

When it is not important which type of slowness model is involved, we will refer to the model abstractly as a vector $\mathbf{s}$ in a vector space. For a block model with n blocks, the model becomes a vector in R^n, the n-dimensional Euclidean vector space.

Fermat's Principle and Traveltime Functionals

The traveltime of a seismic wave is the integral of slowness along a ray path connecting the source and receiver. To make this more precise, we will define two functionals for traveltime.

Let P denote an arbitrary path connecting a given source and receiver in a slowness model $\mathbf{s}$. We will refer to P as a *trial ray path*. We define a functional τ^P which yields the traveltime along path P. Letting $\mathbf{s}$ be the continuous slowness distribution $s(\mathbf{x})$, we have

$$\tau^P(\mathbf{s}) = \int_P s(\mathbf{x})\, dl^P, \tag{1}$$

where dl^P denotes the infinitesimal distance along the path P.

Fermat's principle states that the correct ray path between two points is the one of least overall traveltime, *i.e.*, it minimizes $\tau^P(\mathbf{s})$ with respect to path P. [Actually, Fermat's principle is the weaker condition that the traveltime integral is *stationary* with respect to variations in the ray path, but for traveltime tomography using measured first arrivals it follows that the traveltimes must truly be *minima*.]

Let us define τ^* to be the functional that yields the traveltime along the Fermat (least-time) ray path. Fermat's principle then states

$$\tau^*(\mathbf{s}) = \min_{P \in Paths} \tau^P(\mathbf{s}), \tag{2}$$

where *Paths* denotes the set of all continuous paths connecting the given source and receiver. The particular path that produces the minimum in (2) is denoted P^*. If more than one path produces the same minimum traveltime value, then P^* denotes any particular member in this set of minimizing paths.

To summarize, we have

$$\tau^*(\mathbf{s}) = \int_{P^*} s(\mathbf{x})\, dl^{P^*} = \min_P \int_P s(\mathbf{x})\, dl^P = \min_P \tau^P(\mathbf{s}). \tag{3}$$

The traveltime functional $\tau^*(\mathbf{s})$ is stationary with respect to small variations in the path $P^*(s)$.

Snell's law is a well-known and easily derived consequence of the stationarity of the traveltime functional [Feynman *et al.*, 1963; Whitham, 1974].

Seismic Inversion

Suppose we have a set of observed traveltimes, $t_1, \ldots, t_m$, from m source-receiver pairs in a medium of slowness $s(\mathbf{x})$. Let P_i be the Fermat ray path connecting the ith source-receiver pair. Neglecting observational errors, we can write

$$\int_{P_i} s(\mathbf{x})\, dl^{P_i} = t_i, \quad i = 1, \ldots, m. \tag{4}$$

Given a block model of slowness, let l_{ij} be the length of the ith ray path through the jth cell:

$$l_{ij} = \int_{P_i \cap \mathrm{cell}_j} dl^{P_i}. \tag{5}$$

Given a model with n cells with the jth cell having constant slowness s_j, (4) can then be written

$$\sum_{j=1}^{n} l_{ij} s_j = t_i, \quad i = 1, \ldots, m. \tag{6}$$

Note that for any given i, the ray-path lengths l_{ij} are zero for most cells j, since a given ray path will in general intersect only a few of the cells in the model. Figure **??** illustrates the ray path intersections for a 2-D block model.

We can rewrite (6) in matrix notation by defining the column vectors $\mathbf{s}$ and $\mathbf{t}$ and the matrix $\mathbf{M}$ as follows:

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}, \qquad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mn} \end{pmatrix}. \tag{7}$$

Equation (6) then becomes

$$\mathbf{Ms} = \mathbf{t}. \tag{8}$$

Equation (8) is just the discretized form of (3).

Equation (8) will be become our generic inversion problem in the remainder of this talk. The right hand side is the *data* m-vector $\mathbf{t}$, whose elements are the m traveltime measurements in this application. The left hand side has as one of its factors an $m \times n$ matrix $\mathbf{M}$, which for this talk will be assumed known (but for real applications is often another unknown in the problem). The remaining factor is the *model* n-vector $\mathbf{s}$, which is the model we seek in the inversion process. In practice, either $m < n$ or $m > n$ may occur in real problems. We refer to the case $m < n$ as underdetermined and the case $m > n$ as overdetermined. Most of the remainder of the article will concentrate on solving this equation approximately for $\mathbf{s}$, and simultaneously providing some means of analyzing how well the method of approximate solution has worked.