Math 128A Spring 2002 Handout # 5
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Fixed Point Iteration

Mathematica notebook: `http://math.lbl.gov/~fomel/128A/FixedPoint.nb`

Example Function

We will study fixed-point iteration using the function

$$f(x) = x^2 - x - e^{-x}$$

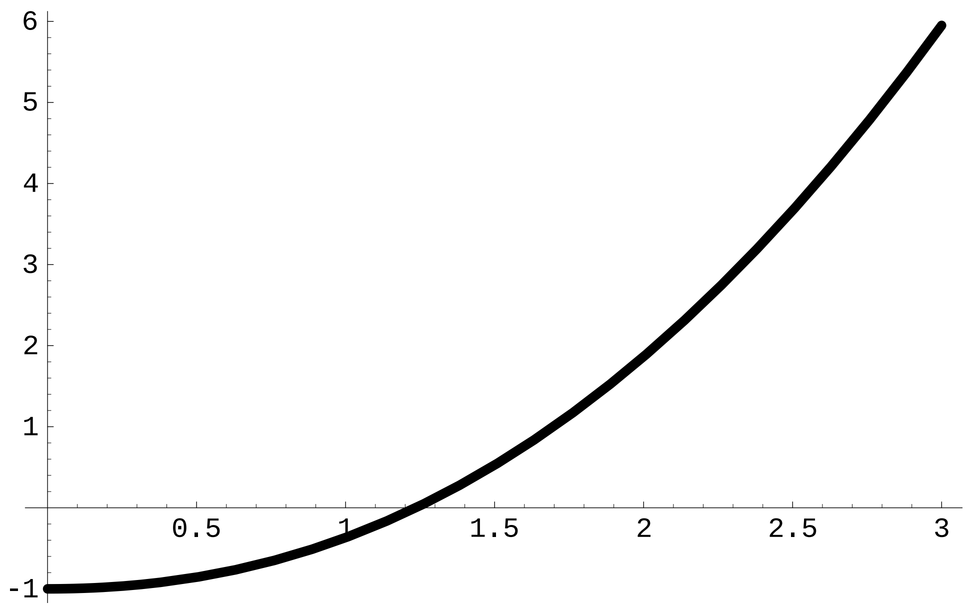


Figure 1: Plotting the function $f(x)$ shows that it has a root around 1.25

There are several ways to transform the equation $f(x) = 0$ to the form $x = g(x)$ suitable for the fixed-point iteration:

$$\begin{aligned} g_1(x) &= -e^{-x} + x^2\,; \\ g_2(x) &= \sqrt{e^{-x} + x}\,; \\ g_3(x) &= -\ln(-x + x^2)\,; \\ g_4(x) &= 1 + \frac{e^{-x}}{x} \end{aligned}$$

Which of these functions cause the fixed-point iteration to converge? Let us study this graphically.

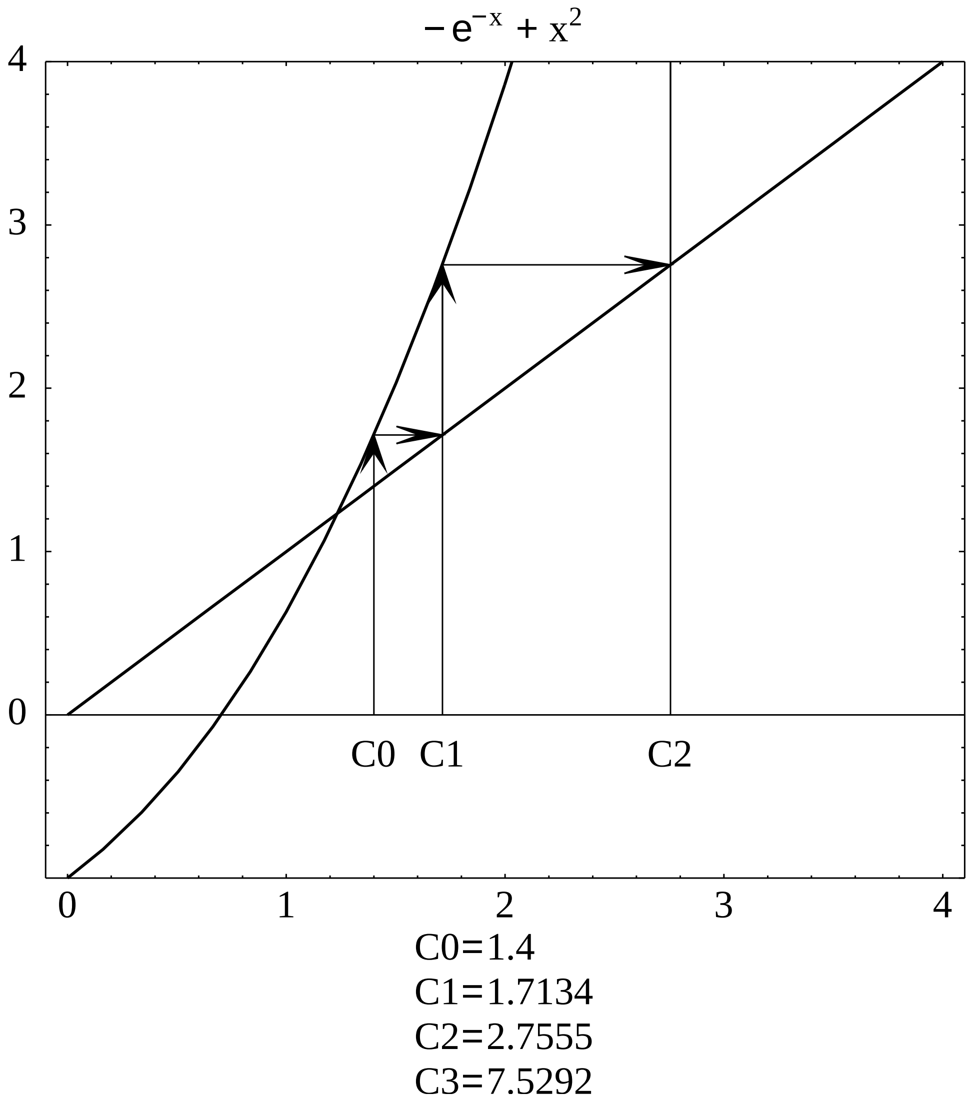


Figure 2: The function $g_1(x)$ clearly causes the iteration to diverge away from the root.

Convergence Analysis

Newton's iteration

Newton's iteration can be defined with the help of the function

$$g_5(x) = x - \frac{f(x)}{f'(x)}$$

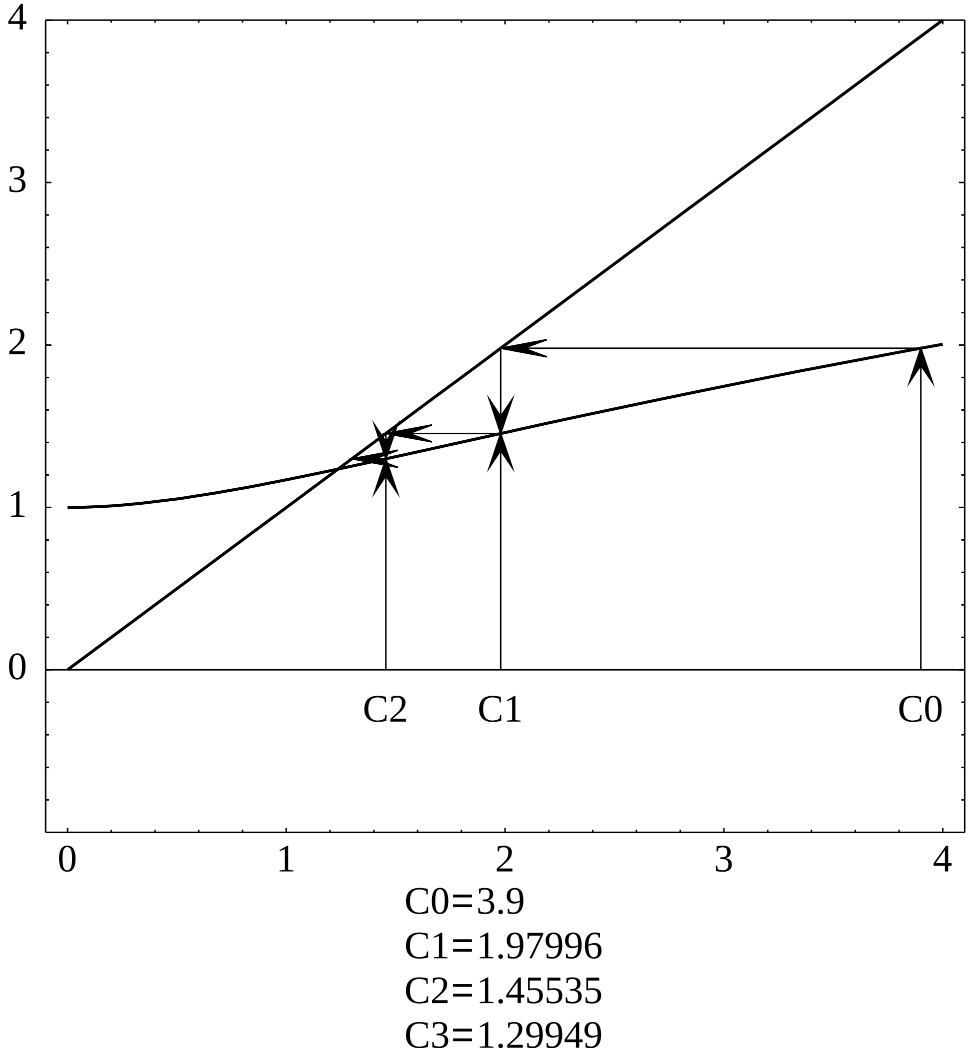


Figure 3: The function $g_2(x)$ leads to convergence, although the rate of convergence is slow.

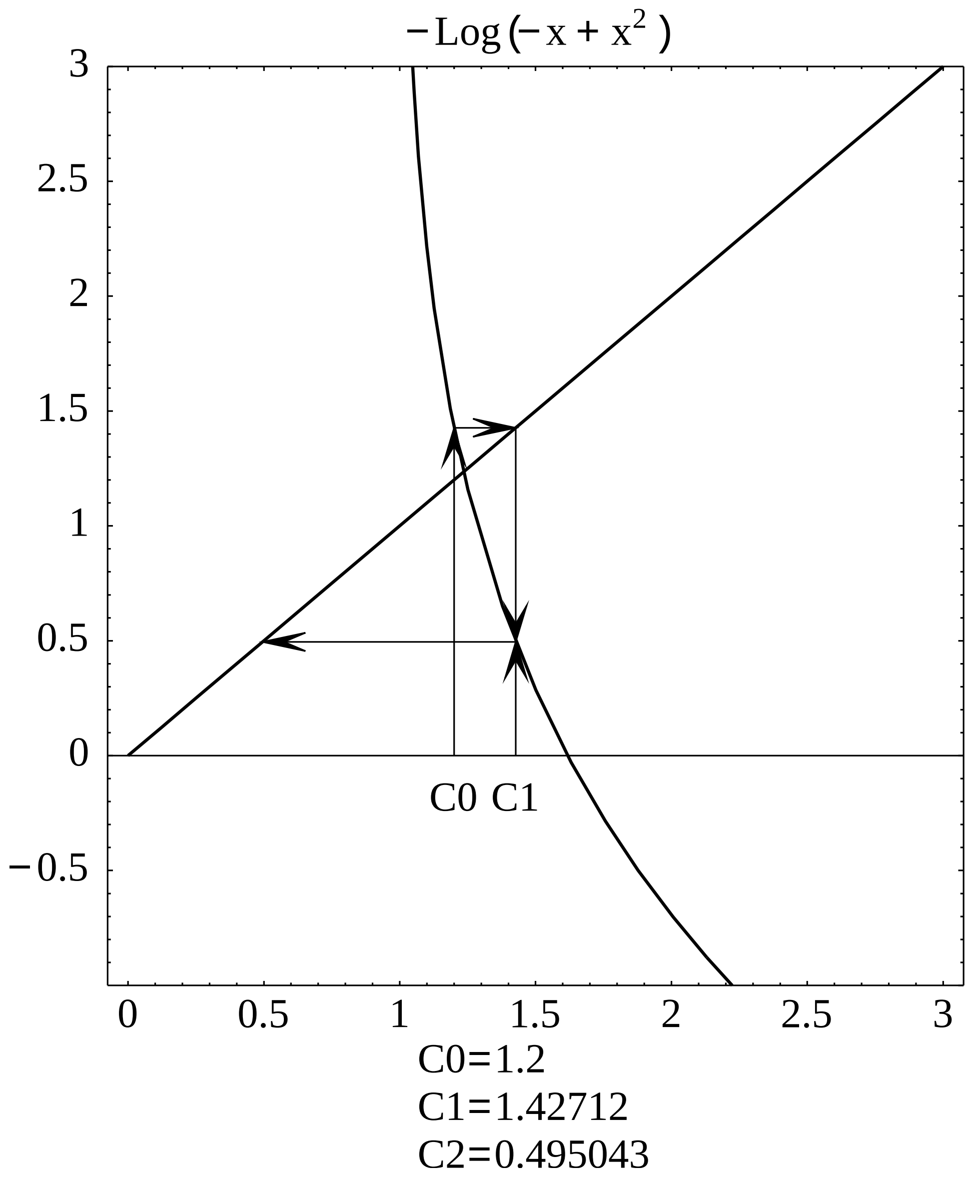
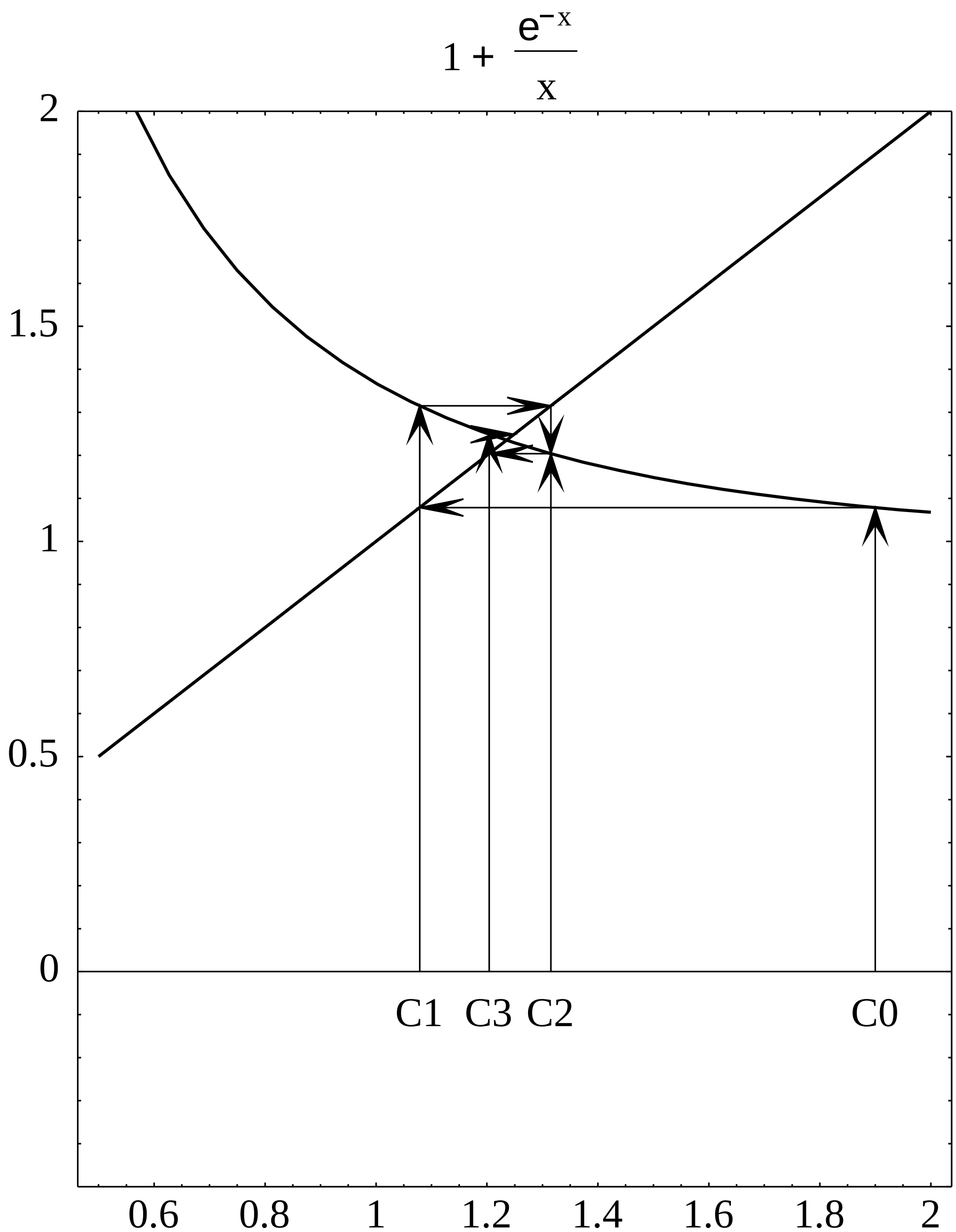


Figure 4: In the case of $g_3(x)$, the iteration diverges, spiraling away from the root.



C0=1.9
C1=1.07872
C2=1.31522
C3=1.20409
C4=1.24912

Figure 5: In the case of $g_4(x)$, the iteration converges, spiraling towards the root.

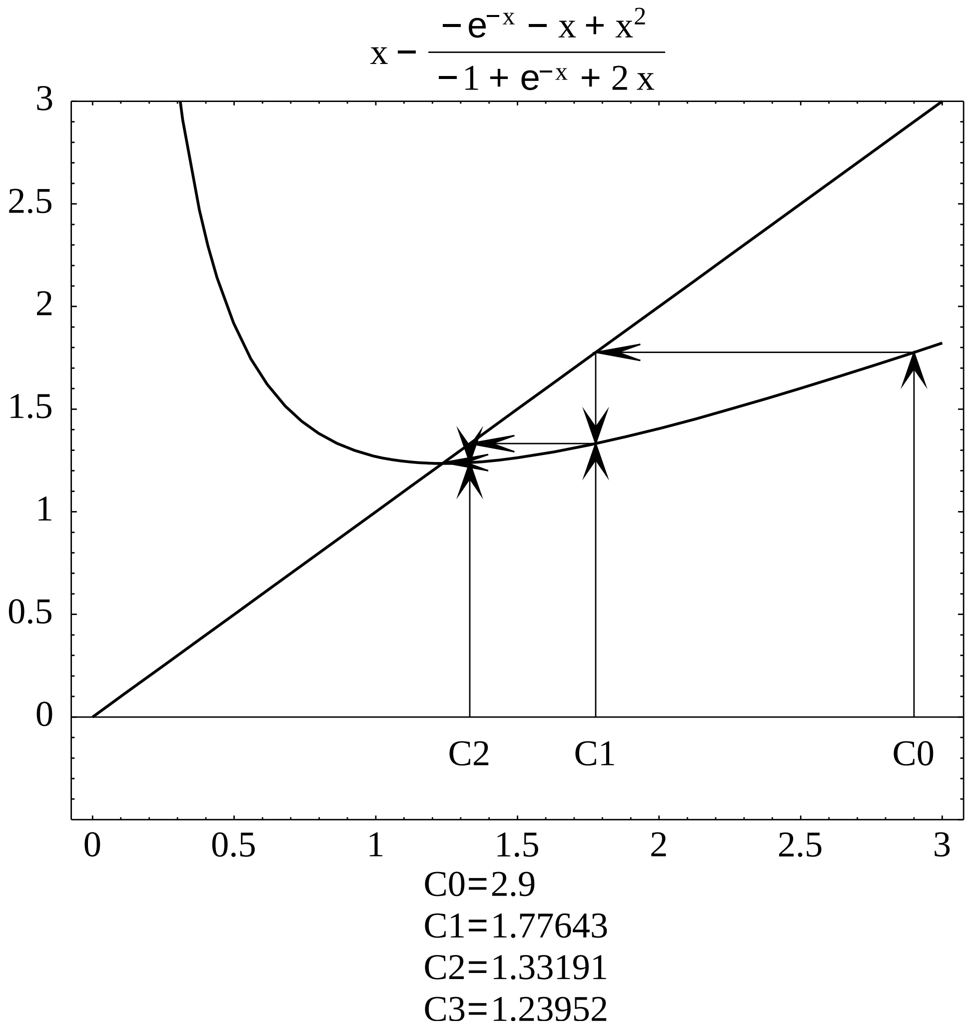


Figure 6: The iteration convergence very fast due to the fact that the function $g_5(x)$ has zero slope around the root.

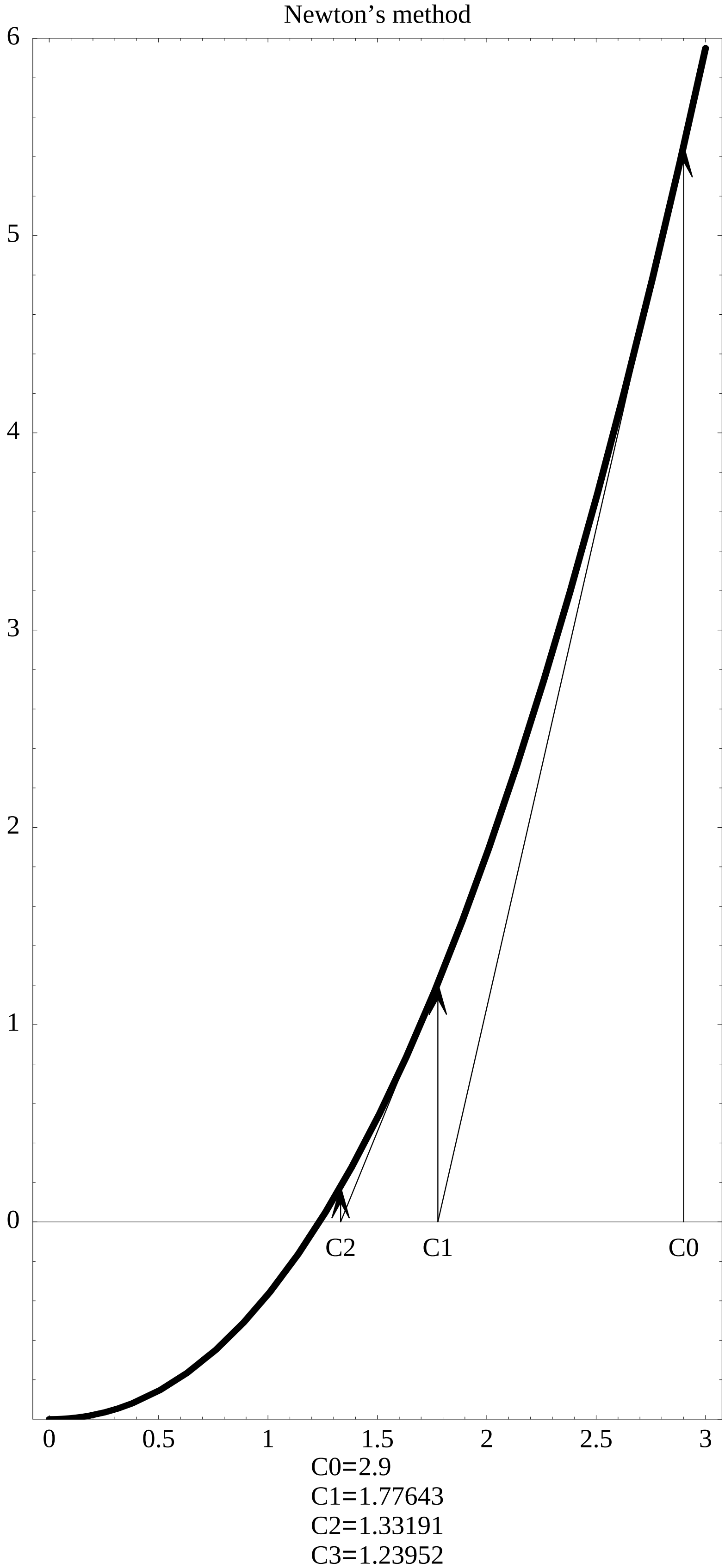


Figure 7: Another way to display the Newton iteration is by using tangent lines.