Math 128A Spring 2002 Handout # 20
Sergey Fomel April 2, 2002

# Homework 8: Numerical Differentiation (due on April 9)

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1. Approximate $f'(x_0)$ and $f''(x_0)$ using the values of $f(x)$ at $x_0-h$, $x_0$ and $x_0+\alpha\,h$ ($\alpha>0$)

   (a) applying the polynomial interpolation method

   (b) applying the Taylor series method

   Assuming $f(x)\in C^3$, evaluate the approximation error using either of the two methods. What is the approximation order?

2. Apply the polynomial interpolation method at $2\,n+1$ regularly spaced points

$$x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n$$

   with $x_k = x_0 + k\,h$, $k=-n,-n+1,\ldots,-1,0,1,\ldots,n-1,n$ to derive the approximation

$$f^{(2n)}(x_0) \approx \frac{\Delta^{2n} f(x_{-n})}{h^{2n}} = \sum_{k=-n}^{n} \frac{(-1)^{k+n}}{h^{2n}} \left( \begin{array}{c} 2\,n \\ k+n \end{array} \right) f(x_k)\,, \tag{1}$$

   where $\Delta f(x) = f(x+h) - f(x)$.

   Hint: Recall Stirling's interpolation formula from Homework 5.

3. Richardson extrapolation can be implemented with the following algorithm:

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RICHARDSON(N(x), h, tol, n)
for k ← 1, 2, ..., n
do
    R_{k,1} ← N(h)
    t ← 1
    for i ← 1, 2, ..., k − 1
    do
        t ← 2t
        R_{k,i+1} ← R_{k,i} + (R_{k,i} − R_{k−1,i})/(t − 1)
    if k > 1 and |R_{k,k} − R_{k−1,k−1}| ≤ tol
       then return R_{k,k}
    h ← h/2
return R_{n,n}
```

The algorithm successively fills the rows of the triangular matrix

$$\begin{bmatrix} R_{1,1} & & & \\ R_{2,1} & R_{2,2} & & \\ \vdots & \vdots & \ddots & \\ R_{n,1} & R_{n,2} & \cdots & R_{n,n} \end{bmatrix} . \tag{2}$$

Modify the algorithm so that only one row of length $n$ is stored in memory instead of the whole matrix.

Hint: Rearrange the matrix in the form

$$\begin{bmatrix} & & & R_{1,1} \\ & & R_{2,1} & R_{2,2} \\ & \iddots & \vdots & \vdots \\ R_{n,1} & \cdots & R_{n,n-1} & R_{n,n} \end{bmatrix} . \tag{3}$$

4. (Programming) In Homework 1, we applied an ancient geometric method to compute the value of $\pi$. The approximation formula is

$$\pi \approx \frac{k\,L_k}{2}\,, \tag{4}$$

where $L_k$ (the side of a regular polygon) satisfies the recursion

$$L_{2k} = \frac{L_k}{\sqrt{2+\sqrt{4-L_k^2}}}\,. \tag{5}$$

starting with $L_6 = 1$.

Implement the Richardson extrapolation algorithm and apply it to accelerate the convergence of the geometric estimation of $\pi$. Start with $k = 6$, take $h = 6/k$, $N(h) = k\,L_k/2$ and output five rows of the Richardson table.

5. (Programming)

(a) Compute the derivative of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ using

i. the forward difference approximation

ii. the central difference approximation

iii. your approximation from problem 1 with $\alpha = 1/2$.

Use the step size $h = 10^{-n}$ for $n = 0, 1, 2, \ldots, 15$. Perform all the computations with double precision and output your results in a table. Explain the difference between the rows and columns of the table.

(b) Compute the derivative of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ using the forward difference approximation and the Richardson extrapolation algorithm. Start with $h = 1$ and find the number of rows in the Richardson table required to estimate the derivative with six significant decimal digits. Output the table.