Chapter 1

Waves in layered media

The gravitational stratification of the earth arouses our interest in **layered media**. There are many kinds of waves that can exist. Here we study waves without assuming knowledge of physics and differential equations. We use only assumptions about the general principles of delay, continuity, and **energy conservation**. The results are directly applicable to sound waves, water waves, light in thin films, normal incident elastic waves of both pressure and shear type, electromagnetic waves, transmission lines, and electrical ladder networks. This chapter seems to concern itself only with vertically incident waves. Actually, with certain precautions, the theory here applies to the set of all waves with a fixed **Snell parameter** $p = \sin\theta/v$.

A lengthier version of this chapter appears in chapter 8 in my book **Fundamental of Geophysical Data Processing** (**FGDP**) (available on the web). The lengthier web version also presents the solution to the more practical problem, to deduce the reflection coefficients given the observed waves.

1.1 REFLECTION AND TRANSMISSION COEFFICIENTS

Consider two halfspaces (deep ocean on top of earth, for example). If a wave of unit amplitude is incident onto the boundary, there is a **transmitted wave** of amplitude t and a **reflected wave** of amplitude c as depicted in Figure 1.1. A very simple relationship exists between t and c. The wave amplitudes have a physical meaning of something like pressure, material displacement, traction, or tangential electric or magnetic fields. These physical variables must be the same value on either side of the boundary. This means the transmitted wave must equal the sum of the incident plus reflected waves.

$$t = 1 + c \tag{1.1}$$

$$t' = 1 + c' \tag{1.2}$$

The **reflection coefficient** c may be positive or negative so the **transmission coefficient** t may be greater than unity. It may seem surprising that t can be greater than unity. This does not

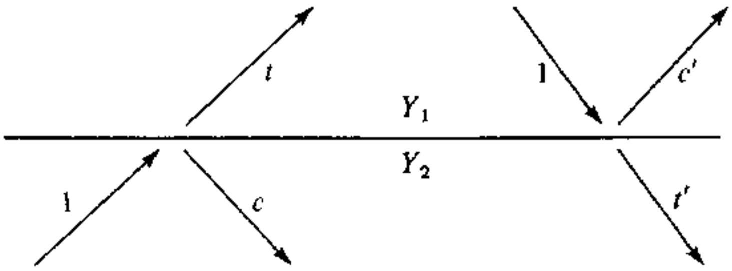

Figure 1.1: Waves incident, reflected c, and transmitted t at an interface. fgdp8-8-1 [NR]

violate any physical laws. At the seashore we see waves approaching the shore and they get larger as they arrive. Energy is not determined by wave height alone. Energy is equal to the squared wave amplitude multiplied by a proportionality factor Y depending upon the medium in which the wave is measured. If we denote the factor of the top medium by Y_1 and the bottom by Y_2, then the statement that the energy before incidence equals the energy after incidence is

$$Y_2 1^2 \quad = \quad Y_2 c^2 + Y_1 t^2 \tag{1.3}$$

solving for c leads us to

$$\begin{aligned} 0 &= -Y_2 + Y_2 c^2 + Y_1(1+c)^2 \\ 0 &= Y_2(c-1) + Y_1(1+c) \\ 0 &= (Y_2+Y_1)c + (Y_1 - Y_2) \\ c &= \frac{Y_2 - Y_1}{Y_2 + Y_1} \end{aligned} \tag{1.4}$$

In acoustics the up- and downgoing wave variables may be normalized to either pressure or velocity. When they measure velocity, the scale factor multiplying velocity squared is called the **impedance** I. When they measure pressure, the scale factor is called the **admittance** Y.

The wave c' which reflects when energy is incident from the other side is obtained from (1.4) if Y_1 and Y_2 are interchanged. Thus

$$c' \quad = \quad -c \tag{1.5}$$

A perfectly reflecting interface is one which does not allow energy through. This comes about not only when $t = 0$ or $c = -1$, but also when $t = 2$ or $c = +1$. To see this, note that on the left in Figure 1.1

$$\begin{aligned} \frac{\text{Energy transmitted}}{\text{Energy incident}} &= \frac{Y_1 t^2}{Y_2 1^2} \quad = \quad \frac{Y_1}{Y_2}\left(1 + \frac{Y_2 - Y_1}{Y_1 + Y_2}\right)^2 \\ &= \frac{Y_1}{Y_2}\left(\frac{2\,Y_2}{Y_1 + Y_2}\right)^2 \quad = \quad \frac{4\,Y_1\,Y_2}{(Y_1 + Y_2)^2} \end{aligned} \tag{1.6}$$

Equation (1.6) says that 100 percent of the incident energy is transmitted when $Y_1 = Y_2$, but the percentage of transmission is very small when Y_1 and Y_2 are very different.

Ordinarily there are two kinds of variables used to describe waves, and both of these can be continuous at a material discontinuity. One is a scalar like pressure, tension, voltage, potential, stress, or temperature. The other is a vector which we use the vertical component. Examples of the latter are velocity, stretch, electric current, dislacement, and heat flow. Occasionally a wave variable is a **tensor**. When a **boundary condition** is the vanishing of one of the motion components, then the boundary is often said to be rigid. When it is the pressure or potential which vanishes, then the boundary is often said to be free. Rigid and free boundaries reflect waves with unit magnitude reflection coefficients.

A goal here is to establish fundamental mathematical properties of waves in layers while minimizing specialization to any particular physical type of waves. Each physical problem has its own differential equations. These equations are Fourier transformed over x and y leading to coupled ordinary differential equations in depth z. This analytical process is explained in more detail in FGDP. The next step is an **eigenvector** analysis that relates the physical variables to our abstract up- and down-going waves U and D. In order to better understand boundary conditions we will examine one concrete example, acoustic waves.

In acoustics we have **pressure** P and vertical component of **parcel velocity** W (not to be confused with wave velocity v). The acoustic pressure P is the sum of U and D. Vertical velocity W obviously changes sign when the z axis changes sign (interchange of up and down) and that accounts for the minus sign in the definition of W. (The eigenvector analysis provides us with the scaling factor Y.)

$$P = D + U \tag{1.7}$$

$$W = (D - U)Y \tag{1.8}$$

These definitions are easily inverted.

$$D = \frac{P + W/Y}{2} \tag{1.9}$$

$$U = \frac{P - W/Y}{2} \tag{1.10}$$

For sound waves in the ocean, the sea surface is nearly a perfect reflector because of the great contrast between air and water. If this interface is idealized to a perfect reflector, then it is a free surface. Since the pressure vanishes on a free surface, we have $D = -U$ at the surface so the reflection coefficient is -1 as shown in Figure 1.2. In principle we should measure velocity W at the water surface. In practice, we generally measure pressure P a few meters below the free surface.

The pressure normally vanishes at the sea surface, but if we wish to initiate an impulsive disturbance, the pressure may momentarily take on some other value, say 1. This 1 denotes a constant function of frequency which is an impulsive function of time at $t = 0$. Ensuing waves are depicted in Figure 1.3 where the upcoming wave $-R(Z)$ is a consequence of both the downgoing 1 and the downgoing $+R(Z)$. The vertical component of velocity W of the sea surface due to the source and to the resulting acoustic wave is $D - U = 1 + 2R(Z)$.

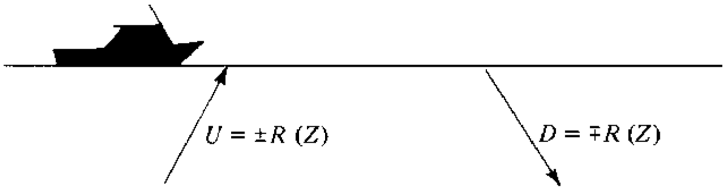

Figure 1.2: A waveform $R(Z)$ reflecting at the surface of the sea. Pressure equal to $U+D$ vanishes at the surface. The vertical velocity of the surface is proportional to $D-U$. Theoretically, waves are observed by measuring W at the surface. fgdp8-8-2 [NR]

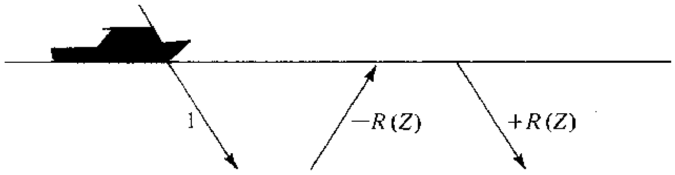

Figure 1.3: An initial downgoing disturbance 1 results in a later upgoing reflected wave $-R(Z)$ which reflects back down as $R(Z)$. The pressure at the surface is $D+U = 1+R-R = 1$. fgdp8-8-3 [NR]

EXERCISES:

1 In a certain application continuity is expressed by saying that $D-U$ is the same on either side of the interface. This implies that $t = 1-c$. Derive an equation like (1.4) for the reflection coefficient in terms of the admittance Y.

2 What are reflection and transmission coefficients in terms of the impedance I? (Clear fractions from your result.)

3 From the principle of energy conservation we showed that $c' = -c$. It may also be deduced from time reversal. To do this, copy Figure 1.1 with arrows reversed. Scale and linearly superpose various figures in an attempt to create a situation where a figure like the right-hand side of Figure 1.1 has $-c'$ for the reflected wave. (HINT: Draw arrows at normal incidence.)

1.2 THE LAYER MATRIX

Consider a layer with waves **reverberating** in it as shown in Figure 1.4. Let the delay operator Z refer to the time delay of a travel path across the layer and back again. The wave seen above the layer is an **infinite sequence** that is expressible as a simple denominator (because $1+X+X^2+X^3+\cdots = 1/(1-X)$).

$$t_2 t_1 [1 + c_1 c_2' Z + (c_1 c_2')^2 Z^2 + \cdots] \quad = \quad \frac{t_2 t_1}{1 - c_1 c_2' Z}$$

It is no accident that the infinite series may be summed. We will soon see that for n layers the waves, which are of infinite duration, may be expressed as simple polynomials of degree

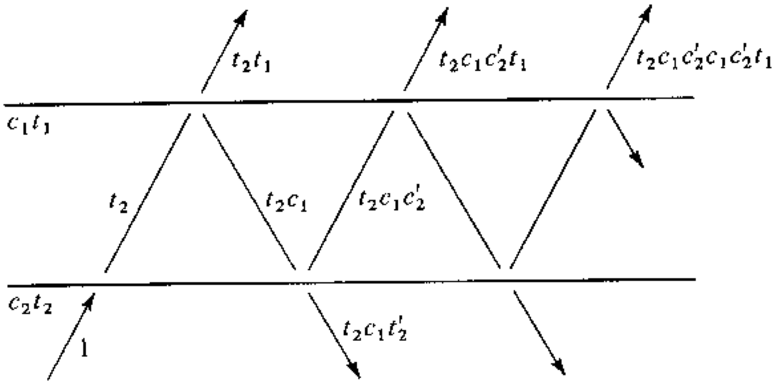

Figure 1.4: Some rays corresponding to resonance in a layer. fgdp8-8-4 [NR]

n. We consider many layers and the general problem of determining waves given reflection coefficients.

Equation (1.11) relates to Figure 1.5 and shows how from the waves U and D' we extrapolate into the future to get U' and D.

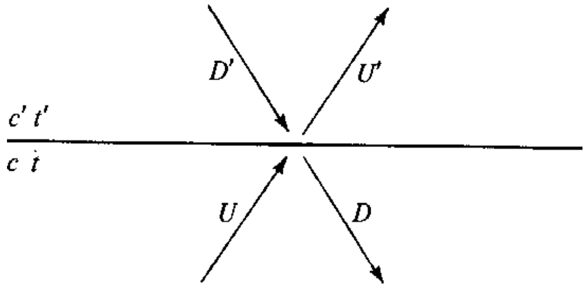

Figure 1.5: Waves incident and reflected from an interface. fgdp8-8-5 [NR]

$$\begin{aligned} U' &= tU + c'D' \\ D &= cU + t'D' \end{aligned} \tag{1.11}$$

Let us rearrange (1.11) to get U' and D' on the right and U and D on the left to get an equation which extrapolates from the primed medium to the unprimed medium. We get

$$\begin{aligned} -tU &= -U' + c'D' \\ -cU + D &= t'D' \end{aligned}$$

Arrange to matrix form

$$\begin{bmatrix} -t & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} -1 & c' \\ 0 & t' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'$$

Premultiply by the inverse of the left-hand matrix

$$\begin{aligned} \begin{bmatrix} U \\ D \end{bmatrix} &= \frac{1}{-t} \begin{bmatrix} 1 & 0 \\ c & -t \end{bmatrix} \begin{bmatrix} -1 & c' \\ 0 & t' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \\ &= \frac{1}{-t} \begin{bmatrix} -1 & c' \\ -c & cc' - tt' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \end{aligned}$$

getting a result we want, an equation to extrapolate the infinitesimal distance from the bottom of one layer to the top of the next.

$$\begin{bmatrix} U \\ D \end{bmatrix} = \frac{1}{t} \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \tag{1.12}$$

Now we consider a layered medium where each layer has the same travel-time thickness [**Goupillaud** medium]. Presumably, any continuously variable medium can be approximated this way. Examine the definitions in the layered medium in Figure 1.6. For this arrangement

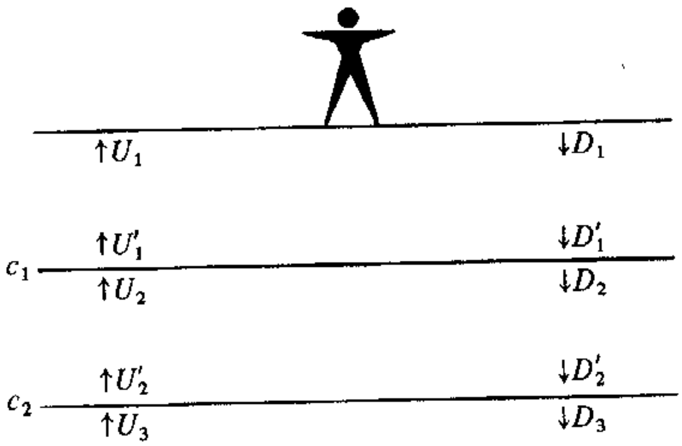

Figure 1.6: Goupillaud-type layered medium (layers have equal travel time). fgdp8-8-6 [NR]

of layers, (1.12) may be written

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{1}{t_k} \begin{bmatrix} 1 & c_k \\ c_k & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'_k$$

Let $Z = e^{i\omega T}$ where T, the two-way travel time, equals the data sampling interval. Clearly, multiplication by $\sqrt{Z}$ is equivalent to delaying a function by $T/2$, the travel time across a layer. This gives in the kth layer a relation between primed and unprimed waves.

$$\begin{bmatrix} U \\ D \end{bmatrix}'_k = \begin{bmatrix} 1/\sqrt{Z} & 0 \\ 0 & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \tag{1.13}$$

Inserting (1.13) into (1.12) we get a ***layer matrix*** that takes us from the top of one layer to the top of the next deeper one.

$$\begin{aligned} \begin{bmatrix} U \\ D \end{bmatrix}_{k+1} &= \frac{1}{t_k} \begin{bmatrix} 1 & c_k \\ c_k & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{Z} & 0 \\ 0 & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\ &= \frac{1}{t_k} \begin{bmatrix} 1/\sqrt{Z} & c_k\sqrt{Z} \\ c_k/\sqrt{Z} & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\ \begin{bmatrix} U \\ D \end{bmatrix}_{k+1} &= \frac{1}{\sqrt{Z}t_k} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \end{aligned} \tag{1.14}$$

1.3 ENERGY FLUX

If there is energy flowing through a stack of layers, there should be the same **energy flow** through the kth layer as through the $(k+1)$st layer. Otherwise, there would be an energy sink or source at the layer boundary. Let us guess that the net upward flow of energy (**energy flux**) at any frequency ω in the $(k+1)$st layer is given by

$$\text{flux}(\omega) \quad = \quad Y_k \left[U(Z)\bar{U}\left(\frac{1}{Z}\right) - D(Z)\bar{D}\left(\frac{1}{Z}\right)\right]_{k+1} \tag{1.15}$$

Inserting equation (1.14) into equation (1.15) (and using a few simplifications all spelled out in **FGDP**) we deduce the principle that the energy flux across any layer matches that across the neighboring layer.

$$Y_{k+1}\left[U\left(\frac{1}{Z}\right)U(Z) - D\left(\frac{1}{Z}\right)D(Z)\right]_{k+1} \quad = \quad Y_k\left[U\left(\frac{1}{Z}\right)U(Z) - D\left(\frac{1}{Z}\right)D(Z)\right]_k \tag{1.16}$$

Since the flux is the same in any two adjacent layers, it must be the same in all the layers. This energy flux theorem leads quickly to some sweeping statements about the waveforms scattered from layered structures.

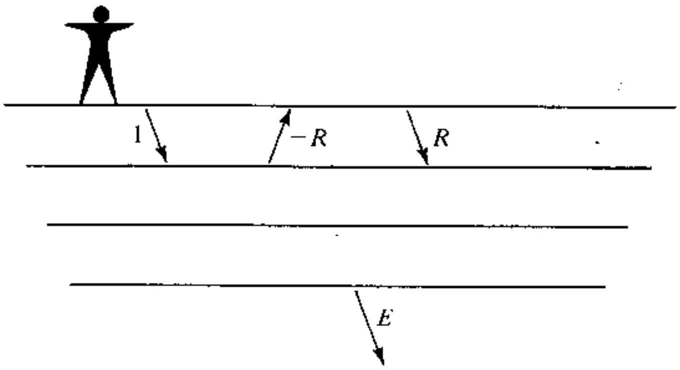

Figure 1.7: The reflection seismology geometry. The person sends an impulse down. The earth sends back $-R(Z)$. Since the surface is perfectly reflective, it sends $R(Z)$ back down. Escaping from the bottom is a wave $E(Z)$ heading to infinity. fgdp8-8-7 [NR]

Figure 1.7 shows the basic geometry of reflection seismology. Applying the energy flux theorem to this geometry we may say that the energy flux in the top layer equals that in the lower halfspace so

$$Y_1\left\{R\left(\frac{1}{Z}\right)R(Z) - \left[1+R\left(\frac{1}{Z}\right)\right][1+R(Z)]\right\} \quad = \quad -Y_k E\left(\frac{1}{Z}\right)E(Z)$$

or

$$1+R\left(\frac{1}{Z}\right)+R(Z) \quad = \quad \frac{Y_k}{Y_1}\,E\left(\frac{1}{Z}\right)E(Z) \tag{1.17}$$

This result is amazing. Nothing in life led me to suspect it before I found it. It says, if we were to observe the escaping wave $E(Z)$, we could by **autocorrelation** construct the waveform seen at the surface. This result has no practical use, but a slightly different arrangement does.

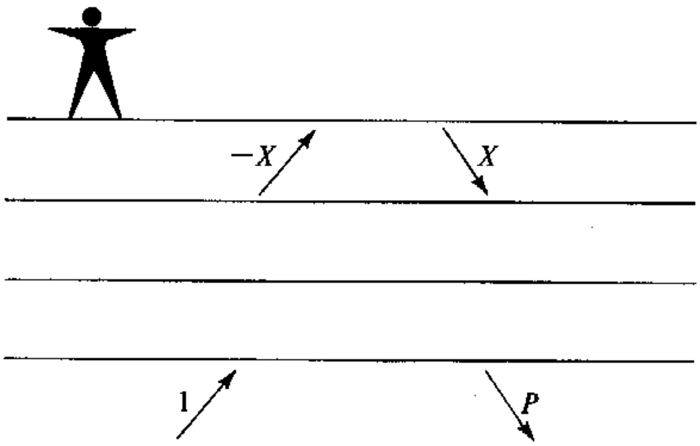

Figure 1.8: Earthquake seismology geometry. An impulse 1 is incident from below. The waveform $X(Z)$ is incident upon the free surface and is reflected back down. The waveform $P(Z)$ scatters back into earth. fgdp8-E8-3-5 [NR]

The escaping wave $E(Z)$ turns out to be the same as the earthquake wave $X(Z)$ in Figure 1.8. This can be shown algebraically (see exercises) or deduced as a consequence of the reciprocity principle (the idea that interchanging source and receiver location yields the same response). Notice that the incident earthquake "1" need not be an impulsive wave; it only need have the **autocorrelation** of an impulsive wave. This conclusion leads to the remarkable hypothesis that the autocorrelation of **natural** noises (microseism) gives reflection seismograms. This concept has proven useful in **solar seismology**, but a variety of practical factors (principally spatial distribution of stations) has limited its utility on the earth so far.

A more obvious result applies to the earthquake seismology geometry in Figure 1.8. Applying the energy flux theorem to this geometry we obtain

$$Y_1\left[X\left(\frac{1}{Z}\right)X(Z)-X\left(\frac{1}{Z}\right)X(Z)\right] \quad = \quad Y_k\left[1-P(Z)P\left(\frac{1}{Z}\right)\right]$$

or

$$1 \quad = \quad P(Z)P\left(\frac{1}{Z}\right) \tag{1.18}$$

The interpretation of the result is that the backscattered waveform $P(Z)$ has the form of an all-pass filter. (An all-pass filter has a constant energy spectrum and an arbitrary phase spectrum.) This result may have been anticipated on physical grounds since all the energy which is incident is ultimately reflected without attenuation; thus the only thing which can happen is that there will be frequency-dependent delay.

1.4 GETTING THE WAVES FROM THE REFLECTION COEFFICIENTS

An important result of the last section was the development of a "layer matrix" (1.14) that is, a matrix which can be used to extrapolate waves observed in one layer to the waves observed in the next layer. This process may be continued indefinitely. To see how to extrapolate from layer 1 to layer 3 substitute (1.14) with $k = 1$ into (1.14) with $k = 2$, obtaining

$$\begin{aligned} \begin{bmatrix} U \\ D \end{bmatrix}_3 &= \frac{1}{\sqrt{Z}t_2}\begin{bmatrix} 1 & Zc_2 \\ c_2 & Z \end{bmatrix} \frac{1}{\sqrt{Z}t_1}\begin{bmatrix} 1 & Zc_1 \\ c_1 & Z \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1 \\ &= \frac{1}{\sqrt{Z}^2 t_1 t_2}\begin{bmatrix} 1+Zc_1c_2 & Zc_1+Z^2c_2 \\ c_2+Zc_1 & Zc_1c_2+Z^2 \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1 \end{aligned} \tag{1.19}$$

Inspection of this product suggests the general form for a product of k layer matrices. We'll call it a **multilayer matrix**.

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{1}{\sqrt{Z}^k \prod_{j=1}^k t_j}\begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1 \tag{1.20}$$

Starting from computer programs that add and multiply polynomials by manipulating their coefficients, it is easy enough to write a computer code to recursively build up the $F(Z)$ and $G(Z)$ polynomials by adding a layer at a time starting from $F(Z) = 1$ and $G(Z) = c_1$.

1.4.1 Layers between two halfspaces

Let a stack of layers be sandwiched in between two halfspaces Figure 1.9. An impulse is

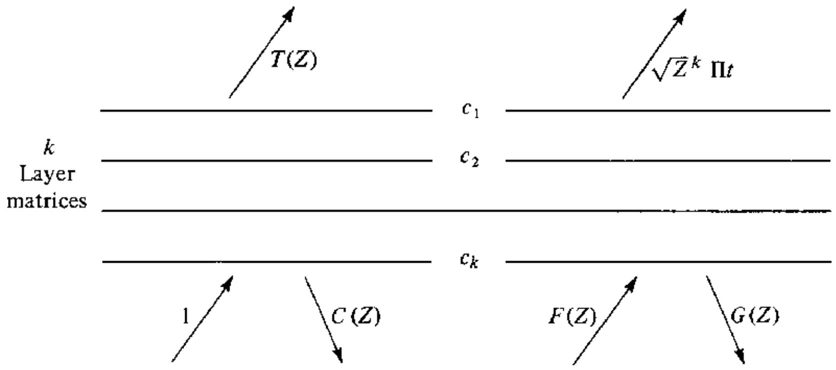

Figure 1.9: Waves incident, reflected, and transmitted from a stack of layers between two half-spaces. fgdp8-8-9 [NR]

incident from below. The backscattered wave is called $C(Z)$ and the transmitted wave is called $T(Z)$. We take these waves and load them into the **multilayer equation** (1.20).

$$\begin{bmatrix} 1 \\ C(Z) \end{bmatrix} = \frac{1}{\sqrt{Z}^k \prod t}\begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix}\begin{bmatrix} T(Z) \\ 0 \end{bmatrix} \tag{1.21}$$

We may solve the first of (1.21) for the transmitted wave $T(Z)$

$$T(Z) = \frac{\sqrt{Z}^k \prod t}{F(Z)} \tag{1.22}$$

and introduce the result back into the second of (1.21) to obtain the backscattered wave

$$C(Z) \quad = \quad \frac{G(Z)T(Z)}{\sqrt{Z}^k \Pi t} \quad = \quad \frac{G(Z)}{F(Z)} \tag{1.23}$$

Let us be clear we understand the meaning of the solutions $T(Z)$ and $C(Z)$. They can be regarded two ways, either as functions of frequency $Z = e^{i\omega}$, or the polynomial coefficients (after dividing out the denominator $F(Z)$) as waveforms, as functions of time. Physical interpretations of $F(Z)$ and $G(Z)$ are on the right side of Figure 1.9.

1.4.2 Reflection seismology geometry

Identify the up and downgoing waves in the reflection seismology geometry in Figure 1.7 and load these waves into the multilayer matrix boundaries, equation (1.20).

$$\begin{bmatrix} 0 \\ E(Z) \end{bmatrix} \quad = \quad \frac{1}{\sqrt{Z}^k \Pi t} \begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \begin{bmatrix} -R \\ 1+R \end{bmatrix} \tag{1.24}$$

From the first equation we may solve for $R(Z)$

$$R(Z) \quad = \quad + \frac{Z^k G(1/Z)}{F(Z) - Z^k G(1/Z)} \tag{1.25}$$

Given $R(Z)$, the second equation in the set (1.24) gives us $E(Z)$. Calculation of the wave fields on the earthquake geometry of Figure 1.8 is reserved for an exercise.

1.4.3 Getting reflection coefficients from the waves

The remainder of chapter 8 in **FGDP** shows how the algebraic techniques developed here can be solved in reverse. Given various waves such as the reflection data $R(Z)$ or the earthquake data $X(Z)$ we may deduce the reflection sequence $(c_1, c_2, c_3, \cdots)$. This appears to be the solution to the practical problem. It turns out however, for various complicated reasons, that this does not seem to be a methodology in widespread use in the seismic exploration business.

EXERCISES:

1 In Figure 1.9 let $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, and $c_3 = \frac{1}{3}$. What are the polynomial ratios $T(Z)$ and $C(Z)$?

2 For a simple interface, we had the simple relations $t = 1 + c$, $t' = 1 + c'$, and $c = -c'$. What sort of analogous relations can you find for the generalized interface of Figure 1.9? [For example, show $1 - T(Z)T'(1/Z) = C(Z)C(1/Z)$ which is analogous to $1 - tt' = c^2$.]

3 Show that $T(Z)$ and $T'(Z)$ are the same waveforms within a scale factor. Deduce that many different stacks of layers may have the same $T(Z)$.

4 Consider the earth to be modeled by layers over a halfspace. Let an impulse be incident from below (Figure 1.8). Given $F(Z)$ and $G(Z)$, elements of the product of the layer matrices, solve for X and for P. Check your answer by showing that $P(Z)\bar{P}(1/Z) = 1$. How is X related to E? This relation illustrates the principle of reciprocity which says source and receiver may be interchanged.

5 Show that $1 + R(1/Z) + R(Z) =$ (scale factor)$X(Z)X(1/Z)$, which shows that one may autocorrelate the transmission seismogram to get the reflection seismogram.

6 Refer to Figure 1.10. Calculate R' from R.

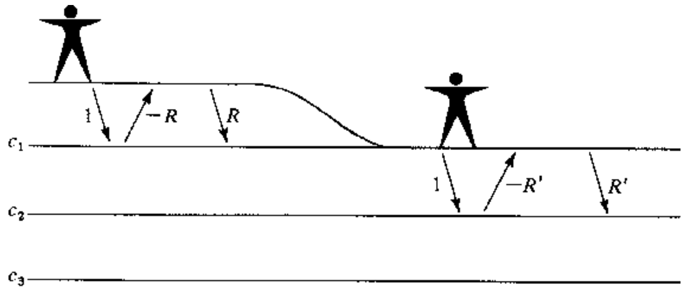

Figure 1.10: Stripping off the surface layer. fgdp8-E8-3-7 [NR]

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