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Homework 4: Interpolation: Polynomial Interpolation (due on February 21)

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1. Prove that the sum of the Lagrange interpolating polynomials

$$L_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i} \tag{1}$$

is one:

$$\sum_{k=1}^{n} L_k(x) = 1 \tag{2}$$

for any real x, integer n, and any set of distinct points $x_1, x_2, \ldots, x_n$.

2. Let $f(x) = x^{n-1}$ for some $n \geq 1$. Find the divided differences
$f[x_1, x_2, \ldots, x_n]$ and $f[x_1, x_2, \ldots, x_n, x_{n+1}]$,
where $x_1, x_2, \ldots, x_n, x_{n+1}$ are distinct numbers.

3. (a) Consider a set of regularly spaced nodes on interval $[a, b]$:

$$h = \frac{b-a}{n}\,, \quad x_k = a + (k-1)\,h\,, \quad k = 1, 2, \ldots, n+1\,. \tag{3}$$

Prove that the polynomial

$$N(x) = (x - x_1)(x - x_2) \cdots (x - x_{n+1}) \tag{4}$$

satisfies

$$|N(x)| \leq n!\, h^{n+1}\,, \quad a \leq x \leq b \tag{5}$$

(b) Using the result of problem (a), prove that if $f(x) = e^x$ and $P_n(x)$ is the interpolating polynomial of order n defined at the $n+1$ regularly spaced nodes

$$x_k = \frac{k-1}{n}\,, \quad k = 1, 2, \ldots, n+1 \tag{6}$$

then the interpolation error

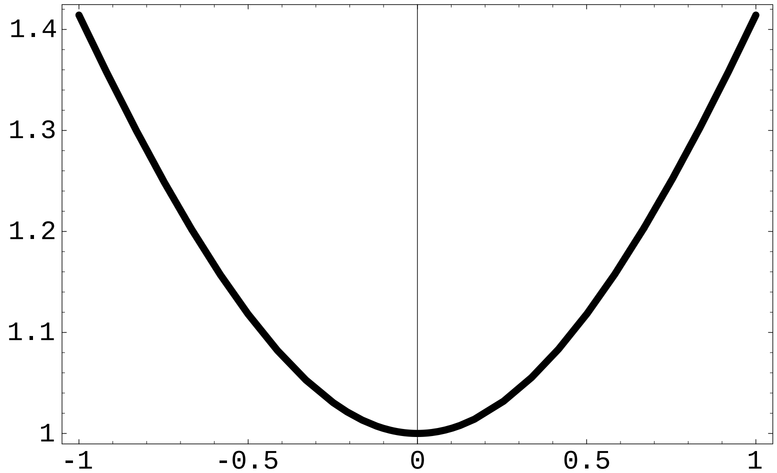
$$e_n = \max_{0 \leq x \leq 1} |f(x) - P_n(x)| \tag{7}$$

goes to zero as n goes to infinity:

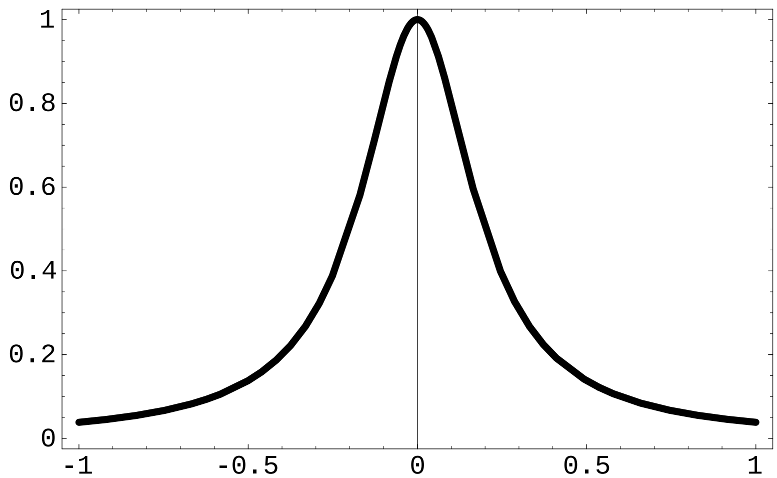
$$\lim_{n \to \infty} e_n = 0 \tag{8}$$

4. (Programming) Implement one of the algorithms for polynomial interpolation and interpolate

(a) hyperbola $f(x) = \sqrt{1+x^2}$



(b) Runge's function $f(x) = \frac{1}{1+25x^2}$



using a set of $n+1$ regularly spaced nodes

$$x_k = -1 + \frac{2(k-1)}{n}, \quad k = 1, 2, \ldots, n+1 .$$

Take $n = 5, 10, 20$ and compute the interpolation polynomial $P_n(x)$ and the error $f(x) - P_n(x)$ at 41 regularly spaced points. You can either plot the error or output it in a table. Does the interpolation accuracy increase with the order n?

5. (Programming) Repeat the experiments of the previous problem replacing the regularly spaced nodes with nodes

$$x_k = \cos\left(\frac{\pi(k-1)}{n}\right), \quad k = 1, 2, \ldots, n+1 .$$

Compare the accuracy.