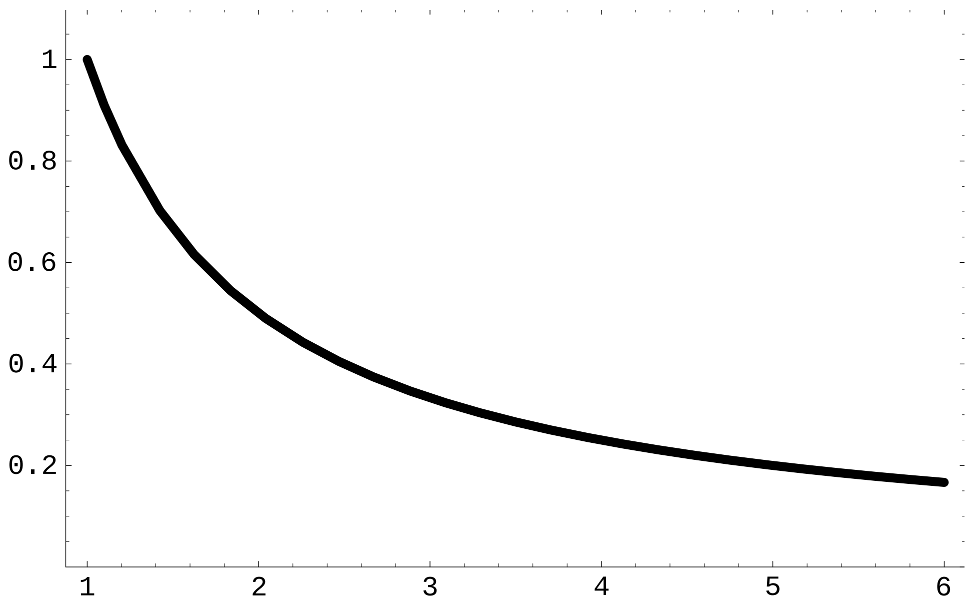
Math 128A Spring 2002 Handout # 9
Sergey Fomel February 19, 2002

Polynomial Interpolation (Mathematica notebook: http://math.lbl.gov/~fomel/128A/Polynomial.nb)

Example Function

An example function for studying polynomial interpolation is

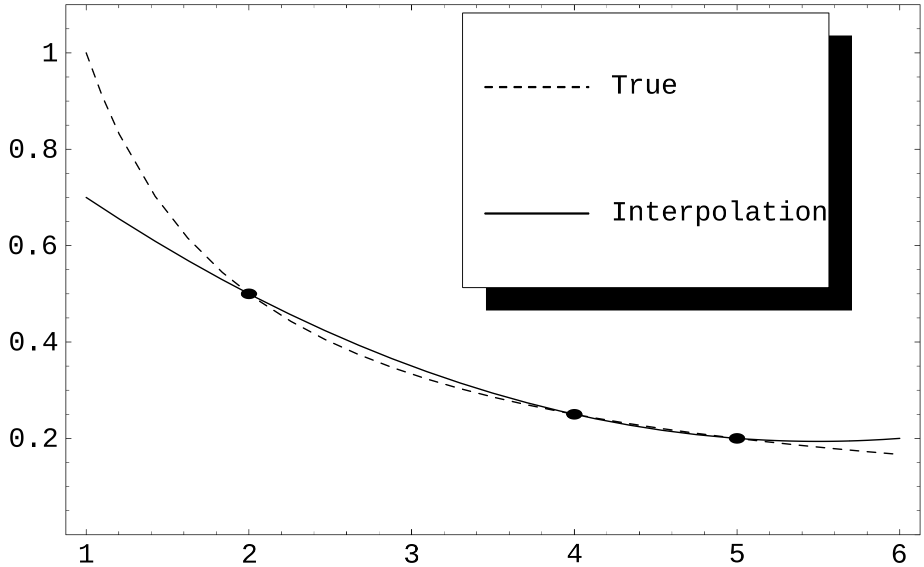
$$f(x) = \frac{1}{x} \,.$$



We select three nodes and find the interpolating polynomial for them.

x	2	4	5
f	0.5	0.25	0.2

The second-order interpolation polynomial does not reconstruct the function exactly but fits the input data.



Lagrange Interpolation

The Lagrange form of the interpolation polynomial is

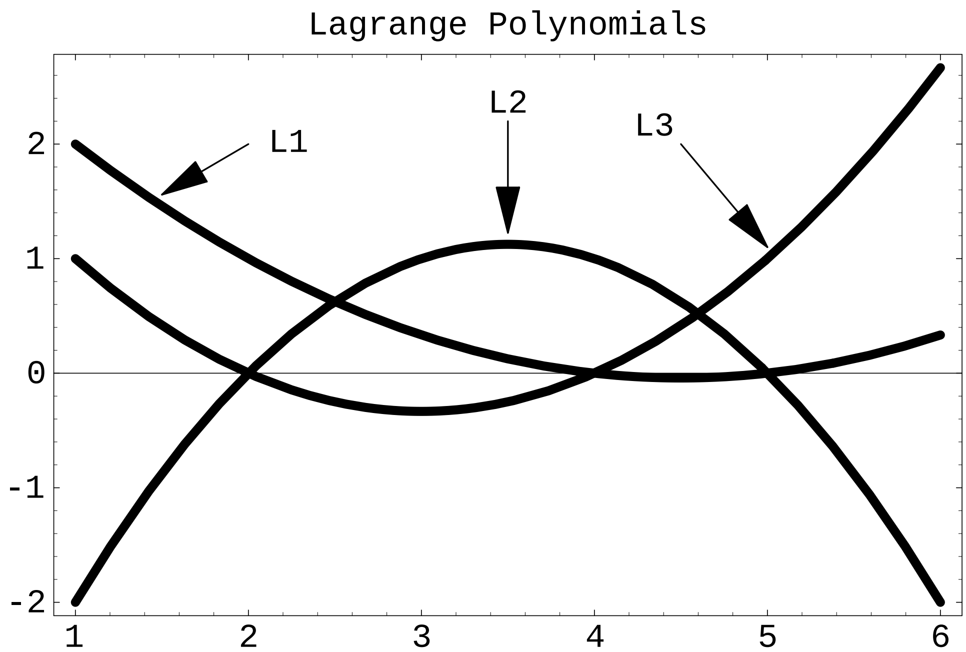
$$P(x) = \sum_{k=1}^{n} f_k \, L_k(x) \, ,$$

where

$$L_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i} \, .$$

In our example,

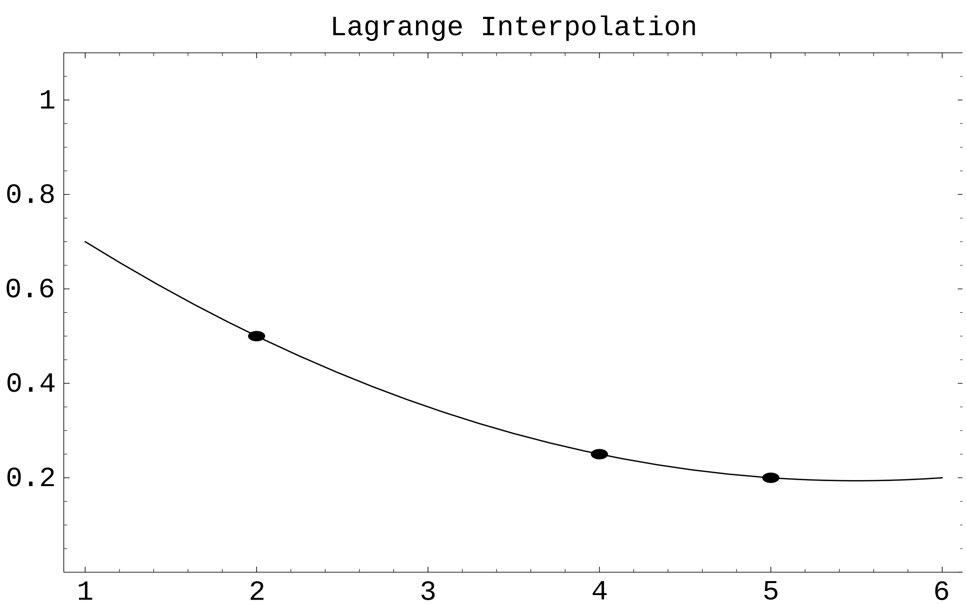
$$\begin{aligned} L_1(x) &= \frac{(x-4)(x-5)}{(2-4)(2-5)} = \frac{(x-4)(x-5)}{6} \\ L_2(x) &= \frac{(x-2)(x-5)}{(4-2)(4-5)} = -\frac{(x-2)(x-5)}{2} \\ L_3(x) &= \frac{(x-2)(x-4)}{(5-2)(5-4)} = \frac{(x-2)(x-4)}{3} \end{aligned}$$



$L_1(x)$ is one at $x_1 = 2$ and zero at $x_2 = 4$ and $x_3 = 5$. Likewise, $L_2(x)$ is one at x_2 and zero at x_1 and x_3. $L_3(x)$ is one at x_3 and zero at x_1 and x_2.

Putting it all together,

$$P(x) = \frac{(x-4)(x-5)}{12} - \frac{(x-2)(x-5)}{8} + \frac{(x-2)(x-4)}{15} = 0.025\,x^2 - 0.275\,x + 0.95\,.$$



Newton Interpolation

The Newton form of the interpolation polynomial is

$$P(x) = \sum_{k=1}^{n} f[x_1\, x_2, \ldots, x_k]\, N_k(x)\,,$$

where

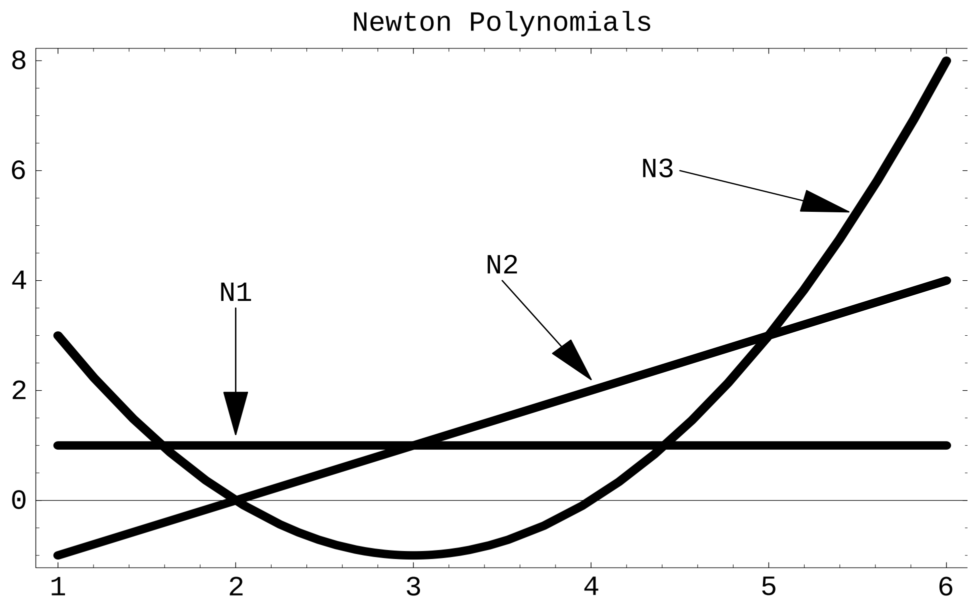
$$N_k(x) = \prod_{i=1}^{k-1}(x - x_i)\,,$$

and $f[x_1\,x_2,\ldots,x_k]$ is the divided difference, evaluated with the help of the recursive relationship

$$\begin{aligned} f[x_k] &= f_k \\ f[x_1\,x_2,\ldots,x_k] &= \frac{f[x_2,\ldots,x_k] - f[x_1\,\ldots,x_{k-1}]}{x_k - x_1} \end{aligned}$$

In our example,

$$\begin{aligned} N_1(x) &= 1 \\ N_2(x) &= x-2 \\ N_3(x) &= (x-2)(x-4) \end{aligned}$$

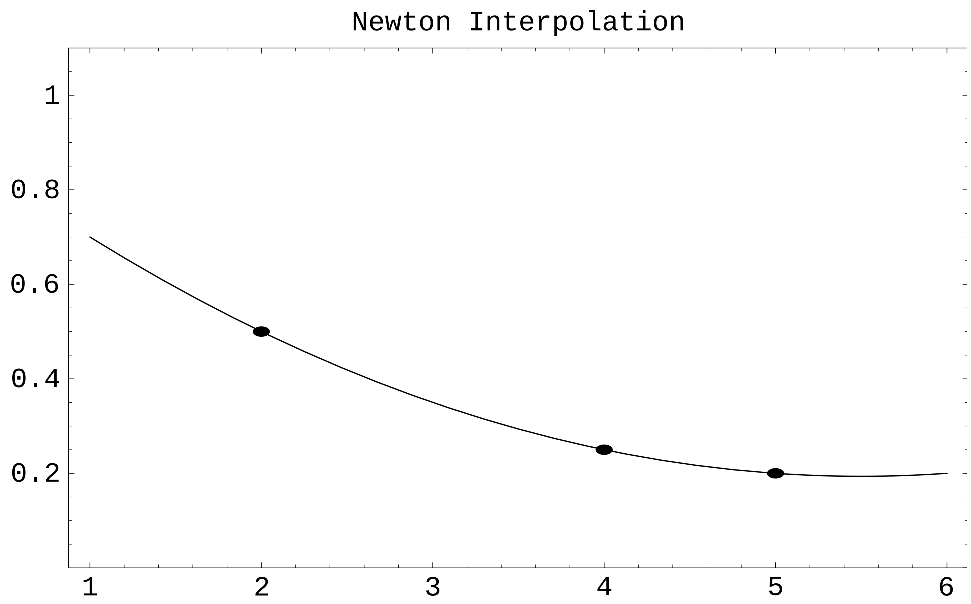


The divided difference table is

$f[x_1] = 0.5$		
$f[x_2] = 0.25$	$f[x_1, x_2] = \frac{0.25-0.5}{4-2} = -0.125$	
$f[x_3] = 0.2$	$f[x_2, x_3] = \frac{0.2-0.25}{5-4} = -0.05$	$f[x_1, x_2, x_3] = \frac{-0.05+0.125}{5-2} = 0.025$

Putting it all together,

$$P(x) = \frac{1}{2} - \frac{x-2}{8} + \frac{(x-2)(x-4)}{40} = 0.025\,x^2 - 0.275\,x + 0.95\,.$$



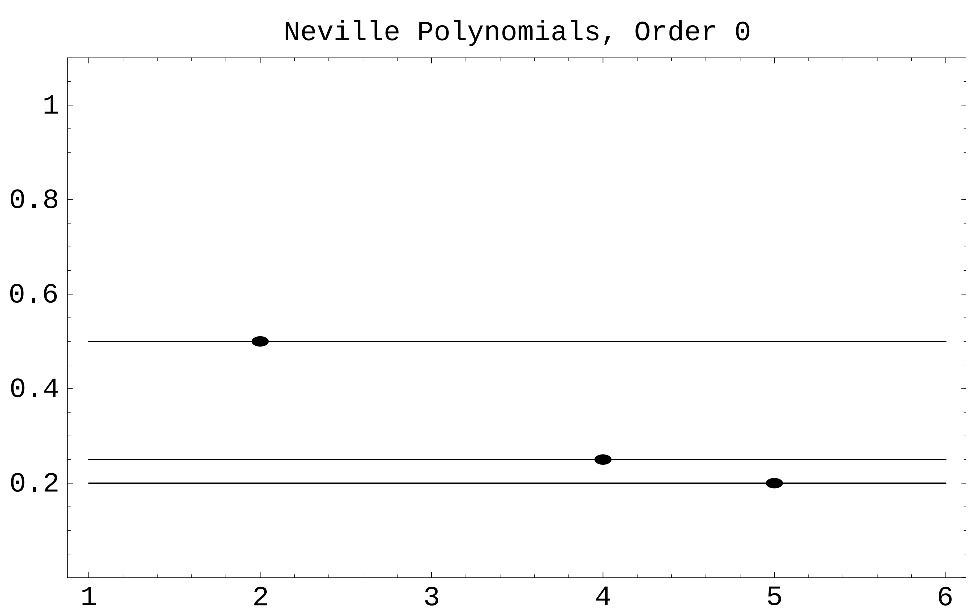
Neville Interpolation

The Neville form of the interpolation polynomial is defined by recursion

$$
\begin{aligned}
P_0(x) &= f_1 \\
P_{k-1}(x) &= \frac{P_{k-2}(x)(x_k - x) - Q_{k-2}(x)(x - x_1)}{x_k - x_1},
\end{aligned}
$$

where $P_{k-1}(x)$ interpolates at nodes $x_1, x_2, \ldots, x_k$, and $Q_{k-2}(x)$ interpolates at nodes $x_2, \ldots, x_k$.

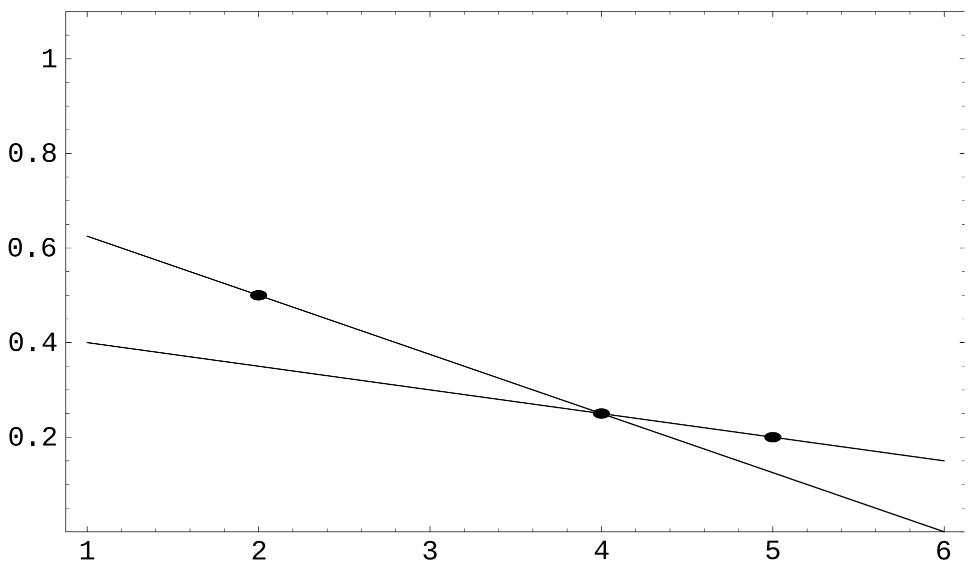
The zeroth-order Neville polynomials are constant functions.



The first-order Neville polynomials are

$$P_1(x) = \frac{\frac{1}{2}(4-x)+\frac{1}{4}(x-2)}{4-2} = \frac{6-x}{8}$$
$$Q_1(x) = \frac{\frac{1}{4}(5-x)+\frac{1}{5}(x-4)}{5-4} = \frac{9-x}{20}$$

Neville Polynomials, Order 1



The second-order Neville polynomial is

$$P_2(x) = \frac{\frac{6-x}{8}(5-x)+\frac{9-x}{20}(x-2)}{5-2} = \frac{(6-x)(5-x)}{24} + \frac{(9-x)(x-2)}{60} = 0.025\,x^2 - 0.275\,x + 0.95\,.$$

Neville Interpolation