Chapter 9

Mathematical physics in stratified media

In stratified media there are many common mathematical aspects in phenomena so physically diverse as acoustics, electromagnetic waves, magnetostatics, gravitational-elastic spherical resonance, heat flow, gas diffusion, electric current in a resistive material, seismic waves, water waves, and atmospheric gravity waves, among many others. We will present the general theory and work out some of the details for the case of simple acoustics.

By a stratified medium we mean one in which material properties, compressibility, conductivity, density, etc., are functions of one spatial coordinate only. The usual situation is cartesian coordinates, but when geophysics is done on a global scale spherical coordinates may be used.

9-1 FROM PHYSICS TO MATHEMATICS

First Step:

The first step is to write down all the basic partial differential equations of classical physics which relate to the problem of interest. Do not write down equations containing second space derivatives which are derived from first-derivative equations. Write down the first-derivative equations. Write each component of vector or matrix equations.

In acoustics we have the gradient pressure p gives rise to an acceleration of mass density ρ. For convenience we restrict motion to the x, z plane. Letting u and w represent x and z components of velocity we have

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{9-1-1}$$

$$\rho \frac{\partial w}{\partial t} \quad = \quad -\frac{\partial p}{\partial z} \tag{9-1-2}$$

Another equation which is important in acoustics is the one that states that the divergence of velocity multiplied by the incompressibility K yields the rate of pressure decrease.

$$\frac{\partial p}{\partial t} \quad = \quad -K \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) + s(x,z,t) \tag{9-1-3}$$

In (9-1-3) we included a pressure source s. This is something to be externally prescribed. The quantity s may be a source of chemical energy such as an explosion; thus it may vanish everywhere except at a point. Distributed sources are also often of interest; for example, radioactive rocks in a heat-flow model of the earth. To be more general, we could also have put momentum sources into (9-1-1) or (9-1-2), but the basic principles will be adequately exemplified with a source only in (9-1-3).

Second step:

The wave disturbance variables are taken to be unknown and the material properties known. Count equations and unknowns. We have three equations; u, w, and p are the three unknowns. We take K, ρ, and s to be known. Notice that the equations are linear in the unknowns. Now we make the stratification assumption; that is, we assume K and ρ are functions of depth z only and that they are constant in x. Since our linear equations now have constant coefficients with respect to x and t, we may always expect sinusoidal solutions in x and t. We do not know what to expect of our solutions in the z coordinate because of the arbitrary z-dependence of the coefficients K and ρ. This leads to step three.

Third step:

Fourier transform time and the space coordinates with constant coefficients. In other words, we make the following substitution into (9-1-1), (9-1-2), and (9-1-3)

$$\begin{bmatrix} u(x,z,t) \\ w(x,z,t) \\ p(x,z,t) \\ s(x,z,t) \end{bmatrix} \quad = \quad \begin{bmatrix} U(k,z,\omega) \\ W(k,z,\omega) \\ P(k,z,\omega) \\ S(k,z,\omega) \end{bmatrix} e^{-i\omega t + ik_x x} \tag{9-1-4}$$

After substitution, cancel the exponential and obtain

$$-i\omega\rho(z)U \quad = \quad -ik_x P \tag{9-1-5a}$$

$$-i\omega\rho(z)W \quad = \quad -\frac{\partial P}{\partial z} \tag{9-1-5b}$$

$$-i\omega P \quad = \quad -K(z) \left(ik_x U + \frac{\partial W}{\partial z} \right) + S \tag{9-1-5c}$$

Fourth step:

Eliminate algebraically the algebraic unknowns. In other words, when you examine (9-1-5) you see terms in $\partial P / \partial z$ but you do not see $\partial U / \partial z$. This means that U is an algebraic variable which can be eliminated by purely algebraic means. We do this by substituting (9-1-5a), into (9-1-5c).

Fifth step:

Bring $\partial/\partial z$ terms to the left, bring all others to the right, and arrange terms into a neat matrix form. We have

$$\begin{aligned} \frac{\partial P}{\partial z} &= i\omega\rho W \\ \frac{\partial W}{\partial z} &= i\left(\frac{\omega}{K} - \frac{k_x^2}{\omega\rho}\right)P + \frac{S}{K} \end{aligned}$$

and then

$$\frac{\partial}{\partial z}\begin{bmatrix} P \\ W \end{bmatrix} = i\begin{bmatrix} 0 & \omega\rho \\ \frac{\omega}{K} - \frac{k_x^2}{\omega\rho} & 0 \end{bmatrix}\begin{bmatrix} P \\ W \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{S}{K} \end{bmatrix} \tag{9-1-6}$$

Sixth step:

Recognize that, no matter the physical problem with which you started, you should have a matrix first-order differential equation of the form

$$\frac{\partial}{\partial z}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathrm{s} \tag{9-1-7}$$

where $\mathbf{x}$ is a vector containing the field variables of interest, $\mathbf{A}$ is a matrix depending on temporal and spatial frequency and on material properties, and $\mathbf{s}$ is a (possibly absent) vector function of the sources.

Before we look into techniques of solving (9-1-7) we can immediately deduce that in a source-free region the field variables $\mathbf{x}$ are smoother functions than the material properties. To see this, consider two homogeneous layers in contact. At the contact the $\mathbf{A}$ matrix has step-function discontinuities. Now let us see whether the wave fields in $\mathbf{x}$ can have step-function discontinuities. Obviously they cannot, since a step discontinuity in $\mathbf{x}$ would imply $d\mathbf{x}/dz = \infty$, whereas (9-1-7) in a source-free region states that $d\mathbf{x}/dz = \mathbf{A}\mathbf{x}$ and both $\mathbf{A}$ and $\mathbf{x}$ are supposed finite. This does not mean that all field variables are always smooth. The algebraic variables eliminated in the *fourth step* can and often will be discontinuous at layer boundaries.

EXERCISES:

1 What form does (9-1-7) take for the heat-flow equation? Include radioactive sources. [Hint: See equations (10-1-1) and (10-1-2).]

2 Using Maxwell's equations, $\boldsymbol{\nabla}\times\mathbf{E} = -\mu\mathbf{H}$, $\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J} + \epsilon\mathbf{E}$, and Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$ where σ is conductivity, set $\partial/\partial y = 0$ and derive (9-1-7).

3 In electrostatics the electric field in the ionosphere may be derived from a potential $\boldsymbol{\nabla}\phi = -\mathbf{E}$, the divergence of electrical current vanishes $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$ and Ohm's law

must have an extra term due to wind (a current source due to differential drag on ions and electrons across the earth's magnetic field) $\mathbf{J} = \boldsymbol{\sigma}\mathbf{E} + \boldsymbol{\tau}\mathbf{V}$. Assume you know $\mathbf{V}$. What form does (9-1-7) take assuming σ and τ to be scalars? Indicate how the calculation proceeds if $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are matrices (assume you have the inverse of any matrix you wish).

4 In magnetostatics **curl** $\mathbf{H} = \mathbf{J}$ and $\operatorname{div} \mathbf{B} = 0$, and $\mathbf{B} = \mu \mathbf{H}$. Taking $\mathbf{J}$ as given, what is the form of (9-1-7)?

5 This exercise illustrates the linearization of nonlinear problems. For acoustic waves in a stratified windy atmosphere we used the trial solutions

$$\begin{bmatrix} P \\ U \\ W \end{bmatrix} = \begin{bmatrix} \bar{P} \\ \bar{U}(z) \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{P}(z) \\ \tilde{U}(z) \\ \tilde{W}(z) \end{bmatrix} e^{-i\omega t + ik_x x}$$

Reduce the partial differential equations to a matrix ordinary differential equation. Hint: The horizontal acceleration terms is

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \\ &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial z} w \end{aligned}$$

with a like term for vertical acceleration. Drop second-order terms in $\tilde{P}$, $\tilde{U}$, and $\tilde{W}$.

6 Two equations come from heat flow: (H_x, H_z) equals the conductivity σ multiplied by the negative of the temperature gradient $(\partial_x, \partial_z)T$. The time derivative of temperature multiplied by the heat capacity c equals the negative of the heat-flow divergence $\partial_x H_x + \partial_z H_z$ gives another equation. Insert the trial solutions

$$\begin{bmatrix} T \\ H_x \\ H_z \end{bmatrix} = \begin{bmatrix} \bar{T}(z) \\ 0 \\ \bar{H}_z \end{bmatrix} + \begin{bmatrix} \tilde{T}(z) \\ \tilde{H}_x(z) \\ \tilde{H}_x(z) \end{bmatrix} e^{-i\omega t + ik_x x}$$

(*a*) First derive steady-state equations for $\bar{T}$ and $\bar{H}$ assuming $\tilde{T}$ and $\tilde{H}$ vanish.

(*b*) Assuming $\bar{T}$ and $\bar{H}$ satisfy part (*a*), find equations for $\tilde{T}$ and $\tilde{H}$.

(*c*) Repeat (*a*) and (*b*) assuming linear temperature dependence of heat capacity and conductivity, i.e.,

$$\begin{aligned} \sigma &= \sigma_0(z) + \sigma_1(z)T \\ c &= c_0(z) + c_1(z)T \end{aligned}$$

You will have to drop squared terms in $\tilde{T}$ and $\tilde{H}$.

7 Consider a compressible liquid sphere pulsating radially under its own gravitational attraction. What is the form of (9-1-6)?

HINTS:
$$\begin{array}{lll} \rho\ddot{\mathbf{v}} & = -\boldsymbol{\nabla} - \rho\mathbf{g} & \text{momentum} \\ \tilde{\rho} + \bar{\rho}\boldsymbol{\nabla}\cdot\mathbf{v} & = 0 & \text{mass} \\ p + K\boldsymbol{\nabla}\cdot\mathbf{v} & = 0 & \text{state} \\ \boldsymbol{\nabla}\cdot\mathbf{g} & = 4\pi\gamma\rho & \text{gravity} \end{array}$$

9-2 NUMERICAL MATRIZANTS

A differential equation relates field variables at a point to field variables at neighboring points. A matrizant relates field variables at one depth in a stratified material to variables at some other depth. A matrizant may also be regarded as the integral of the matrix differential equation (9-1-7). First we will show how to get the matrizant of (9-1-7) by numerical means. That is, we will solve the problem for arbitrary depth variations in density and in compressibility. Then we will come back and develop analytical solutions for the special case of constant material properties. We have

$$\frac{\partial \mathbf{Z}}{\partial z} \approx \frac{\mathbf{X}(z+\triangle z) - \mathbf{X}(z)}{\triangle z} \approx \mathbf{A}(z)\mathbf{X}(z) + \mathbf{S}(z)$$

or

$$\mathbf{X}(z+\triangle z) = (\mathbf{I} + \mathbf{A}\triangle z)\mathbf{X}(z) + \mathbf{S}(z)\triangle z \tag{9-2-1}$$

Given $\mathbf{X}$ for some particular z it is clear that (9-2-1) may be used recursively to get $\mathbf{X}$ for any z. For simplicity we may take $\triangle z = 1$ and use subscripts to indicate the z coordinate. Let $[\mathbf{I} + \mathbf{A}(z)\triangle z]$ be denoted by $\mathbf{Q}(z)$, then (9-2-1) becomes

$$\mathbf{X}_{k+1} = \mathbf{Q}_k\mathbf{X}_k + \mathbf{S}_k$$

or

$$\mathbf{X}_1 = \mathbf{Q}_0\mathbf{X}_0 + \mathbf{S}_0$$

hence

$$\begin{aligned} \mathbf{X}_2 &= \mathbf{Q}_1(\mathbf{Q}_0\mathbf{X}_0 + \mathbf{S}_0) + \mathbf{S}_1 \\ &= \mathbf{Q}_1\mathbf{Q}_0\mathbf{X}_0 + \mathbf{Q}_1\mathbf{S}_0 + \mathbf{S}_1 \end{aligned}$$

hence

$$\begin{aligned} \mathbf{X}_3 &= \mathbf{Q}_2(\mathbf{Q}_1\mathbf{Q}_0\mathbf{X}_0 + \mathbf{Q}_1\mathbf{S}_0 + \mathbf{S}_1) + \mathbf{S}_2 \\ &= \mathbf{Q}_2\mathbf{Q}_1\mathbf{Q}_0\mathbf{X}_0 + \mathbf{Q}_2(\mathbf{Q}_1\mathbf{S}_0 + \mathbf{S}_1) + \mathbf{S}_2 \end{aligned}$$

likewise

$$\mathbf{X}_4 = \mathbf{M}\mathbf{X}_0 + \mathbf{T} \tag{9-2-2}$$

So we have in general a numerically determinable matrix $\mathbf{M}$ (called the matrizant) and a vector $\mathbf{T}$ which relates the field variables at the top of the strata to those on the bottom by

$$\mathbf{X}_{z_{\text{top}}} = \mathbf{M}\mathbf{X}_{z_{\text{bot}}} + \mathbf{T} \tag{9-2-3}$$

The matrix $\mathbf{M}$ is also called an integral matrix. Physical problems present themselves in different ways with different boundary conditions. For the acoustic problem discussed earlier $\mathbf{X}$ is a two-component vector involving pressure and of the stratified medium. Thus (9-2-3) represents two equations for four unknowns. The solution to the problem comes only when two boundary conditions are introduced. If we are talking about sound waves in the ocean, (simplified) boundary conditions would be to prescribe zero pressure at the surface and zero vertical displacement at the sea floor. Then these boundary conditions with (9-2-3) would be two equations and two unknowns and consequently could be solved for surface displacement and bottom pressure. From these, pressure and displacement could be determined everywhere. Proper determination of boundary conditions is often the trickiest part of a problem; we will return to it for some other problems in a later section.

If portions of the material have constant material properties and contain no sources, then it is possible to find an analytical expression for the matrizant. A matrizant which takes one across such a layer of constant properties is called, appropriately enough, a *layer matrix*. It may be verified by substitution that

$$\mathbf{X}_z = e^{[\mathbf{A}(z-z_0)]}\mathbf{X}_{z_0} \tag{9-2-4}$$

is the solution to $(\partial/\partial z)\mathbf{X} = \mathbf{A}\mathbf{X}$ where

$$e^{\mathbf{A}(z-z_0)} = \mathbf{I} + \mathbf{A}(z-z_0) + \frac{\mathbf{A}^2(z-z_0)^2}{2!} + \cdots \tag{9-2-5}$$

in a region of space where $\mathbf{A}$ is constant with z. Thus $e^{\mathbf{A}(z-z_0)}$ is the required matrizant. The matrix exponential could be computed numerically either by the method of (9-2-2) or the method of (9-2-5) or the method of Sylvester's theorem described in Chap. 5. In the next section we will see how Sylvester's theorem leads directly to the ideas of up- and downgoing waves.

EXERCISES:

1 What is $\mathbf{Q}_k$ for the improved central difference approximation?

$$\mathbf{X}(z+\triangle z) - \mathbf{X}(z) = \frac{\triangle z\mathbf{A}[\mathbf{X}(z+\triangle z)+\mathbf{X}(z)]}{2}$$

9-3 UP- AND DOWNGOING WAVES

We have seen a host of examples of how physical problems in stratified source-free media reduce to the form

$$\frac{d}{dz}\mathbf{X} \quad = \quad \mathbf{AX} \tag{9-3-1}$$

Where $\mathbf{X}$ is a vector of physical variables and $\mathbf{A}$ is a matrix which depends on z if material properties depend upon z. An important set of new variables in the vector $\mathbf{V}$ is defined by multiplying the vector of physical variables $\mathbf{X}$ by a square matrix $\mathbf{R}$

$$\mathbf{V} \quad = \quad \mathbf{RX} \tag{9-3-2}$$

where $\mathbf{R}$ is the matrix of row eigenvectors of the matrix $\mathbf{A}$. Inverse to $\mathbf{R}$ is the matrix $\mathbf{C}$ of column eigenvectors of $\mathbf{A}$. Premultiplying (9-3-2) by $\mathbf{C}$ and using $\mathbf{CR} = \mathbf{I}$ we get the inverse relation to (9-3-2) which is useful to find the physical variables $\mathbf{X}$ from the new variables $\mathbf{V}$.

$$\mathbf{X} \quad = \quad \mathbf{CV} \tag{9-3-3}$$

Inserting (9-3-3) into (9-3-1) we obtain

$$(\mathbf{CV}_z) \quad = \quad \mathbf{ACV} \tag{9-3-4}$$

$$\mathbf{CV}_z \quad = \quad \mathbf{ACV} - \mathbf{C}_z\mathbf{V} \tag{9-3-5}$$

Premultiplying by $\mathbf{R}$ and using $\mathbf{RC} = \mathbf{I}$ we obtain

$$\mathbf{V}_z \quad = \quad (\mathbf{RAC})\mathbf{V} - \mathbf{RC}_z\mathbf{V} \tag{9-3-6}$$

Since we have supposed $\mathbf{R}$ and $\mathbf{C}$ to be row and column eigenvector matrices of $\mathbf{A}$ we can replace $\mathbf{RAC}$ by the diagonal matrix of eigenvalues $\mathbf{\Lambda}$, that is

$$\mathbf{V}_z \quad = \quad \mathbf{\Lambda V} - \mathbf{RC}_z\mathbf{V} \tag{9-3-7}$$

In any region of physical space where the material is homogeneous then $\mathbf{A}$, hence $\mathbf{C}$, will be independent of z and (9-3-5) will reduce to

$$\frac{d}{dz}\mathbf{V} \quad = \quad \mathbf{\Lambda V} \tag{9-3-8}$$

But the only matrix in (9-3-6) is a diagonal matrix, and so the problem for the different variables in the vector $\mathbf{V}$ decouples into a separate problem for each component. In wave problems it will be seen to be appropriate to call the components of $\mathbf{V}$ upgoing and downgoing wave variables. These variables flow up and down in homogeneous regions without interacting with each other. Let us consider an example.

In Sec. 9-1 we deduced that the matrix first-order differential equation for the acoustic problem in a region of no sources takes the form

$$\frac{d}{dz}\begin{bmatrix} P \\ W \end{bmatrix} \quad = \quad \begin{bmatrix} 0 & ia^2 \\ ib^2 & 0 \end{bmatrix}\begin{bmatrix} P \\ W \end{bmatrix} \tag{9-3-9}$$

where

$$a^2 = \omega\rho \tag{9-3-10}$$

$$b^2 = \frac{\omega}{K} - \frac{k_x^2}{\omega\rho} \tag{9-3-11}$$

The matrix of column eigenvectors **C** and the matrix of row eigenvectors of the matrix of (9-3-7) are readily verified to be

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ -\frac{b}{a} & \frac{b}{a} \end{bmatrix} \tag{9-3-12}$$

$$\mathbf{R} = \frac{1}{2}\begin{bmatrix} 1 & -\frac{a}{b} \\ 1 & \frac{a}{b} \end{bmatrix} \tag{9-3-13}$$

It is also readily verified that the vectors are normalized, namely $\mathbf{RC} = \mathbf{CR} = \mathbf{I}$ and that

$$\Lambda = \mathbf{RAC} = \begin{bmatrix} -iab & 0 \\ 0 & +iab \end{bmatrix}$$

The downgoing wave variable D is associated with the iab eigenvalue and the upcoming wave variable U is associated with the $-iab$. We have definitions for up- and downgoing waves as

$$\begin{bmatrix} U \\ D \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\frac{a}{b} \\ 1 & \frac{a}{b} \end{bmatrix}\begin{bmatrix} P \\ W \end{bmatrix} \tag{9-3-12a}$$

Of course a row eigenvector may contain an arbitrary multiplicative scaling factor if the scaling factor is divided from the corresponding column eigenvector. This means that the definition (9-3-12a) is not unique. As it happens, the present scale factors give the up- and downgoing waves the physical dimensions of P. The physical variables P and W are found from U and D by the inverse relation

$$\begin{bmatrix} P \\ W \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{b}{a} & \frac{b}{a} \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix} \tag{9-3-12b}$$

from which we see that the pressure P is the downgoing wave plus the upcoming wave and the vertical velocity is b/a times the difference. Equation (9-3-5) governing the propagation of U and D is

$$\frac{d}{dz}\begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} -iab & 0 \\ 0 & iab \end{bmatrix}\begin{bmatrix} U \\ C \end{bmatrix} - \frac{1}{2}\frac{(b/2)_z}{b/a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix} \tag{9-3-13}$$

In any region of space where b/a is not a function of z we are left with the simple uncoupled equations

$$\frac{d}{dz}\begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} -iab & \\ & iab \end{bmatrix}\begin{bmatrix} U \\ C \end{bmatrix} \tag{9-3-14}$$

Strictly, to justify the definitions of U and D as up- and downgoing waves we will have to be sure that the downgoing solution takes the form

$$d(z,t) \quad = \quad e^{-i\omega t + i k_z z} \tag{9-3-15}$$

where ω and k_z must agree in sign so that constant phase is maintained as both z and t increase. The opposite sign must apply to U. In other words $k_z = ab$ must take the sign of ω. To see that this happens we take the square root of the product of (9-3-8) and (9-3-9).

$$k_z \quad = \quad ab \quad = \quad \omega \left(\frac{\rho}{K} - \frac{k_x^2}{\omega^2} \right)^{1/2} \tag{9-3-16}$$

For vertically propagating waves we have $k_x = 0$ so that $k_z = ab$ specializes to $k_z = \omega(\rho/K)^{1/2}$. Substituting this value into (9-3-15), we see that the phase angle of the exponential is constant if $z/t = (K/\rho)^{1/2}$, making it clear that the material's intrinsic velocity is given by

$$v \quad = \quad \left(\frac{K}{\rho} \right)^{1/2} \tag{9-3-17}$$

Reference to Figure 9-1 shows that the angle θ between the vertical and a ray is defined by

$$\sin\theta \quad = \quad \frac{k_x v}{\omega} \tag{9-3-18}$$

Inserting (9-3-17) and (9-3-18) into (9-3-16) we obtain

$$k_z \quad = \quad ab \quad = \quad \frac{\omega}{v}\cos\theta \tag{9-3-19}$$

The time function (9-3-15) is complex. To get a real time function the expression (9-3-15) must be summed or integrated to include both positive and negative frequencies. Then, as we saw in the chapters on time series analysis, we must have $D(\omega) \quad = \quad \bar{D}(-\omega)$.

The quantity b/a will turn out to be the material's characteristic admittance Y. Taking the square root of the ratio of (9-3-9) over (9-3-8) we have

$$Y \quad = \quad \frac{b}{a} \quad = \quad \frac{(1 - v^2 k_x^2/\omega^2)^{1/2}}{\rho v} \tag{9-3-20}$$

$$Y \quad = \quad \frac{b}{a} \quad = \quad \frac{\cos\theta}{\rho v} \tag{9-3-21}$$

$$I \quad = \quad \frac{b}{a} \quad = \quad \frac{\rho v}{\cos\theta} \tag{9-3-22}$$

We shall now verify that this definition of impedance is the same as the one in the previous chapter. To do this we take a careful look at the matrizant to cross a layer $\exp[\mathbf{A}_1(z_2 - z_1] = \exp(\mathbf{A}\,\triangle z)$. By Sylvester's theorem we have for the matrizant

$$\exp(\mathbf{A}\,\triangle z) = \mathbf{C} \begin{bmatrix} e^{-ik_z\,\triangle z} & 0 \\ 0 & e^{+ik_z\,\triangle z} \end{bmatrix} \mathbf{R} \tag{9-3-23}$$

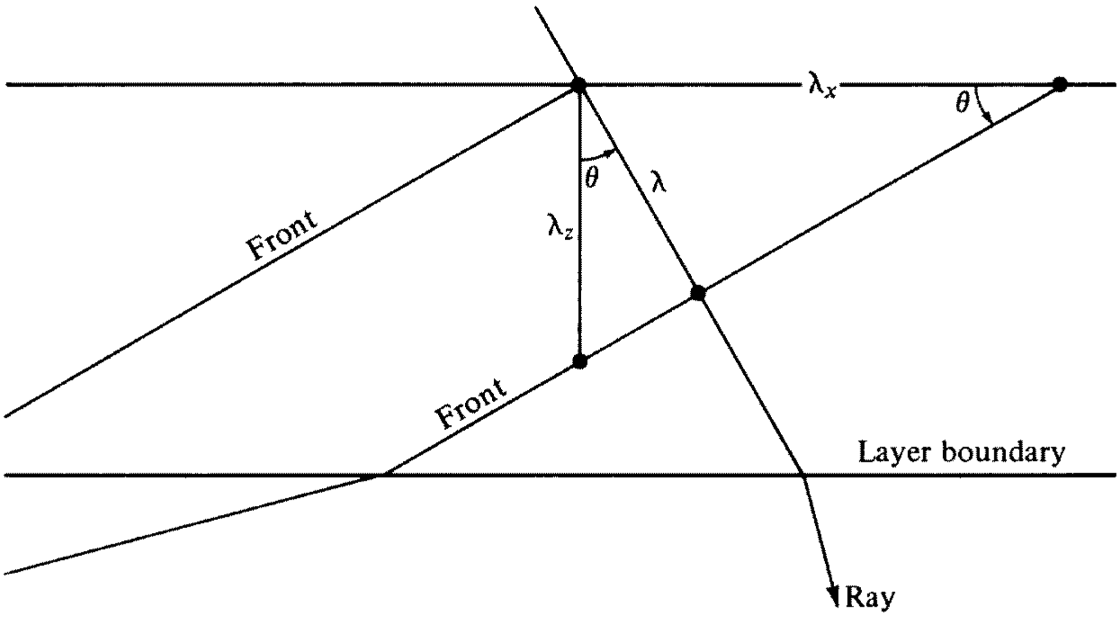

Figure 9.1: Rays and wavefronts in a layer. The wavelength λ_x seen on the x axis and the wavelength λ_z seen on the z axis are both greater than the wavelength λ seen along the ray. Clearly, $\lambda/\lambda_x = \sin\theta$ and $\lambda/\lambda_z = \cos\theta$ so the spatial frequencies $k_x = 2\pi/\lambda_x$ and $k_z = 2\pi/\lambda_z$ satisfy $k_x^2 + k_z^2 = (2\pi/\lambda)^2 = \omega^2/v^2$, which, besides being the Pythagorean theorem (since $\sin\theta = k_x\, v/\omega$), is the Fourier transform of the wave equation. Snell's law that $(\sin\theta)/v$ is the same from layer to layer is thus equivalent to saying that k_x/ω is the same in each layer. That the spatial frequency k_x is the same constant in each layer is essential to the satisfaction of continuity conditions at the layer interfaces. $\boxed{\text{c9-9-1}}$ [NR]

The matrizant relates the wave variables at the top z_1 of a layer to those at the bottom z_2. Thus (9-3-23) enables us to write

$$\begin{bmatrix} P \\ W \end{bmatrix}_2 = \mathbf{C}_1 \exp(\mathbf{\Lambda}_1 \triangle z) \mathbf{R}_1 \begin{bmatrix} P \\ W \end{bmatrix}_1 \tag{9-3-24}$$

Equation (9-3-24) which seems to have jumped at us from the mysteries of Sylvester's theorem actually has a simple interpretation. Starting on the right, we interpret the multiplication of $\mathbf{R}_1$ into the P and W variables as a conversion to up- and downgoing variables. Then the multiplication by $\exp(\mathbf{\Lambda_1} \triangle z)$ carries these across the layer and the multiplication by $\mathbf{C}_1$ converts back to P and W variables which are continuous crossing an interface. Multiplying (9-3-24) through by R_2 and noting (9-3-11) and (9-3-12) we have

$$\begin{bmatrix} U \\ D \end{bmatrix}_2 = \mathbf{R}_2 \mathbf{C}_1 \exp(\mathbf{\Lambda} \triangle z) \begin{bmatrix} U \\ D \end{bmatrix}_1 \tag{9-3-25}$$

In (9-3-25) we have now defined the up- and downgoing waves just beneath the interface as we did in the previous chapter. We should now be able to recognize the matrix as having the same form. It is

$$\frac{1}{2} \begin{bmatrix} 1 & -\frac{1}{Y_2} \\ 1 & \frac{1}{Y_2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -Y_1 & Y_1 \end{bmatrix} \begin{bmatrix} e^{-iab \triangle z} & 0 \\ 0 & e^{iab \triangle z} \end{bmatrix} \tag{9-3-26}$$

Defining the Z transform variable by

$$Z = \exp\left(\frac{2i\omega \triangle z}{v \cos\theta}\right)$$

Now we recognize that the travel time across the layer is $\triangle t = \triangle z / v \cos\theta$. The layer matrix (9-3-26) is

$$\frac{Y_1 + Y_2}{2Y_2} \begin{bmatrix} 1 & \frac{Y_2 - Y_1}{Y_2 + Y_1} \\ \frac{Y_2 - Y_1}{Y_2 + Y_1} & 1 \end{bmatrix} \begin{bmatrix} Z^{-1/2} & \\ & Z^{1/2} \end{bmatrix} \tag{9-3-27}$$

which may be compared to the matrix of (8-2-4) namely,

$$\frac{1}{t} \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{bmatrix} Z^{-1/2} & 0 \\ 0 & Z^{1/2} \end{bmatrix}$$

establishing that the definition $Y = b/a$ has led to the familiar definition of reflection coefficient

$$\begin{aligned} c &= \frac{Y_2 - Y_1}{Y_2 + Y_1} = \frac{I_1 - I_2}{I_1 + I_2} \\ &= \frac{\rho_1 v_1 / \cos\theta_1 - \rho_2 v_2 / \cos\theta_2}{\rho_1 v_1 / \cos\theta_1 + \rho_2 v_2 / \cos\theta_2} \end{aligned} \tag{9-3-28}$$

EXERCISES:

1 Redefine the eigenvectors so that $W + D + U$ and $P = (D - U)/Y$. This transformation would be useful if we wanted $t = 1 + c$ to refer to vertical velocity normalized variables instead of pressure variables as in Chap. 8. Deduce changes to all the equations of this section.

2 Write the matrizant which crosses a layer in terms of a, b, and layer thickness h.

9-4 SOURCE-RECEIVER RECIPROCITY

The principle of reciprocity states that a source and receiver may (under some conditions) be interchanged and the same waveform will be observed. This principle is often used to advantage in calculations and may also be used to simplify data collection. It is somewhat amazing that this principle applies t other earth with its complicated inhomogeneities. Intuitively, the main reason for validity of the reciprocal principle is that energy propagates equally well along a given ray in either direction. Either way, it goes at the same speed with the same attenuation. This is true for all common types of waves.

Little more would need to be said if all waves were scalar phenomena with scalar sources and scalar receivers as, for example, acoustic pressure waves with explosive sources and pressure-sensitive receivers. The situation becomes more complicated when the sources or receivers are moving diaphragms, because then their orientations become important. The directional properties of the source and receiver are often referred to as radiation patterns. To apply the reciprocity principle it is necessary to regard the radiation patterns as attached to the medium, not as being attached to the source and receiver. Thus, when source and receiver are said to be interchanged, it is only a scalar magnitude which is interchanged; the radiation patterns stay fixed at the same place. These general ideas are made more precise in the following derivation. It will be seen that the notion of rays actually turns out to be irrelevant. Reciprocity also works in diffusion and potential problems.

Theoretical treatments are often somewhat hard to read. They often begin by specifying that the differential operator along with suitable boundary conditions should constitute a self-adjoint problem. This means that when you reexpress the differential equations in difference form you discover that the matrix of coefficients is symmetric. Let us take the example of acoustic waves in one dimension. Newton's equation says that mass density ρ times accelerations $\partial_{tt}u$ equals the negative of the pressure gradient $-\partial_x p$ plus the external force F_x. Utilizing $e^{-i\omega t}$ time dependence we have

$$-\rho\omega^2 u \quad = \quad -\partial_x p + F_x$$

which, defining $F = -F_x$, may be written

$$\rho\omega^2 u - \partial_x p \quad = \quad F \tag{9-4-1}$$

The other important equation of acoustics says that the incompressibility K^{-1} multiplied by the pressure p plus the divergence of displacement $\partial_x u$ equals the external (relative) volume injection V, that is

$$K^{-1} p + \partial_x u \quad = \quad V \tag{9-4-2}$$

We will now combine (9-4-1) and (9-4-2) in a finite difference form with, for convenience, $\triangle x = 1$. In practice, one might like to use many grid points to approximate the behavior of continuous functions, but for the sake of illustration we only need use a few grid points. Luckily, in this case reciprocity will be exactly true despite the small number of grid points. We have

$$\begin{bmatrix} \frac{1}{I_0} & 1 & & & & & \\ 1 & \rho\omega^2 & -1 & & & & \\ & -1 & K^{-1} & 1 & & & \\ & & 1 & \rho\omega^2 & -1 & & \\ & & & -1 & K^{-1} & 1 & \\ & & & & 1 & I_n & \end{bmatrix} \begin{bmatrix} p_0 \\ u_0 \\ p_1 \\ u_1 \\ p_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} V_0 \\ F_0 \\ V_1 \\ F_1 \\ V_2 \\ F_2 \end{bmatrix} \tag{9-4-3}$$

The first and last rows of (9-4-3) requires some special comment. The quantities I_0 and I_n are called impedances. If they vanish, we have zero pressure end conditions; if they are infinite, we have zero motion end conditions.

Now with all this fuss we have gone through to obtain the matrix (9-4-3), the only thing we want from it is to observe that the matrix is indeed a symmetric matrix (even if ρ and K^{-1} were functions of x). In the exercises it is shown that a symmetric matrix may also be attained in two dimensions. That the matrix is symmetric is partly a result of the physical nature of sound and partly a result of careful planning on the part of the author. To obtain the correct statement of reciprocity in other situations you may have to do some careful planning too. The essence of reciprocity is that since the matrix of (9-4-3) is symmetric then the inverse matrix will also be symmetric. Premultiplying (9-4-3) through by the inverse matrix we get the responses as a result of matrix multiplication on the external excitations.

$$\begin{bmatrix} p_0 \\ u_0 \\ p_1 \\ u_1 \\ p_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & A & B & \cdot & \cdot \\ \cdot & \cdot & C & D & \cdot & \cdot \\ A & C & \cdot & \cdot & \cdot & \cdot \\ B & D & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} V_0 \\ F_0 \\ V_1 \\ F_1 \\ V_2 \\ F_2 \end{bmatrix} \tag{9-4-4}$$

The letters A, B, C, and D indicate the symmetry of the matrix of (9-4-4). Now if all external sources vanish except on one end where there is a unit strength volume source $V_0 = 1$, then according to (9-4-4) the pressure in the middle p_1 will equal A.

If in a second experiment all the external sources vanish except the middle volume source $V_1 = 1$, then according to (9-4-4) the pressure response p_0 at the end will also equal A. This is the reciprocal principle. Note that with the letter D in (9-4-4) a like statement applies to the forces and the displacements. A mixed statement applies with the letters C and B.

In a realistic experiment it may not be possible to have a pure volume source or a pure external force. In other words, the external source may have some finite, nonzero impedance. Then the first experiment we would perform would be with the excitation at the middle, getting for the end response:

$$\begin{bmatrix} p_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ F_1 \end{bmatrix} \tag{9-4-5}$$

Interchanging source and receiver locations, we have

$$\begin{bmatrix} p_1 \\ u_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_0 \\ F_0 \end{bmatrix} \tag{9-4-6}$$

The notable feature of (9-4-5) and (9-4-6) is that the matrices are transposes of one another. This feature would not be lost if we were to consider a more elaborate experiment where the vectors in (9-4-5) and (9-4-6) contained more elements. For example, a vector in (9-4-5) or (9-4-6) could contain elements of an array of physically separated volume sources or pressure sensors. In fact, if the reader is able to frame elastic, electromagnetic, diffusion, or potential problems as symmetric algebraic equations like (9-4-3), then the matrices like (9-4-5) and (9-4-6) will still be transposes of one another. The setting up of symmetric equations like (9-4-3) is often not difficult, although it may get somewhat complicated in multidimensional noncartesian geometry.

In such a more general case we may denote the right-hand vectors in (9-4-5) or (9-4-6) by $\mathbf{E}$ to denote excitation and the left-hand vectors by $\mathbf{R}$ to denote response. Using $\mathbf{M}$ for the matrix of (9-4-5) and $\mathbf{M}^T$ for the transposed matrix, (9-4-5) and (9-4-6) would be

$$\mathbf{R}_0 = \mathbf{M}\mathbf{E}_1 \tag{9-4-7}$$

$$\mathbf{R}_1 = \mathbf{M}^T\mathbf{E}_0 \tag{9-4-8}$$

Now let us deduce a physical statement from (9-4-7) and (9-4-8). First take the inner product of (9-4-7) with $\mathbf{E}_0^T$

$$\mathbf{E}_0^T\mathbf{R}_0 = \mathbf{E}_0^T\mathbf{M}\mathbf{E}_1$$

The right-hand side, which is a scalar, may be transposed

$$\mathbf{E}_0^T\mathbf{R}_0 = (\mathbf{E}_0^T\mathbf{M}\mathbf{E}_1)^T = \mathbf{E}_1\mathbf{M}^T\mathbf{E}_0$$

substituting from (9-4-8) we have

$$\mathbf{E}_0^T \mathbf{R}_0 \quad = \quad \mathbf{E}_1^T \mathbf{R}_1 \tag{9-4-9}$$

Equation (9-4-9) is the basic statement of reciprocity; the inner product of the excitation vector and the response vector at place 0 equals their inner product at place 1. Notice that the inner products are between vectors which occur in *different experiments.*

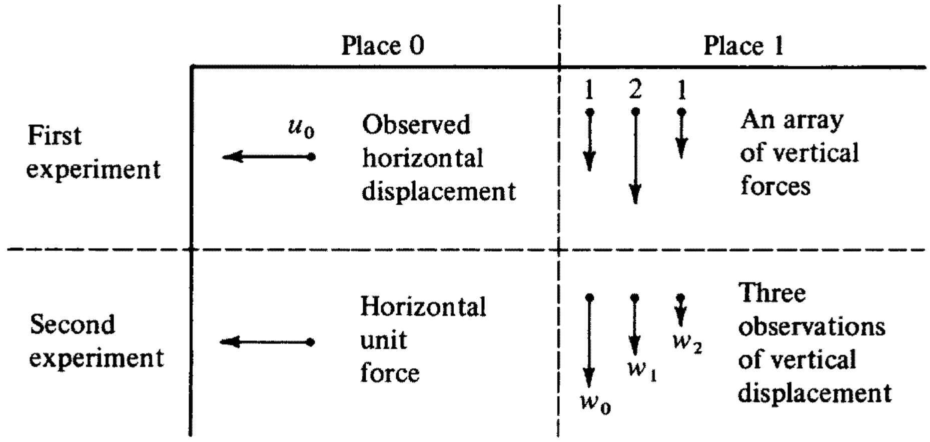

Figure 9.2: A reciprocity example. Reciprocity says that $u_0 = w_0 + 2w_1 + w_2$. $\boxed{\text{c9-9-2}}$ [NR]

An example of an elastic system with vector-directed displacement and force vectors is depicted in Figure 9-2. A laboratory example by J. E. White which combines electromagnetic, solid, liquid, and gaseous media is down in Figure 9-3. A geophone is a spring pendulum couped to an induction coil. The first geophone is mounted on a pipe which rests on the bottom of a glass desiccator. The second geophone is attached to the glass with a chunk of modeling clay, below the water level. The top pair of traces shows the (source) current into the first geophone and the (open circuit) voltage at the second; the bottom traces show the current in the second geophone and the voltage at the first.

EXERCISES:

1 Consider Poisson's equation $\partial_{xx} R = -E$ on five grid points where the boundary conditions are that the end points are zero. A unit excitation at the third grid point gives the solution $(0, \frac{1}{2}, 1, \frac{1}{2}, 0)$. Find the solution with a unit excitation in the second grid point. Observe reciprocity if you do it right.

2 Write an equation like (9-4-3) for the heat-flow equation. How will the introduction of imaginary numbers change the statement of the reciprocal principle?

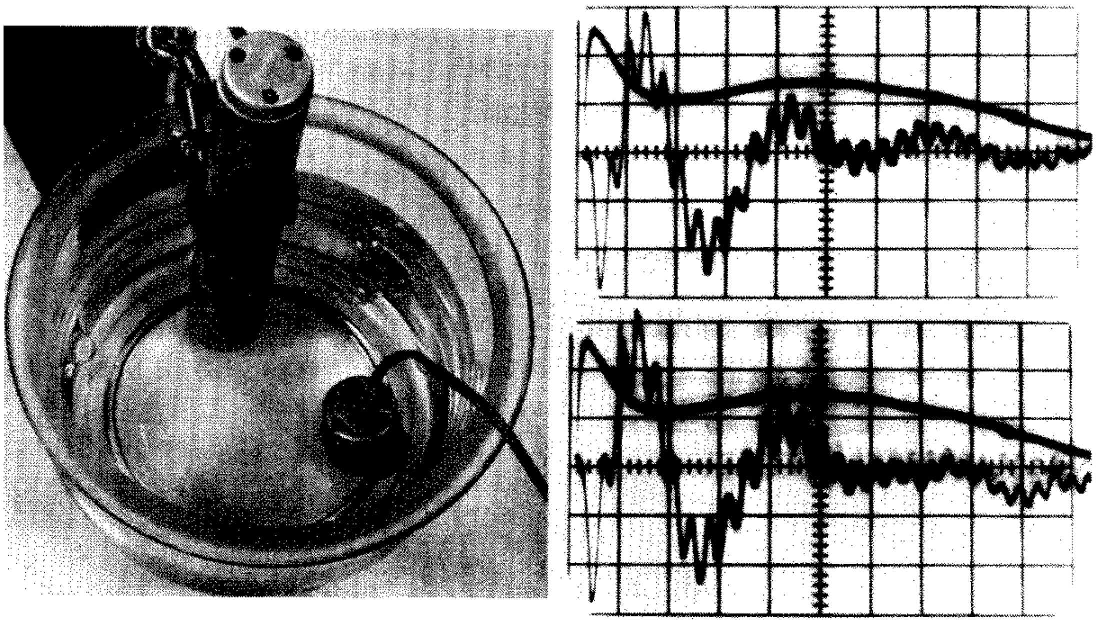

Figure 9.3: .
] An example of the reciprocal principle in a combined electromagnetic, solid, liquid, and gaseous system [J.E. White]. $\boxed{\text{c9-9-3}}$ [NR]

3 Write the three first-order partial differential equations of acoustics in two-dimensional cartesian geometry. Observe the gridding arrangement below.

Write a set of 27×27 equations for the vector $(U_1, P_1, W_1, U_2, P_2, W_2, \cdots, U_9, P_9, W_9)$. Make it come out symmetric and in an obviously orderly form.

4 In Sec 8-3, Exercises 5 and 6 taken together illustrate the reciprocity theorem which states, "If source and receiver are interchanged, the same waveform will be observed." Solve the problem of a surface source with a receiver in the middle of the layers and solve the same problem with interchanged source and receiver to test the reciprocity theorems.

9-5 CONSERVATION PRINCIPLES AND MODE ORTHOGONALITY

We showed earlier how problems in stratified media reduce to a first-order matrix differential equation of the form

$$\frac{\partial}{\partial z}\mathbf{y} \quad = \quad \mathbf{A}\mathbf{y} + \mathbf{s} \qquad (9\text{-}5\text{-}1)$$

It turns out that many problems in the form of (9-5-1) can be reformulated into what we will call the *Atkinson* form. It is

$$\mathbf{J}\frac{\partial}{\partial z}\mathbf{y} \quad = \quad [\mathbf{G}(z) + \lambda\mathbf{H}(z)]\mathbf{y} \tag{9-5-2}$$

where $\mathbf{J}$ is a skew-Hermitian matrix ($\mathbf{J}^* = -\mathbf{J}$) independent of z, $\mathbf{G}(z)$ and $\mathbf{H}(z)$ are Hermitian matrices ($\mathbf{H}^* = \mathbf{H}$), and λ is a scalar which will come to play the role of an eigenvalue. For example, in acoustics we have

$$\frac{\partial}{\partial z}\begin{bmatrix} P \\ W \end{bmatrix} \quad = \quad i\begin{bmatrix} 0 & \omega\rho \\ \left(+\frac{\omega}{K} - \frac{k_x^2}{\omega\rho}\right) & 0 \end{bmatrix}\begin{bmatrix} P \\ W \end{bmatrix} + \begin{bmatrix} s_p \\ s_w \end{bmatrix} \tag{9-5-3}$$

which can be premultiplied by a skew-Hermitian matrix to give

$$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix} P \\ W \end{bmatrix} \quad = \quad \begin{bmatrix} \left(+\frac{\omega}{K} - \frac{k_x^2}{\omega\rho}\right) & 0 \\ 0 & \omega\rho \end{bmatrix}\begin{bmatrix} P \\ W \end{bmatrix} + \begin{bmatrix} s_p \\ s_w \end{bmatrix} \tag{9-5-4}$$

The significant thing about (9-5-4) is that the operators are self-adjoint, meaning that the right-hand matrix is Hermitian and so is the left-hand operator. To understand why $\mathbf{J}(\partial/\partial z)$ is Hermitian, write it out as a difference approximation

$$\begin{aligned} \mathbf{J}\frac{\partial}{\partial z} \quad &= \quad \frac{i}{\triangle z}\begin{bmatrix} 0 & \delta_z \\ \delta_z & 0 \end{bmatrix} \\ &= \quad \frac{1}{\triangle z}\left[\begin{array}{cccc|cccc} & & & & i & -i & & \\ & & & & & i & -i & \\ & & & & & & i & -i \\ \hline i & -i & & & & & & \\ & i & -i & & & & & \\ & & i & -i & & & & \end{array}\right] \end{aligned} \tag{9-5-5}$$

Inspecting (9-5-5) we see that it is two rows short of being square. Choosing two boundary conditions will be like obtaining two more rows. Clearly (9-5-5) is so close to being Hermitian that two more rows can be chosen to make it Hermitian. For example, the two rows

$$\begin{bmatrix} & & & & & & & i \\ -i & & & & & & & \end{bmatrix}$$

could be squeezed between the top and bottom halves of (9-5-5). Since the operator (9-5-5) can be made Hermitian by choice of suitable boundary conditions and since the other operators in (9-5-4) are already Hermitian, it seems that the Atkinson form applies to physical problems in which the reciprocity principle is applicable. Reciprocity does apply to most geophysical prospecting problems. A simple physical situation in which reciprocity does *not* apply is sound waves in a windy atmosphere. Physically it is because waves go more slowly upwind than downwind, and mathematically it is because no $\mathbf{J}$ matrix can be found to convert (9-5-1) into the form (9-5-2). Only in a source-free region can we convert (9-5-1) to (9-5-2). If we choose to let ω play the role of the eigenvalue, then taking source terms to be zero we split (9-5-4) into

$$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \frac{\partial}{\partial z} \begin{bmatrix} P \\ W \end{bmatrix} = +\omega \begin{bmatrix} \left(\frac{1}{K} - \frac{k_x^2}{\omega^2 \rho}\right) & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} P \\ W \end{bmatrix} \tag{9-5-6}$$

Here $\mathbf{G}(z)$ has turned out to vanish and k_x^2/ω^2, which is proportional to the sine of the incident angle, is to be regarded as a constant for variable values of the eigenvalue ω. Alternatively, we could choose $-k_x^2$ to be the eigenvalue, and then (9-5-4) would become

$$\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \frac{\partial}{\partial z} \begin{bmatrix} P \\ W \end{bmatrix} = \begin{bmatrix} \frac{\omega}{K} & 0 \\ 0 & \omega\rho \end{bmatrix} \begin{bmatrix} P \\ W \end{bmatrix} - k^2 \begin{bmatrix} \frac{1}{\omega\rho} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P \\ W \end{bmatrix} \tag{9-5-7}$$

Obviously, still another possibility is to let the angle variable $-k_x^2/\omega^2$ be the eigenvalue for fixed ω.

The Atkinson form (9-5-2) leads directly to various conservation principles. Let us computer the vertical derivative of the quadratic form $\mathbf{y}^*\mathbf{J}\mathbf{y}$.

$$\begin{aligned} \frac{\partial}{\partial z}\mathbf{y}^*\mathbf{J}\mathbf{y} &= \mathbf{y}_z^*\mathbf{J}\mathbf{y} + \mathbf{y}^*\mathbf{J}\mathbf{y}_z \\ &= -\mathbf{y}_z^*\mathbf{J}\mathbf{y}^* + \mathbf{y}^*\mathbf{J}\mathbf{y}_z \\ &= -(\mathbf{J}\mathbf{y}_z)^*\mathbf{y} + \mathbf{y}^*(\mathbf{J}\mathbf{y}_z) \\ &= -(\mathbf{G}\mathbf{y} + \lambda\mathbf{H}\mathbf{y})^*\mathbf{y} + \mathbf{y}^*(\mathbf{G}\mathbf{y} + \lambda\mathbf{H}\mathbf{y}) \\ &= (\lambda - \lambda^*)\mathbf{y}^*\mathbf{H}\mathbf{y} \end{aligned} \tag{9-5-8}$$

Very often we take the eigenvalues ω, $-k_x^2$, or $-k_x^2/\omega^2$ to be real, and in such a case we have $\lambda - \lambda^* = 0$ and (9-5-8) shows the $\mathbf{y}^*\mathbf{J}\mathbf{y}$ is a quadratic function of the wave variables which is invariant with z. In the acoustic example, this quadratic invariant is proportional to the energy flux. Specifically

$$\begin{aligned} \mathbf{y}^*\mathbf{J}\mathbf{y} &= -i[P^* \quad W^*] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P \\ W \end{bmatrix} \\ &= -i(P^*W + W^*P) \quad = \quad -2i\,\mathrm{Re}(P^*W) \end{aligned} \tag{9-5-9}$$

If we wish to consider a complex frequency $\omega = \omega_r + i\omega_i$, then in the first acoustic example (9-5-6) equation (9-5-8) becomes

$$-\frac{\partial}{\partial z}\mathrm{Re}(P^*W) = +\omega_i\Big[\Big(\frac{1}{K} - \frac{k_x^2}{\omega^2\rho}\Big)P^*P + \rho W^*W\Big] \tag{9-5-10}$$

Noting that if P and W have time dependence $\exp[-i(\omega_r + i\omega_i)t] = \exp(-i\omega_r t + \omega_x t)$, then quadratics like P^*P and W^*W have time dependence $e^{2\omega_i t}$ and we see that the multiplier $2\omega_i$ can be regarded as a time derivative. Hence (9-5-10) becomes

$$-\frac{\partial}{\partial z}\mathrm{Re}(P^*W) = +\frac{\partial}{\partial t}\Big\{\frac{1}{2}\Big(\frac{1}{K} - \frac{k_x^2}{\omega^2\rho}\Big)P^*P + \rho W^*W\Big\} = \frac{\partial}{\partial t}E \tag{9-5-11}$$

Equation (9-5-11) is interpreted as saying that the time derivative of the energy density E at a point is proportional to the negative of the divergence of energy flux at that point. In other problems the quadratic forms need not always turn out to involve energy. Sometimes momentum in involved.

A well-known theorem in matrix theory is that Hermitian matrices have real eigenvalues. Why then did we consider the possibility of a complex eigenvalue in (9-5-8)? The answer is that the finite difference operator matrix need not be chosen to have boundary conditions which make the operators Hermitian. In particular, for $\partial E/\partial t$ to be nonzero, energy must leak in or out at a boundary.

Now, let us suppose boundary conditions have been chosen to make $\mathbf{J}\partial/\partial z$ symmetric so the eigenvalues become real. Let $y_n(z)$ be a solution to (9-5-2) with eigenvalue λ_n, and let $y_m(z)$ be another solution with a different eigenvalue λ_m. The reasoning which led up to (9-5-8) can be used to obtain

$$\frac{\partial}{\partial z}(\mathbf{y}_m^* \mathbf{J} \mathbf{y}_n)(\lambda_n - \lambda_m)\mathbf{y}_m^* \mathbf{H} \mathbf{y}_n \tag{9-5-12}$$

Integrating through z from z_a to z_b, we have

$$\mathbf{y}_m^* \mathbf{J} \mathbf{y}_n \Big|_{z_a}^{z_b} \quad = \quad (\lambda_n - \lambda_m) \int_a^b \mathbf{y}_m^*(z, \lambda_m) \mathbf{H}(z) \mathbf{y}(z, \lambda_n)\, dz \tag{9-5-13}$$

If boundary conditions have been chosen so that no energy gets in or out at z_a and z_b, then the left-hand side vanishes. Since by hypothesis $\lambda_n \neq \lambda_m$ we must have the right-hand integral vanishing. This states the orthogonality of the two solutions (called the two *modes*) and the idea is the same as the orthogonality of eigenvectors of the Hermitian difference operator matrices. The orthogonality of these functions is frequently useful in theoretical and computational work.

EXERCISES:

1 Show that application of (9-5-8) to (9-5-7) leads to a definition of horizontal energy flux. You may wish to take $k_x = k_r + ik_i$ and assume $|k_r| \gg |k_i|$.

9-6 ELASTIC WAVES

It is now presumed that the reader has a general knowledge of classical elasticity theory. Few textbooks, if any, develop the special subject of stratified media which is so important in seismology. Many papers on that subject may be found in the *Bulletin of the Seismological Society of America (BSSA)*. For those readers unfamiliar with the *BSSA*, we now present the results of applying the general methods of this chapter to the equations of isotropic elasticity.

The conventions in elasticity are (u, w) displacements in x and z directions, τ is the stress matrix, λ and μ are Lame's constants and ρ is density. Hooke's law and Newton's law with $e^{-i\omega t}$ time dependence leads to

$$\frac{\partial}{\partial z}\begin{bmatrix} U \\ \tau_{zz} \\ W \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\partial_x & \frac{1}{\mu} \\ 0 & 0 & -\rho\omega^2 & \partial_x \\ \frac{-\lambda\partial_x}{(\lambda+2\mu)} & \frac{1}{\lambda+2\mu} & 0 & 0 \\ -\rho\omega^2 - \gamma & \frac{-\partial_x\lambda}{\lambda+2\mu} & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ \tau_{zz} \\ W \\ \tau_{zx} \end{bmatrix} \tag{9-6-1}$$

where

$$\gamma = \partial_x \frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} \partial_x \tag{9-6-2}$$

Define also

$$\begin{aligned} \alpha^2 &= \tfrac{\lambda+2\mu}{\rho} \\ \beta^2 &= \tfrac{\mu}{\rho} \\ m^2 &= \tfrac{-\omega^2}{\alpha^2} - \partial_{xx} \\ n^2 &= \tfrac{-\omega^2}{\beta^2} - \partial_{xx} \\ l^2 &= \tfrac{-\omega^2}{\beta^2} - 2\partial_{xx} \end{aligned} \tag{9-6-3}$$

If material properties do not vary in the x direction, we have the row eigenvector transformation $\mathbf{R}$ to up- and downgoing wave variables.

$$\begin{bmatrix} p^+ \\ s^+ \\ p^- \\ s^- \end{bmatrix} = \frac{\Lambda^{-1}}{2\omega^2\rho} \begin{bmatrix} 2\mu m\,\partial_x & m & \mu l^2 & \partial_x \\ \mu l^2 & -\partial_x & -2\mu n\,\partial_x & n \\ -2\mu m\,\partial_x & -m & \mu l^2 & \partial_x \\ -\mu l^2 & \partial_x & -2\mu n\,\partial_x & n \end{bmatrix} \begin{bmatrix} u \\ \tau_{zz} \\ w \\ \tau_{zx} \end{bmatrix} \tag{9-6-4}$$

and the column eigenvector inverse transform $\mathbf{C}$

$$\begin{bmatrix} u \\ \tau_{zz} \\ w \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} -\partial_x & -n & -\partial_x & -n \\ -\mu l^2 & 2\mu n\,\partial_x & -\mu l^2 & 2\mu n\,\partial_x \\ -m & \partial_x & m & -\partial_x \\ -2\mu m\,\partial_x & -\mu l^2 & 2\mu m\,\partial_x & \mu l^2 \end{bmatrix} \begin{bmatrix} p^+ \\ s^+ \\ p^- \\ s^- \end{bmatrix} \tag{9-6-5}$$

where

$$\Lambda = \begin{bmatrix} m & & & \\ & n & & \\ & & -m & \\ & & & -n \end{bmatrix}$$

The matrices partition nicely into 2×2 blocks. The reader may verify that $\mathbf{CR} = \mathbf{RC} = \mathbf{I}$ and $\mathbf{C\Lambda R} = \mathbf{A}$.