Homework 10: Numerical Solution of ODE: One-Step Methods (due on April 23)

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1. (a) Which of the following functions satisfy the Lipschitz condition on y? For those that do, find the Lipschitz constant.
   i. $f(x,y) = \sqrt{x^2+y^2}$ for $x \in [-1,1]$
   ii. $f(x,y) = |y|$
   iii. $f(x,y) = \sqrt{|y|}$
   iv. $f(x,y) = |y|/x$ for $x \in [-1,1]$

   (b) Prove that the function $f(x,y) = -\sqrt{|1-y^2|}$ does not satisfy the Lipschitz condition and find two different solutions of the initial-value problem

$$\left\{ \begin{array}{rcl} y'(x) & = & -\sqrt{|1-y^2(x)|} \\ y(0) & = & 1 \end{array} \right. \tag{1}$$

   on the interval $x \in [0,\pi]$.

2. Consider the initial-value problem

$$\left\{ \begin{array}{rcl} y''(x) & = & y(x) \\ y(0) & = & y_0 \\ y'(0) & = & y_1 \end{array} \right. \tag{2}$$

   Write it as a system of two first-order differential equations with the appropriate initial conditions. Prove that Euler's method applied to this system can be unstable for a large step size.

   Hint: Take the special case $y_1 = -y_0$.

3. Consider the initial-value problem

$$\left\{ \begin{array}{rcl} y'(x) & = & \lambda\, y(x) \\ y(0) & = & y_0 \end{array} \right. \tag{3}$$

   (a) Prove that the Taylor series method for this problem takes the form

$$y(x_{k+1}) \approx y_{k+1} = \left[1 + \lambda\, h + \frac{(\lambda\, h)^2}{2} + \cdots + \frac{(\lambda\, h)^n}{n!}\right] y_k\,, \tag{4}$$

   where $h = x_{k+1} - x_k$, and n is the order of the method.

   (b) Prove that every second-order Runge-Kutta method for this problem is equivalent to the second-order Taylor method.

   (c) Prove that the second-order Taylor method can be unstable for large negative λ and find the stability region for the step size h.

4. (Programming) The exact solution of the initial-value problem

$$\begin{cases} y'(x) &= f(x,y) = y^2(x)\,e^{-x} \\ y(0) &= 1 \end{cases} \tag{5}$$

is

$$y(x) = e^x \;. \tag{6}$$

Solve the problem numerically on the interval $x \in [0,1]$ using

(a) Euler's method

$$y_{k+1} = y_k + h\; f(x_k, y_k) \tag{7}$$

(b) Second-order Taylor method

$$y_{k+1} = y_k + h\; f(x_k, y_k) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\; f(x_k, y_k)\right] \tag{8}$$

(c) Midpoint method

$$y_{k+1} = y_k + h\; f\left[x_k + \frac{h}{2}, y_k + \frac{h}{2}\; f(x_k, y_k)\right] \tag{9}$$

Take the step size $h = 0.1$ and output the error at all steps of the computation.

5. (Programming) In 1926, Volterra developed a mathematical model for predator-prey systems. If R is the population density of prey (rabbits), and F is the population density of predators (foxes), then Volterra's model for the population growth is the system of ordinary differential equations

$$R'(t) \;=\; a\,R(t) - b\,R(t)\,F(t)\,; \tag{10}$$
$$F'(t) \;=\; d\,R(t)\,F(t) - c\,F(t)\,, \tag{11}$$

where t is time, a is the natural growth rate of rabbits, c is the natural death rate of foxes, b is the death rate of rabbits per one unit of the fox population, and d is the growth rate of foxes per one unit of the rabbit population.

Adopt the midpoint method for the solution of this system. Take $a = 0.03$, $b = 0.01$, c=0.01, and $d = 0.01$, the interval $t \in [0, 500]$, the step size $h = 1$ and the initial values

(a)

$$\begin{aligned} R(0) &= 1.0\,; \\ F(0) &= 2.0 \end{aligned}$$

(b)

$$\begin{aligned} R(0) &= 1.0\,; \\ F(0) &= 4.0 \end{aligned}$$

Plot the solution: functions $R(t)$ and $F(t)$.