Chapter 1

Two-dimensional Fourier transform

1.1 SLOW PROGRAM FOR INVERTIBLE FT

Typically, signals are real valued. But the programs in this chapter are for complex-valued signals. In order to use these programs, copy the real-valued signal into a complex array, where the signal goes into the real part of the complex numbers; the imaginary parts are then automatically set to zero.

1.1.1 The slow FT code

The `slowft()` routine exhibits features found in many physics and engineering programs. For example, the time-domain signal (which I call "`tt()`"), has `nt` values subscripted, from `tt(1)` to `tt(nt)`. The first value of this signal `tt(1)` is located in real physical time at `t0`. The time interval between values is `dt`. The value of `tt(it)` is at time `t0+(it-1)*dt`. I do not use "`if`" as a pointer on the frequency axis because `if` is a keyword in most programming languages. Instead, I count along the frequency axis with a variable named `ie`.

```
subroutine slowft( adj, add, t0,dt,tt,nt,  f0,df,  ff,nf)
integer it,ie,     adj, add,          nt,             nf
complex cexp, cmplx,                tt(nt),         ff(nf)
real pi2, freq, time, scale, t0,dt,         f0,df
call adjnull(      adj, add,        tt,2*nt,        ff,2*nf)
                        pi2   = 2. * 3.14159265;
                        scale = 1./sqrt( 1.*nt)
                        df    = (1./dt) / nf
                        f0    =   - .5/dt

do ie = 1, nf {   freq= f0 + df*(ie-1)
do it = 1, nt {   time= t0 + dt*(it-1)

        if( adj == 0 )
                ff(ie)= ff(ie) + tt(it) * cexp(cmplx(0., pi2*freq*time)) * scale
        else
```

```
                tt(it)= tt(it) + ff(ie) * cexp(cmplx(0.,-pi2*freq*time)) * scale
        }}
return; end
```

The total frequency band is 2π radians per sample unit or $1/\Delta t$ Hz. Dividing the total interval by the number of points `nf` gives Δf. We could choose the frequencies to run from 0 to 2π radians/sample. That would work well for many applications, but it would be a nuisance for applications such as differentiation in the frequency domain, which require multiplication by $-i\omega$ including the **negative frequencies** as well as the positive. So it seems more natural to begin at the most negative frequency and step forward to the most positive frequency.

1.2 TWO-DIMENSIONAL FT

Let us review some basic facts about **two-dimensional Fourier transform**. A two-dimensional function is represented in a computer as numerical values in a matrix, whereas a one-dimensional Fourier transform in a computer is an operation on a vector. A 2-D Fourier transform can be computed by a sequence of 1-D Fourier transforms. We can first transform each column vector of the matrix and then each row vector of the matrix. Alternately, we can first do the rows and later do the columns. This is diagrammed as follows:

$$\begin{array}{ccc} p(t,\, x) & \longleftrightarrow & P(t,\, k_x) \\ \updownarrow & & \updownarrow \\ P(\omega,\, x) & \longleftrightarrow & P(\omega,\, k_x) \end{array}$$

The diagram has the notational problem that we cannot maintain the usual convention of using a lower-case letter for the domain of physical space and an upper-case letter for the Fourier domain, because that convention cannot include the mixed objects $P(t,k_x)$ and $P(\omega,x)$. Rather than invent some new notation, it seems best to let the reader rely on the context: the arguments of the function must help name the function.

An example of **two-dimensional Fourier transform**s on typical deep-ocean data is shown in Figure 1.1. In the deep ocean, sediments are fine-grained and deposit slowly in flat, regular, horizontal beds. The lack of permeable rocks such as sandstone severely reduces the potential for petroleum production from the deep ocean. The fine-grained shales overlay irregular, igneous, **basement rock**s. In the plot of $P(t,k_x)$, the lateral continuity of the sediments is shown by the strong spectrum at low k_x. The igneous rocks show a k_x spectrum extending to such large k_x that the deep data may be somewhat **spatially aliased** (sampled too coarsely). The plot of $P(\omega,x)$ shows that the data contains no low-frequency energy. The dip of the sea floor shows up in (ω,k_x)-space as the energy crossing the origin at an angle.

Altogether, the **two-dimensional Fourier transform** of a collection of seismograms involves only twice as much computation as the one-dimensional Fourier transform of each

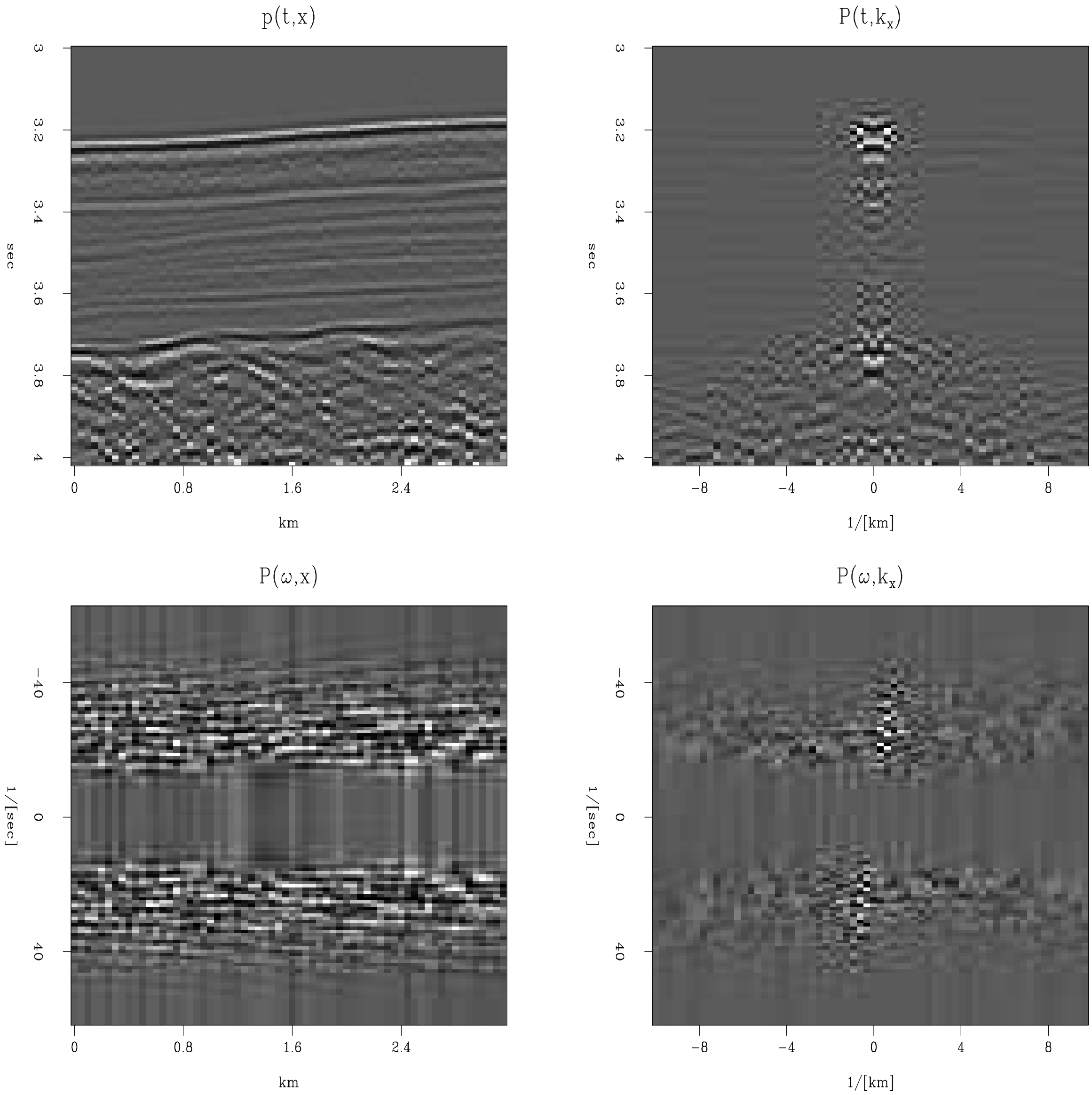

Figure 1.1: A deep-marine dataset $p(t,x)$ from Alaska (U.S. Geological Survey) and the *real* part of various Fourier transforms of it. Because of the long traveltime through the water, the time axis does not begin at $t=0$. dft-plane4 [ER]

seismogram. This is lucky. Let us write some equations to establish that the asserted procedure does indeed do a 2-D Fourier transform. Say first that any function of x and t may be expressed as a superposition of sinusoidal functions:

$$p(t,x) \quad = \quad \int\int e^{-i\omega t + ik_x x} \ P(\omega, k_x) \, d\omega \, dk_x \tag{1.1}$$

The double integration can be nested to show that the temporal transforms are done first (inside):

$$\begin{aligned} p(t,x) \quad &= \quad \int e^{i\,k_x x} \left[\int e^{-i\omega t} \ P(\omega, k_x) \, d\omega \right] dk_x \\ &= \quad \int e^{i\,k_x x} \ P(t, k_x) \, dk_x \end{aligned}$$

The quantity in brackets is a Fourier transform over ω done for each and every k_x. Alternately, the nesting could be done with the k_x-integral on the inside. That would imply rows first instead of columns (or vice versa). It is the separability of $\exp(-i\omega t \, + \, i\,k_x x)$ into a product of exponentials that makes the computation this easy and cheap.

1.2.1 Signs in Fourier transforms

In Fourier transforming t-, x-, and z-coordinates, we must choose a sign convention for each coordinate. Of the two alternative **sign convention**s, electrical engineers have chosen one and physicists another. While both have good reasons for their choices, our circumstances more closely resemble those of physicists, so we will use their convention. For the *inverse* Fourier transform, our choice is

$$p(t,x,z) \quad = \quad \int\int\int e^{-i\omega t \, + ik_x x \, + ik_z z} \ P(\omega, k_x, k_z) \, d\omega \, dk_x \, dk_z \tag{1.2}$$

For the *forward* Fourier transform, the space variables carry a *negative* sign, and time carries a *positive* sign.

Let us see the reasons why electrical engineers have made the opposite choice, and why we go with the physicists. Essentially, engineers transform only the time axis, whereas physicists transform both time and space axes. Both are simplifying their lives by their choice of sign convention, but physicists complicate their time axis in order to simplify their many space axes. The engineering choice minimizes the number of minus signs associated with the time axis, because for engineers, d/dt is associated with $i\omega$ instead of, as is the case for us and for physicists, with $-i\omega$. We confirm this with equation (1.2). Physicists and geophysicists deal with many more independent variables than time. Besides the obvious three space axes are their mutual combinations, such as midpoint and offset.

You might ask, why not make *all* the signs positive in equation (1.2)? The reason is that in that case waves would not move in a positive direction along the space axes. This would be especially unnatural when the space axis was a radius. Atoms, like geophysical sources, always radiate from a point to infinity, not the other way around. Thus, in equation (1.2) the sign of the spatial frequencies must be opposite that of the temporal frequency.

The only good reason I know to choose the engineering convention is that we might compute with an array processor built and microcoded by engineers. Conflict of sign convention is not a problem for the programs that transform complex-valued time functions to complex-valued frequency functions, because there the sign convention is under the user's control. But sign conflict does make a difference when we use any program that converts real-time functions to complex frequency functions. The way to live in both worlds is to imagine that the frequencies produced by such a program do not range from 0 to $+\pi$ as the program description says, but from 0 to $-\pi$. Alternately, we could always take the complex conjugate of the transform, which would swap the sign of the ω-axis.

1.2.2 Examples of 2-D FT

An example of a **two-dimensional Fourier transform** of a pulse is shown in Figure 1.2.

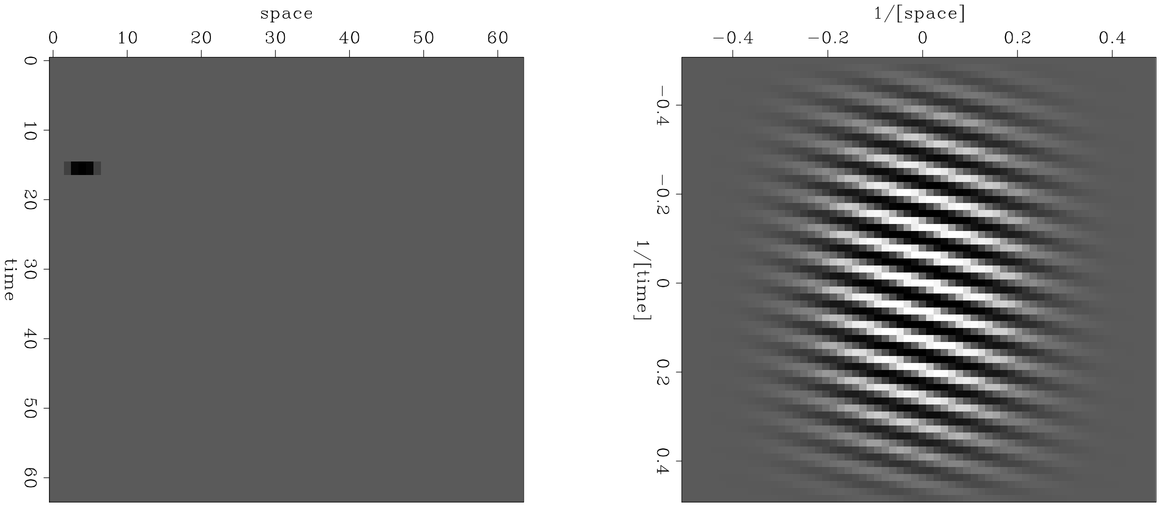

Figure 1.2: A broadened pulse (left) and the real part of its FT (right). dft-ft2dofpulse [ER]

Notice the location of the pulse. It is closer to the time axis than the frequency axis. This will affect the real part of the FT in a certain way (see exercises). Notice the broadening of the pulse. It was an impulse smoothed over time (vertically) by convolution with (1,1) and over space (horizontally) with (1,4,6,4,1). This will affect the real part of the FT in another way.

Another example of a two-dimensional Fourier transform is given in Figure 1.3. This example simulates an impulsive air wave originating at a point on the x-axis. We see a wave propagating in each direction from the location of the source of the wave. In Fourier space there are also two lines, one for each wave. Notice that there are other lines which do not go through the origin; these lines are called "**spatial alias**es." Each actually goes through the origin of another square plane that is not shown, but which we can imagine alongside the one shown. These other planes are periodic replicas of the one shown.

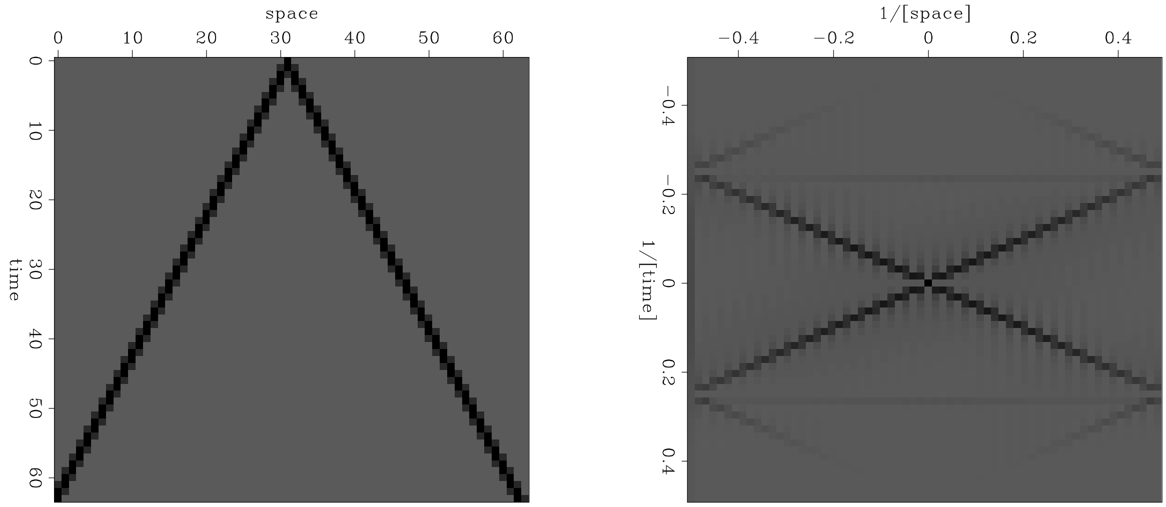

Figure 1.3: A simulated air wave (left) and the amplitude of its FT (right). dft-airwave [ER]

EXERCISES:

1 Most time functions are real. Their imaginary part is zero. Show that this means that $F(\omega,k)$ can be determined from $F(-\omega,-k)$.

2 What would change in Figure 1.2 if the pulse were moved (a) earlier on the t-axis, and (b) further on the x-axis? What would change in Figure 1.2 if instead the time axis were smoothed with (1,4,6,4,1) and the space axis with (1,1)?

3 What would Figure 1.3 look like on an earth with half the earth velocity?

4 Numerically (or theoretically) compute the two-dimensional spectrum of a plane wave $[\delta(t-px)]$, where the plane wave has a randomly fluctuating amplitude: say, rand(x) is a random number between ± 1, and the randomly modulated plane wave is $[(1+.2\,\mathrm{rand}(x))\,\delta(t-px)]$.

5 Explain the horizontal "layering" in Figure 1.1 in the plot of $P(\omega,x)$. What determines the "layer" separation? What determines the "layer" slope?

Index