Chapter 1

Downward continuation of waves

1.1 DIPPING WAVES

We begin with equations to describe a dipping plane wave in a medium of constant velocity. Figure 1.1 shows a ray moving down into the earth at an angle θ from the vertical. Perpendicular to the ray is a **wavefront**. By elementary geometry the angle between the

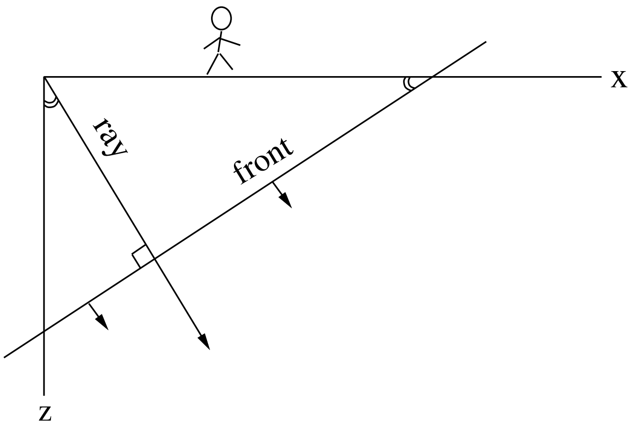

Figure 1.1: Downgoing ray and wavefront. dwnc-front [NR]

wave**front** and the earth's surface is also θ. The **ray** increases its length at a speed v. The speed that is observable on the earth's surface is the intercept of the wavefront with the earth's surface. This speed, namely $v/\sin\theta$, is faster than v. Likewise, the speed of the intercept of the wavefront and the vertical axis is $v/\cos\theta$. A mathematical expression for a straight line like that shown to be the wavefront in Figure 1.1 is

$$z \;=\; z_0 - x \tan\theta \tag{1.1}$$

In this expression z_0 is the intercept between the wavefront and the vertical axis. To make the intercept move downward, replace it by the appropriate velocity times time:

$$z \;=\; \frac{v\,t}{\cos\theta} - x \tan\theta \tag{1.2}$$

Solving for time gives

$$t(x,z) \;=\; \frac{z}{v}\,\cos\theta \;+\; \frac{x}{v}\,\sin\theta \tag{1.3}$$

Equation (1.3) tells the time that the wavefront will pass any particular location (x, z). The expression for a shifted waveform of arbitrary shape is $f(t - t_0)$. Using (1.3) to define the time shift t_0 gives an expression for a wavefield that is some waveform moving on a **ray**.

$$\text{moving wavefield} \;=\; f\left(t \;-\; \frac{x}{v}\,\sin\theta \;-\; \frac{z}{v}\,\cos\theta \right) \tag{1.4}$$

1.1.1 Snell waves

In reflection seismic surveys the velocity contrast between shallowest and deepest reflectors ordinarily exceeds a factor of two. Thus depth variation of velocity is almost always included in the analysis of field data. Seismological theory needs to consider waves that are just like plane waves except that they bend to accommodate the velocity stratification $v(z)$. Figure 1.2 shows this in an idealized geometry: waves radiated from the horizontal flight of a supersonic airplane. The airplane passes location x at time $t_0(x)$ flying horizontally at a constant speed.

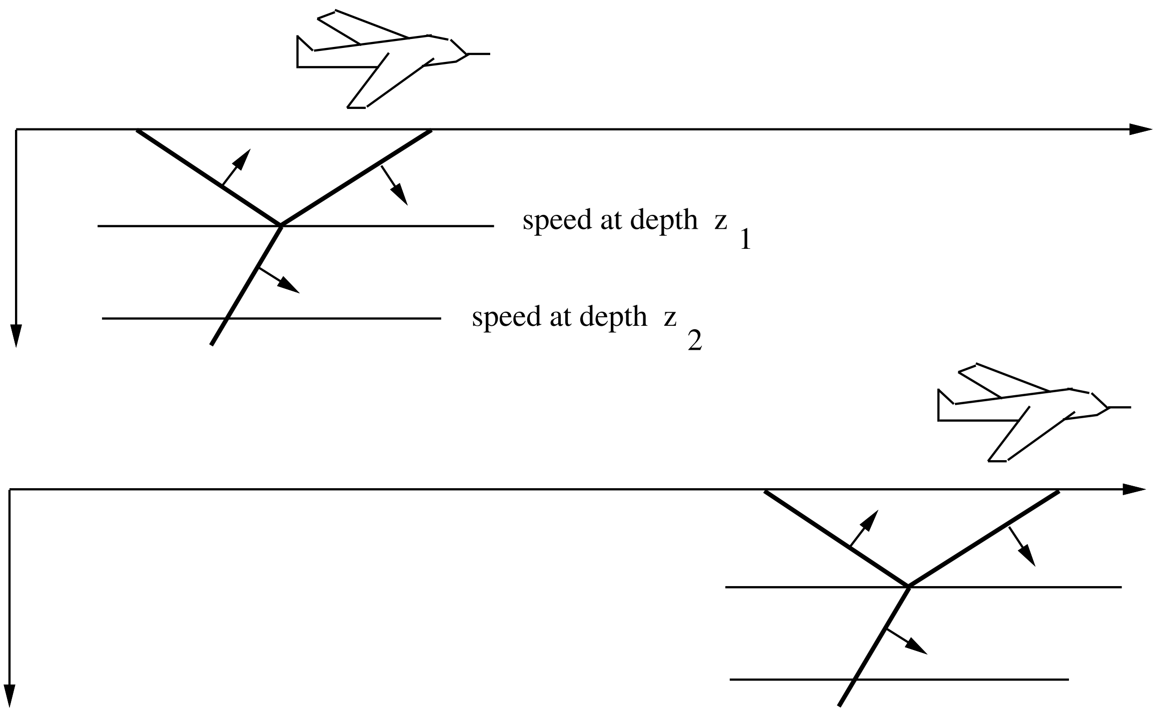

Figure 1.2: Fast airplane radiating a sound wave into the earth. From the figure you can deduce that the horizontal speed of the wavefront is the same at depth z_1 as it is at depth z_2. This leads (in isotropic media) to Snell's law. dwnc-airplane [NR]

Imagine an earth of horizontal plane layers. In this model there is nothing to distinguish any point on the x-axis from any other point on the x-axis. But the seismic velocity varies from layer to layer. There may be reflections, head waves, shear waves, converted waves, anisotropy, and multiple reflections. Whatever the picture is, it moves along with the airplane. A picture of the wavefronts near the airplane moves along with the airplane. The top of the picture and the bottom of the picture both move laterally at the same speed even if the earth velocity increases with depth. If the top and bottom didn't go at the same speed, the picture would become distorted, contradicting the presumed symmetry of translation. This horizontal speed, or rather its inverse $\partial t_0/\partial x$, has several names. In practical work it is called the ***stepout***. In theoretical work it is called the ***ray parameter***. It is very important to note that $\partial t_0/\partial x$ does not change with depth, even though the seismic velocity does change with depth. In a constant-velocity medium, the angle of a wave does not change with depth. In a stratified medium, $\partial t_0/\partial x$ does not change with depth.

Figure 1.3 illustrates the differential geometry of the wave. The diagram shows that

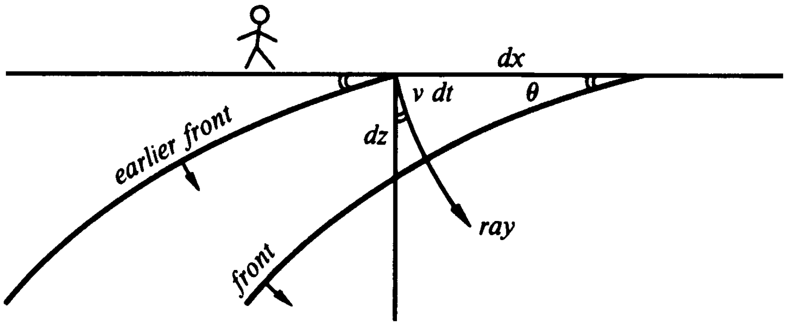

Figure 1.3: Downgoing fronts and rays in stratified medium $v(z)$. The wavefronts are horizontal translations of one another. dwnc-frontz [NR]

$$\frac{\partial t_0}{\partial x} = \frac{\sin\theta}{v} \tag{1.5}$$

$$\frac{\partial t_0}{\partial z} = \frac{\cos\theta}{v} \tag{1.6}$$

These two equations define two (inverse) speeds. The first is a horizontal speed, measured along the earth's surface, called the *horizontal* ***phase velocity***. The second is a vertical speed, measurable in a borehole, called the *vertical phase velocity*. Notice that both these speeds *exceed* the velocity v of wave propagation in the medium. Projection of wave *fronts* onto coordinate axes gives speeds larger than v, whereas projection of *rays* onto coordinate axes gives speeds smaller than v. The inverse of the phase velocities is called the ***stepout*** or the ***slowness***.

Snell's law relates the angle of a wave in one layer with the angle in another. The constancy of equation (1.5) in depth is really just the statement of Snell's law. Indeed, we have just derived Snell's law. All waves in seismology propagate in a velocity-stratified medium. So they cannot be called plane waves. But we need a name for waves that are near to plane waves. A ***Snell wave*** will be defined to be the generalization of a plane wave to a stratified medium $v(z)$. A plane wave that happens to enter a medium of depth-variable velocity $v(z)$ gets changed into a Snell wave. While a plane wave has an angle of propagation, a Snell wave has instead a ***Snell parameter*** $p = \partial t_0/\partial x$.

It is noteworthy that Snell's parameter $p = \partial t_0/\partial x$ is directly observable at the surface, whereas neither v nor θ is directly observable. Since $p = \partial t_0/\partial x$ is not only observable, but constant in depth, it is customary to use it to eliminate θ from equations (1.5) and (1.6):

$$\frac{\partial t_0}{\partial x} = \frac{\sin\theta}{v} = p \tag{1.7}$$

$$\frac{\partial t_0}{\partial z} = \frac{\cos\theta}{v} = \sqrt{\frac{1}{v(z)^2} - p^2} \tag{1.8}$$

1.1.2 Evanescent waves

Suppose the velocity increases to infinity at infinite depth. Then equation (1.8) tells us that something strange happens when we reach the depth for which p^2 equals $1/v(z)^2$. That is the depth at which the ray turns horizontal. Below this critical depth the seismic wavefield damps exponentially with increasing depth. Such waves are called **evanescent**. For a physical example of an evanescent wave, forget the airplane and think about a moving bicycle. For a bicyclist, the slowness p is so large that it dominates $1/v(z)^2$ for all earth materials. The bicyclist does not radiate a wave, but produces a ground deformation that decreases exponentially into the earth. To radiate a wave, a source must move faster than the material velocity.

1.2 DOWNWARD CONTINUATION

Given a vertically upcoming plane wave at the earth's surface, say $u(t,x,z=0) = u(t)\mathrm{const}(x)$, and an assumption that the earth's velocity is vertically stratified, i.e. $v = v(z)$, we can presume that the upcoming wave down in the earth is simply time-shifted from what we see on the surface. (This assumes no multiple reflections.) Time shifting can be represented as a linear operator in the time domain by representing it as convolution with an impulse function. In the frequency domain, time shifting is simply multiplying by a complex exponential. This is expressed as

$$u(t,z) = u(t,z=0) * \delta(t + z/v) \tag{1.9}$$

$$U(\omega,z) = U(\omega,z=0)\, e^{-i\omega z/v} \tag{1.10}$$

1.2.1 Continuation of a dipping plane wave.

Next consider a plane wave **dip**ping at some angle θ. It is natural to imagine continuing such a wave back along a ray. Instead, we will continue the wave straight down. This requires the assumption that the plane wave is a perfect one, namely that the same waveform is observed at all x. Imagine two sensors in a vertical well bore. They should record the same signal except for a time shift that depends on the angle of the wave. Notice that the arrival time difference between sensors at two different depths is greatest for vertically propagating waves, and the time difference drops to zero for horizontally propagating waves. So the time shift Δt is $v^{-1}\cos\theta\,\Delta z$ where θ is the angle between the wavefront and the earth's surface (or the angle between the well bore and the ray). Thus an equation to **downward continue** the wave is

$$U(\omega,\theta,z+\Delta z) = U(\omega,\theta,z)\,\exp(-i\omega\,\Delta t) \tag{1.11}$$

$$U(\omega,\theta,z+\Delta z) = U(\omega,\theta,z)\,\exp\left(-i\omega\frac{\Delta z\cos\theta}{v}\right) \tag{1.12}$$

Equation (1.12) is a downward continuation formula for any angle θ. We can generalize the method to media where the velocity is a function of depth. Evidently we can apply equation (1.12) for each layer of thickness Δz, and allow the velocity vary with z. This is a well known approximation that handles the timing correctly but keeps the amplitude constant (since $|e^{i\phi}| = 1$) when in real life, the amplitude should vary because of reflection and transmission coefficients. Suffice it to say that in practical earth imaging, this approximation is almost universally satisfactory.

In a stratified earth, it is customary to eliminate the angle θ which is depth variable, and change it to the Snell's parameter p which is constant for all depths. Thus the downward continuation equation for any Snell's parameter is

$$U(\omega,p,z+\Delta z) = U(\omega,p,z)\,\exp\left(-\frac{i\omega\Delta z}{v(z)}\sqrt{1-p^2v(z)^2}\right) \tag{1.13}$$

It is natural to wonder where in real life we would encounter a **Snell wave** that we could downward continue with equation (1.13). The answer is that any wave from real life can be regarded as a sum of waves propagating in all angles. Thus a field data set should first be decomposed into Snell waves of all values of p, and then equation (1.13) can be used to downward continue each p, and finally the components for each p could be added. This process akin to Fourier analysis. We now turn to Fourier analysis as a method of downward continuation which is the same idea but the task of decomposing data into Snell waves becomes the task of decomposing data into sinusoids along the x-axis.

1.2.2 Downward continuation with Fourier transform

One of the main ideas in Fourier analysis is that an impulse function (a delta function) can be constructed by the superposition of sinusoids (or complex exponentials). In the study of time series this construction is used for the *impulse response* of a filter. In the study of functions of space, it is used to make a physical point source that can manufacture the downgoing waves that initialize the reflection seismic experiment. Likewise observed upcoming waves can be Fourier transformed over t and x.

Recall that a plane wave carrying an arbitrary waveform is given by equation (1.4). Specializing the arbitrary function to be the real part of the function $\exp[-i\omega(t-t_0)]$ gives

$$\text{moving cosine wave} = \cos\left[\omega\left(\frac{x}{v}\sin\theta + \frac{z}{v}\cos\theta - t\right)\right] \tag{1.14}$$

Using Fourier integrals on time functions we encounter the ***Fourier kernel*** $\exp(-i\omega t)$. To use Fourier integrals on the space-axis x the spatial angular frequency must be defined. Since we will ultimately encounter many space axes (three for shot, three for geophone, also the midpoint and offset), the convention will be to use a subscript on the letter k to denote the axis being Fourier transformed. So k_x is the angular spatial frequency on the x-axis and $\exp(ik_x x)$ is its Fourier kernel. For each axis and Fourier kernel there is the question of the sign before the i. The sign convention used here is the one used in most physics books, namely, the one that agrees with equation (1.14). With this convention, a wave moves in the *positive* direction along the space axes. Thus the Fourier kernel for (x,z,t)-space will be taken to be

$$\text{Fourier kernel} \;=\; e^{i\,k_x x}\; e^{i\,k_z z}\; e^{-i\omega t} \;=\; \exp[i(k_x x \;+\; k_z z \;-\; \omega t)] \tag{1.15}$$

Now for the whistles, bells, and trumpets. Equating (1.14) to the real part of (1.15), physical angles and velocity are related to Fourier components. The Fourier kernel has the form of a plane wave. These relations should be memorized!

Angles and Fourier Components	
$\sin\theta = \dfrac{v\,k_x}{\omega}$	$\cos\theta = \dfrac{v\,k_z}{\omega}$

(1.16)

A point in (ω, k_x, k_z)-space is a plane wave. The one-dimensional Fourier kernel extracts frequencies. The multi-dimensional Fourier kernel extracts (monochromatic) plane waves.

Equally important is what comes next. Insert the angle definitions into the familiar relation $\sin^2\theta + \cos^2\theta = 1$. This gives a most important relationship:

$$k_x^2 \;+\; k_z^2 \;=\; \frac{\omega^2}{v^2} \tag{1.17}$$

The importance of (1.17) is that it enables us to make the distinction between an arbitrary function and a chaotic function that actually is a wavefield. Imagine any function $u(t,x,z)$. Fourier transform it to $U(\omega, k_x, k_z)$. Look in the (ω, k_x, k_z)-volume for any nonvanishing values of U. You will have a wavefield if and only if all nonvanishing U have coordinates that satisfy (1.17). Even better, in practice the (t,x)-dependence at $z=0$ is usually known, but the z-dependence is not. Then the z-dependence is found by assuming U is a wavefield, so the z-dependence is inferred from (1.17).

Equation (1.17) also achieves fame as the "dispersion relation of the scalar **wave equation**," a topic developed more fully in IEI.

1.2.3 Linking Snell waves to Fourier transforms

To link **Snell wave**s to Fourier transforms we merge equations (1.5) and (1.6) with equations (1.16). To do this, move with the wave by setting the phase to be constant, say zero, so

$0 = i(k_x x + k_z z - \omega t)$ and form the derivative of t by x and by z.

$$\frac{k_x}{\omega} \quad = \quad \frac{\partial t_0}{\partial x} \quad = \quad \frac{\sin\theta}{v} \quad = \quad p \tag{1.18}$$

$$\frac{k_z}{\omega} \quad = \quad \frac{\partial t_0}{\partial z} \quad = \quad \frac{\cos\theta}{v} \quad = \quad \frac{\sqrt{1-p^2v^2}}{v} \tag{1.19}$$

The basic **downward continuation** equation for upcoming waves in Fourier space follows from equation (1.13) by eliminating p by using equation (1.18). For analysis of real seismic data we introduce a minus sign because equation (1.19) refers to downgoing waves and observed data is made from up-coming waves.

$$U(\omega, k_x, z+\Delta z) \quad = \quad U(\omega, k_x, z)\ \exp\left(-\ \frac{i\omega\Delta z}{v} \sqrt{1 - \frac{v^2 k_x^2}{\omega^2}} \right) \tag{1.20}$$

In Fourier space we delay signals by multiplying by $e^{i\omega\Delta t}$, analogously, equation (1.20) says we downward continue signals into the earth by multiplying by $e^{ik_z\Delta z}$. Multiplication in the Fourier domain means convolution in time which can be depicted by the engineering diagram in Figure 1.4.

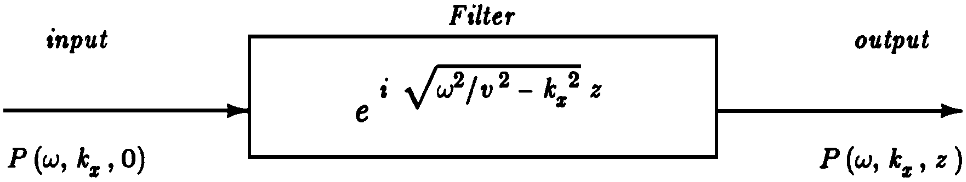

Figure 1.4: Downward continuation of a downgoing wavefield. dwnc-inout [NR]

Downward continuation is a product relationship in both the ω-domain and the k_x-domain. Thus it is a convolution in both time and x. What does the filter look like in the time and space domain? It turns out like a cone, that is, it is roughly an impulse function of $x^2 + z^2 - v^2t^2$. More precisely, it is the Huygens secondary wave source that was exemplified by ocean waves entering a gap through a storm barrier. Adding up the response of multiple gaps in the barrier would be convolution over x.

Migration is more than downward continuation of wave fields. You start with a surface data set $d(t,x)$ and downward continue it to all depth z getting $d(t,x,z)$. Then you have the earth image by selecting the wavefield at time $t = 0$ by the **exploding reflector** concept.

In practice, the downward continuation is done in the frequency domain. Recall to find the zero frequency component $F(\omega = 0)$ we sum over all time t. Interchanging time domain and frequency domain, we realize that to find the exploding reflector signal at zero time $f(t = 0)$ we can simply sum over all frequencies ω.

A nuisance of using Fourier transforms in migration and modeling is that spaces become periodic. This is demonstrated in Figure 1.5. Anywhere an event exits the frame at a side, top, or bottom boundary, the event immediately emerges on the opposite side. In practice, the unwelcome effect of **periodicity** is generally ameliorated by **padding** zeros around the data and the model.

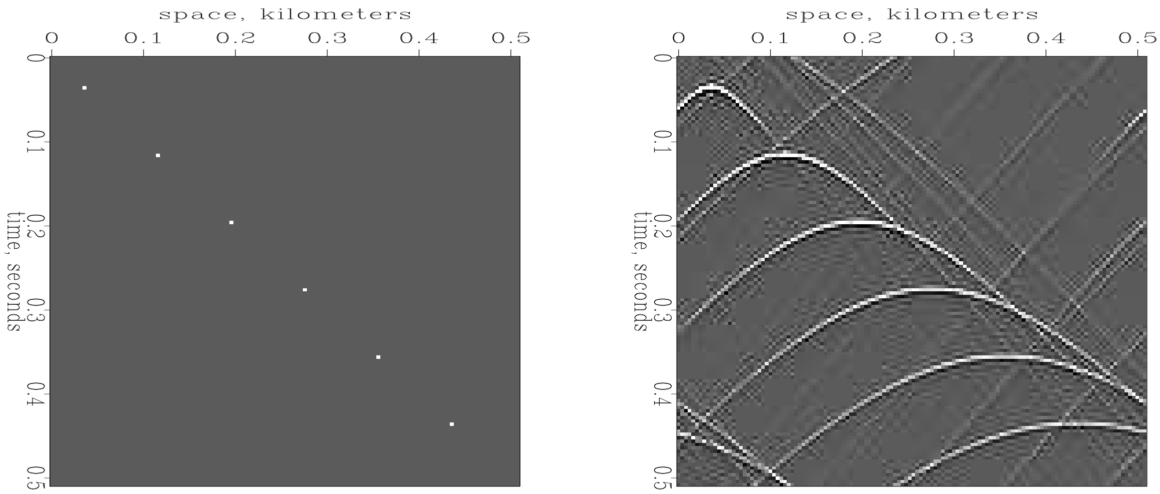

Figure 1.5: A reflectivity model on the left and synthetic data using a Fourier method on the right. $\boxed{\text{dwnc-diag}}$ [ER]

1.3 DOWNWARD CONTINUATION AND IMAGING

By now you are probably wondering what good are downward-continued seismic waves. The answer is that once we know the wavefields inside the earth we can do many seismic data analysis tasks. The obvious one is migration. We can do it better than Kirchhoff. Given the upcoming wave at the earth surface Wave$(t,x,z=0)$, pushing it down to all z in the earth gives us Wave(t,x,z). The exploding reflector idea is that the source is on the reflectors at $t=0$. Thus we construct images with

$$\text{Image}\ (x,z) \quad = \quad \text{Wave}\ (t=0,x,z) \tag{1.21}$$

There is a curious symmetry between the input and output of the imaging process:

$$\begin{array}{ccc} \text{observations} & & \text{model} \\ \boxed{\begin{array}{c} z=0 \\ \\ \text{all } t \end{array}} & \begin{array}{c} \text{migration} \\ \longrightarrow \\ \longleftarrow \\ \text{diffraction} \end{array} & \boxed{\begin{array}{c} t=0 \\ \\ \text{all } z \end{array}} \end{array}$$

Diffraction is sometimes regarded as the natural process that creates and enlarges hyperboloids. *Migration* is the computer process that does the reverse.

Compared to the **Kirchhoff** method, **downward continuation** methods offer several advantages: (1) we handle $v(z)$ media exactly without calculating any ray paths; (2) we get the correct amplitude versus angle; (3) we automatically include various phase shifts and wave phenomena; (4) interpolation and aliasing artifacts are much reduced.

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