Homework 7: Approximation: Polynomial Approximation (due on March 21)

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1. (a) Is the collection of functions $\phi_1(x) = 1$, $\phi_2(x) = x$, and $\phi_3(x) = \sin x$ orthogonal with respect to the inner product

$$<f,g> = \int\limits_{-\pi}^{\pi} f(x)\,g(x)\,dx \quad ? \tag{1}$$

If not, find the corresponding orthogonal functions using the Gram-Schmidt orthogonalization process

$$\hat{\phi}_1(x) = \phi_1(x)\,; \tag{2}$$

$$\hat{\phi}_k(x) = \phi_k(x) - \sum_{i=1}^{k-1} \frac{<\phi_k,\hat{\phi}_i>}{<\hat{\phi}_i,\hat{\phi}_i>}\,\hat{\phi}_i(x)\,, \quad k = 2,3,\ldots \tag{3}$$

(b) Using the Gram-Schmidt process, find the first three orthogonal polynomials with respect to the inner product

$$<f,g> = \int\limits_{0}^{\infty} w(x)\,f(x)\,g(x)\,dx\,, \tag{4}$$

where $w(x) = e^{-a\,x}$ $(a > 0)$.

Hint: Use the equality (for integer n)

$$\int\limits_{0}^{\infty} x^n\,e^{-a\,x}\,dx = \frac{n!}{a^{n+1}}\,.$$

2. (a) Prove that the constant function $f(x) = a$ that fits inconsistent measurements $f_1, f_2, \ldots, f_n$ in the least-squares sense corresponds to the mean value

$$a = \frac{1}{n}\sum_{k=1}^{n} f_k\,. \tag{5}$$

(b) Prove that, if the measurements $f_1, f_2, \ldots, f_n$ are taken at the integer values $x_k = k$, $k = 1,2,\ldots,n$, the linear function $f(x) = a\,x + b$ that fits the data in the least-squares sense corresponds to the values

$$a = \frac{6}{n\,(n^2-1)}\left[2\sum_{k=1}^{n} k\,f_k - (n+1)\sum_{k=1}^{n} f_k\right]; \tag{6}$$

$$b = \frac{2}{n\,(n-1)}\left[(2\,n+1)\sum_{k=1}^{n} f_k - 3\sum_{k=1}^{n} k\,f_k\right]. \tag{7}$$

3. Chebyshev polynomials $T_k(x)$ can be defined by the recursive relationship

$$\begin{aligned} T_0(x) &= 1 \quad (8) \\ T_1(x) &= x \quad (9) \\ T_{k+1}(x) &= 2\,x\,T_k(x) - T_{k-1}(x)\,, \quad k = 1,2,\ldots \quad (10) \end{aligned}$$

One can evaluate the Chebyshev polynomial representation

$$f(x) = \sum_{k=0}^{n} c_k\, T_k(x) \tag{11}$$

efficiently with the algorithm

CHEBYSHEV SUM$(x, c_0, c_1, \ldots, c_n)$

$\hat{c}_1 \leftarrow 0$
$\hat{c}_0 \leftarrow c_n$
for $k \leftarrow n-1, n-2, \ldots, 0$
do
    $t \leftarrow \hat{c}_1$
    $\hat{c}_1 \leftarrow \hat{c}_0$
    $\hat{c}_0 \leftarrow c_k + 2\,x\,\hat{c}_0 - t$
return $(\hat{c}_0 - x\,\hat{c}_1)$

Design an analogous algorithm for the *Hermite* polynomial representation

$$f(x) = \sum_{k=0}^{n} c_k\, H_k(x)\,. \tag{12}$$

Hermite polynomials $H_k(x)$ satisfy the recursive relationship

$$\begin{aligned} H_0(x) &= 1 \quad (13) \\ H_1(x) &= 2\,x \quad (14) \\ H_{k+1}(x) &= 2\,x\,H_k(x) - 2\,k\,H_{k-1}(x)\,, \quad k = 1,2,\ldots \quad (15) \end{aligned}$$

4. (Programming)

   (a) Write a program for evaluating Chebyshev polynomial representation using the algorithm above. Test your program by approximating the infinite sum

$$\frac{1 - t\,x}{1 - 2\,t\,x + t^2} = \sum_{k=0}^{\infty} t^k\, T_k(x) \tag{16}$$

   with the finite sum

$$\sum_{k=0}^{n} t^k\, T_k(x)\,. \tag{17}$$

   Plot (or tabulate) the absolute error on the interval $-1 \le x \le 1$ for $t = 1/2$ and $n = 5, 10, 15$.

(b) Write a program for evaluating Hermite polynomial representation using your algorithm from problem 3. Test your program by approximating the infinite sum

$$e^{t\,(2x-t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, H_k(x) \tag{18}$$

with the finite sum

$$\sum_{k=0}^{n} \frac{t^k}{k!}\, H_k(x)\,. \tag{19}$$

Plot (or tabulate) the absolute error on the interval $0 \le x \le 1$ for $t = 1/2$ and $n = 5, 10, 15$.

5. (Programming) The following table contains the number of medals won by the United States at the winter Olympic games:

Year	Location	Gold	Silver	Bronze	Total	Points
1924	CHAMONIX	1	2	1	4	8
1928	SAINT MORITZ	3	2	2	7	15
1932	LAKE PLACID	6	4	2	12	28
1936	GARMISH PARTENKIRCHEN	1	0	3	4	6
1948	SAINT MORITZ	3	5	2	10	21
1952	OSLO	4	6	1	11	25
1956	CORTINA D'AMPEZZO	2	3	2	7	14
1960	SQUAW VALLEY	3	4	3	10	20
1964	INNSBRUCK	1	2	4	7	11
1968	GRENOBLE	1	4	1	6	12
1972	SAPPORO	3	2	3	8	16
1976	INNSBRUCK	3	3	4	10	19
1980	LAKE PLACID	6	4	2	12	28
1984	SARAJEVO	4	4	0	8	20
1988	CALGARY	2	1	3	6	11
1992	ALBERTVILLE	5	4	2	11	25
1994	LILLEHAMMER	6	5	2	13	30
1998	NAGANO	6	3	4	13	28
2002	SALT LAKE CITY	10	13	11	34	67

The points are computed with the formula

$$\text{Points} = 3 \times \text{Gold} + 2 \times \text{Silver} + \text{Bronze}\,.$$

Using the method of least squares, find linear trends of the form $f(x) = a + b\,x$ for the functions

(a) Points(Year)

(b) Points(Gold)

In each case, find a, b and the Olympic games with the largest and smallest least-squares errors.