Chapter 8

Layers revealed by scattered wave filtering

Waves occur in almost all branches of physics. We are going to study waves, but here we will not assume knowledge of physics and differential equations. We will use only assumptions about the general principles of delay, continuity, and energy conservation. The results will be directly applicable to sound waves, water waves, light in thin films, normal incident elastic waves of both pressure and shear type, electromagnetic waves, transmission lines, electrical ladder networks, and other such things. The methods can also be applied to diffusion problems. Our first main objective is to solve the problem of calculating wave fields given reflection coefficients. Our second main objective is to gain the ability to calculate the reflection coefficients given the observed waves.

8.1 REFLECTION AND TRANSMISSION COEFFICIENTS

Consider two halfspaces (the sky above, the earth below). If a wave of unit amplitude is incident onto the boundary, there will be a transmitted wave of amplitude t and a reflected wave of amplitude c as depicted in Figure 8.1.

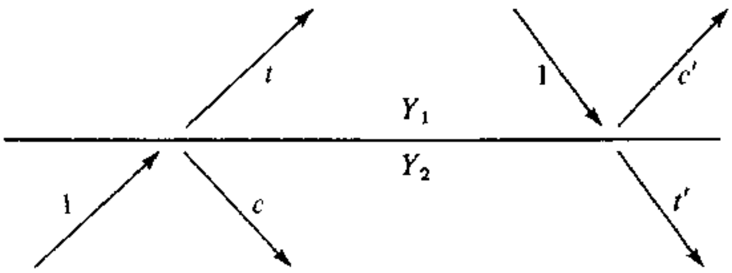

Figure 8.1: Waves incident, reflected c, and transmitted t at an interface. $\boxed{\text{c8-8-1}}$ [NR]

A very simple relationship exists between t and c. The wave amplitudes have a physical meaning of something like pressure, material displacement, or tangential electric or magnetic fields; and these physical variables must be the same on either side of the boundary. Thus, we must have

$$t = 1 + c \tag{8.1}$$

$$t' = 1 + c' \tag{8.2}$$

It may be surprising that t may be greater than unity. However, this phenomenon may easily be seen at the ocean, where waves get larger as they approach the shore (until they break). Energy is not determined by wave height alone. Energy is equal to the squared wave amplitude multiplied by a proportionality factor Y depending upon the medium in which the wave is measured. If we denote the factor of the top medium by Y_1 and the bottom by Y_2, then the statement that the energy before incidence equals the energy after incidence is

$$Y_2 1^2 = Y_2 c^2 + Y_1 t^2 \tag{8.3}$$

solving for c we get

$$\begin{aligned} 0 &= -Y_2 + Y_2 c^2 + Y_1 (1+c)^2 \\ 0 &= (c-1) Y_2 + (c+1) Y_1 \\ 0 &= c(Y_2 + Y_1) + (Y_1 - Y_2) \\ c &= \frac{Y_2 - Y_1}{Y_2 + Y_1} \end{aligned} \tag{8.4}$$

In acoustics the up- and downgoing wave variables may be normalized to either pressure or velocity. When they measure velocity, the scale factor multiplying velocity squared is called the impedance I. When they measure pressure, the scale factor is called the admittance Y.

The wave c' which reflects when energy is incident from the other side is obtained from (8.4) if Y_1 and Y_2 are interchanged. Thus

$$c' = -c \tag{8.5}$$

A perfectly reflecting interface is one which does not allow energy through. This comes about not only when $t = 0$ or $c = -1$, but also when $t = 2$ or $c = +1$. To see this, note that on the left in Figure 8.1

$$\begin{aligned} \frac{\text{Energy transmitted}}{\text{Energy incident}} &= \frac{Y_1 t^2}{Y_2 1^2} = \frac{Y_1}{Y_2}\left(1 + \frac{Y_2 - Y_1}{Y_1 + Y_2}\right)^2 \\ &= \frac{Y_1}{Y_2}\left(\frac{2\,Y_2}{Y_1 + Y_2}\right)^2 = \frac{4\,Y_1\,Y_2}{(Y_1 + Y_2)^2} \end{aligned} \tag{8.6}$$

Equation (8.6) says that 100 percent of the incident energy is transmitted when $Y_1 = Y_2$, but the percentage of transmission is very small when Y_1 and Y_2 are very different.

A word of caution: Occasionally special applications are described by authors who do not define reflection and transmission coefficients in terms of some variable which is continuous at a boundary. This is usually an oversight which unfortunately obscures the relationship of the special application to wave theory in general and this chapter in particular. It is almost never an essential feature of the special application that $t \neq 1+c$ but just a result of an unwise choice of variables in the description. For example, material density is an unwise variable in acoustics because it suffers a discontinuity at a material boundary. Pressure or normal velocity are better descriptors of wave strength.

Ordinarily there are two kinds of variables used to describe waves, and both of these can be continuous at a material discontinuity. One is a scalar like pressure, tension, voltage, potential, stress, or temperature. The other is a vector which we use the vertical component. Examples of the latter are velocity, stretch, electric current, dislacement, and heat flow. Occasionally a wave variable will be a tensor. When a boundary condition is the vanishing of one of the motion components, then the boundary is often said to be rigid. When it is the pressure or potential which vanishes, then the boundary is often said to be free. Rigid and free boundaries reflect waves with unit magnitude reflection coefficients.

The purpose of this chapter is to establish fundamental mathematical properties of waves in layers and to avoid specialization to any particular physical type of waves. That will be done in the next chapter. However, so as not to disguise the physical aspect of the mathematics, a precise definition of upgoing wave U and downgoing wave D will now be given in terms of classical acoustics. In acoustics one deals with pressure P and vertical component of parcel velocity W (not to be confused with wave velocity v). One possible definition for U and D (which will be developed in Chapter 9, Section 3) is

$$D = \frac{P + W/Y}{2} \tag{8.7}$$

$$U = \frac{P - W/Y}{2} \tag{8.8}$$

with the inverse relations

$$P = D + U \tag{8.9}$$

$$W = (D - U)Y \tag{8.10}$$

Other definitions with different scale factors and signs are possible. With this definition, the relation $t = 1 + c$ is readily seen to be associated with (8.9) and continuity of pressure at an interface. The minus signs in (8.8) and (8.10) are associated with the direction of the z axis. Reversal of the z axis changes W to $-W$ and switches the roles of U and D.

We notice that a downgoing wave D all by itself with U vanishing provides a moving disturbance of both pressure P and velocity W, and the vanishing of U

assures us that the ratio between the two $W/P = Y$ is characteristic admittance Y of the material. The energy, we have said, is proportional to either YP^2 or IW^2 from which the ratio $W/P = Y$ allows us to deduce that the impedance of a material is the inverse of its admittance $I = 1/Y$.

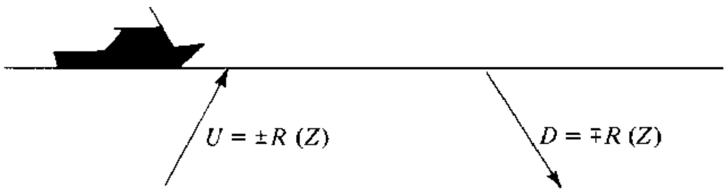

Figure 8.2: A waveform $R(Z)$ reflecting at the surface of the sea. Pressure equal to $U + D$ vanishes at the surface. The vertical velocity of the surface is proportional to $D - U$. Theoretically, waves are observed by measuring W at the surface; however, as a practical matter P is often observed a fraction of a wavelength below the surface. $\boxed{\text{c8-8-2}}$ [NR]

For sound waves in the ocean the sea surface is a nearly perfect reflector because of the great contrast between air and water. If this interface is idealized to a perfect reflector, then it is a free surface. Since the pressure vanishes on a free surface, we have that $D = -U$ at the surface so the reflection coefficient is -1. If a wave is to be seen at the surface, it is necessary to measure not pressure but something proportional to velocity. In geophysical exploration practice, pressure sensing hydrophones are used. They must be kept at a suitable distance below the sea surface. The situation can be depicted as in Figure 8.2. The pressure normally vanishes at the sea surface, but if we wish to initiate an impulsive disturbance, the pressure may momentarily take on some other value, say 1. This is depicted in Figure 8.3. The total vertical component of velocity of the sea surface due to the source and to the resulting acoustic wave is $D - U = 1 + 2R(Z)$.

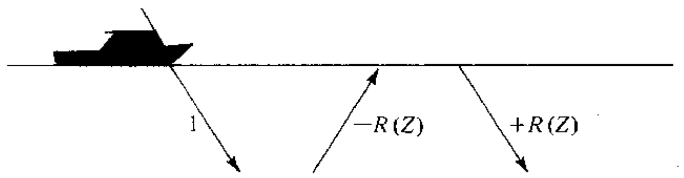

Figure 8.3: An initial downgoing disturbance 1 results in a later upgoing reflected wave $-R(Z)$ which reflects back down as $R(Z)$. The pressure at the surface is $D + U = 1 + R - R = 1$. $\boxed{\text{c8-8-3}}$ [NR]

EXERCISES:

1 Compute t in terms of Y_1 and Y_2.

2 In a certain application continuity is expressed by saying that $D - U$ is the same on either side of the interface. This implies that $t = 1 - c$. Derive an equation like (8.4) for the reflection coefficient in terms of the admittance Y.

3 What are reflection and transmission coefficients in terms of the impedance I? (Clear fractions from your result.)

4 From the principle of energy conservation we showed that $c' = -c$. It may also be deduced from time reversal. To do this, copy Figure 8.1 with arrows reversed. Scale and linearly superpose various figures in an attempt to create a situation where a figure like the right-hand side of Figure 8.1 has $-c'$ for the reflected wave. (HINT: Draw arrows at normal incidence.)

8.2 ENERGY FLUX IN LAYERED MEDIA

First consider wave resonance in a layer. Let the travel time through the layer and back again be given by the delay operator Z. The situation is shown in Figure 8.4. The wave seen above the layer has the form

$$t_2 t_1 [1 + c_1 c_2^{'} Z + (c_1 c_2^{'})^2 Z^2 + \cdots] \quad = \quad \frac{t_2 t_1}{1 - c_1 c_2^{'} Z}$$

It is no accident that the infinite series may be summed. We will soon see that for n layers the waves, which are of infinite duration, may be expressed as simple polynomials of degree n. We will consider many layers and the general problem of determining waves given reflection coefficients and determining reflection coefficients given waves.

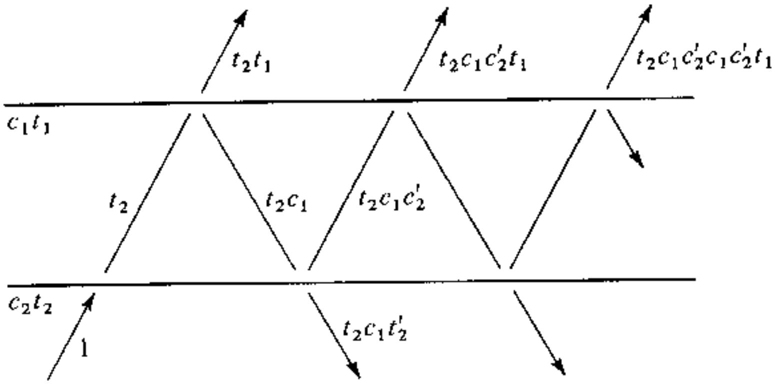

Figure 8.4: Some rays corresponding to resonance in a layer. $\boxed{\text{c8-8-4}}$ [NR]

The reflection and transmission coefficients show one how to calculate the waves resulting from a wave impinging on a layer. Equation (8.11) relates to Figure 8.5 and shows how from the waves U an D' one extrapolates into the future to get U' and D.

$$\begin{aligned} U' &= tU + c'D' \\ D &= cU + t'D' \end{aligned} \tag{8.11}$$

Let us rearrange (8.11) to get U' and D' on the right and U and D on the left. Then we will have an equation which extrapolates from the primed medium to the

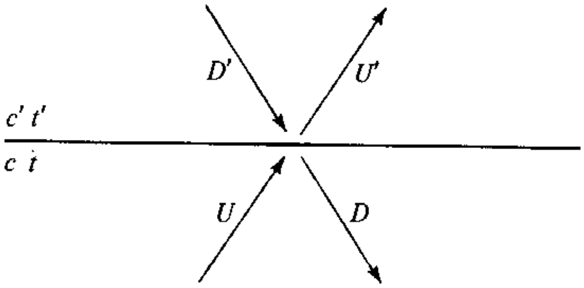

Figure 8.5: Waves incident and reflected from an interface. $\boxed{\text{c8-8-5}}$ [NR]

unprimed medium. We get

$$\begin{aligned} -tU &= -U' + c'D' \\ -cU + D &= t'D' \end{aligned}$$

which may be arranged in the matrix form

$$\begin{bmatrix} -t & 0 \\ -c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} -1 & c' \\ 0 & t' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'$$

Now premultiplying by the inverse of the left-hand matrix

$$\begin{aligned} \begin{bmatrix} U \\ D \end{bmatrix} &= \frac{1}{-t} \begin{bmatrix} 1 & 0 \\ c & -t \end{bmatrix} \begin{bmatrix} -1 & c' \\ 0 & t' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \\ &= \frac{1}{-t} \begin{bmatrix} -1 & c' \\ -c & cc' - tt' \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \end{aligned}$$

finally getting the result, an equation to extrapolate from the primed medium to the unprimed medium.

$$\begin{bmatrix} U \\ D \end{bmatrix} = \frac{1}{t} \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}' \tag{8.12}$$

Now let us consider the Goupillaud type layered medium shown in Figure 8.6. For this arrangement of layers, (8.12) may be written

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{1}{t_k} \begin{bmatrix} 1 & c_k \\ c_k & 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}'_k$$

Let $Z = e^{i\omega T}$ where T, the two-way travel time, equals the data sampling interval. Clearly, multiplication by $\sqrt{Z}$ is equivalent to delaying a function by $T/2$, the travel time across a layer. This gives in the kth layer a relation between primed and unprimed waves.

$$\begin{bmatrix} U \\ D \end{bmatrix}'_k = \begin{bmatrix} 1/\sqrt{Z} & 0 \\ 0 & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \tag{8.13}$$

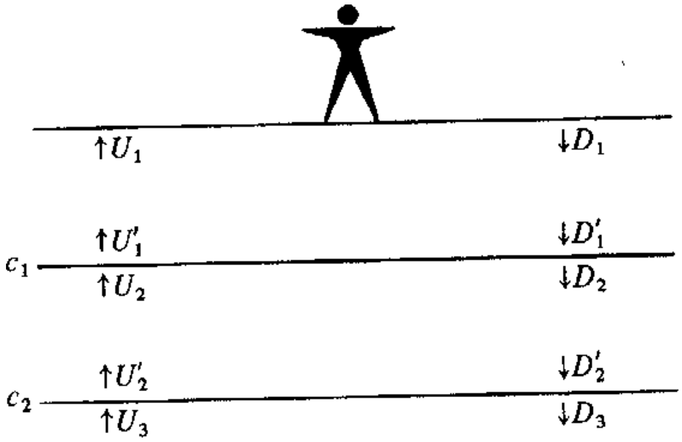

Figure 8.6: Goupillaud-type layered medium (layers have equal travel time). $\boxed{\text{c8-8-6}}$ [NR]

Inserting (8.13) into (8.12) we get a *layer matrix*

$$
\begin{aligned}
\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} &= \frac{1}{t_k} \begin{bmatrix} 1 & c_k \\ c_k & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{Z} & 0 \\ 0 & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\
&= \frac{1}{t_k} \begin{bmatrix} 1/\sqrt{Z} & c_k\sqrt{Z} \\ c_k/\sqrt{Z} & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\
\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} &= \frac{1}{\sqrt{Z} t_k} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k
\end{aligned}
\tag{8.14}
$$

If there is energy flowing through a stack of layers, there must be the same total flow through the kth layer as through the $(k+1)$st layer. Otherwise, there is an energy sink or source at the layer boundary. The net upward flow of energy (energy flux) at any frequency ω in the kth layer is given by

$$
\text{flux}(\omega) \quad = \quad Y_k \left[U(Z)\bar{U}\left(\frac{1}{Z}\right) - D(Z)\bar{D}\left(\frac{1}{Z}\right) \right]_k \tag{8.15}
$$

To establish that this is indeed independent of k, we take the Hermitian conjugate (transpose and conjugate with respect to real ω) of (8.14).

$$
\left[U\left(\frac{1}{Z}\right) D\left(\frac{1}{Z}\right) \right]_{k+1} \quad = \quad \frac{1}{t_k} \left[U\left(\frac{1}{Z}\right) D\left(\frac{1}{Z}\right) \right]_k \begin{bmatrix} \sqrt{Z} & c_k\sqrt{Z} \\ c_k/\sqrt{Z} & 1/\sqrt{Z} \end{bmatrix} \tag{8.16}
$$

Now combine (8.14) with (8.16) in the form

$$
\begin{aligned}
&\left[U\left(\frac{1}{Z}\right) D\left(\frac{1}{Z}\right) \right]_{k+1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} U(Z) \\ D(Z) \end{bmatrix}_{k+1} \quad = \\
&= \frac{1}{{t_k}^2} [\bar{U} \quad \bar{D}]_k \begin{bmatrix} \sqrt{Z} & \sqrt{Z}c_k \\ c_k/\sqrt{Z} & 1/\sqrt{Z} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{Z} & \sqrt{Z}c_k \\ c_k/\sqrt{Z} & \sqrt{Z} \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \\
&= \frac{1}{{t_k}^2} [\bar{U} \quad \bar{D}]_k \begin{bmatrix} 1 - {c_k}^2 & 0 \\ 0 & {c_k}^2 - 1 \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k
\end{aligned}
\tag{8.17}
$$

Since $(1 - {c_k}^2)/{t_k}^2 = t'_k/t_k = Y_k/Y_{k+1}$ this may be rewritten as the desired result, namely

$$Y_{k+1} \left[U\left(\frac{1}{Z}\right) U(Z) - D\left(\frac{1}{Z}\right) D(Z) \right]_{k+1} \quad = \quad Y_k \left[U\left(\frac{1}{Z}\right) U(Z) - D\left(\frac{1}{Z}\right) D(Z) \right]_k \tag{8.18}$$

Equation (8.18) says that at each frequency ω the energy flowing through the kth layer equals the energy flowing through the $(k+1)$st layer.

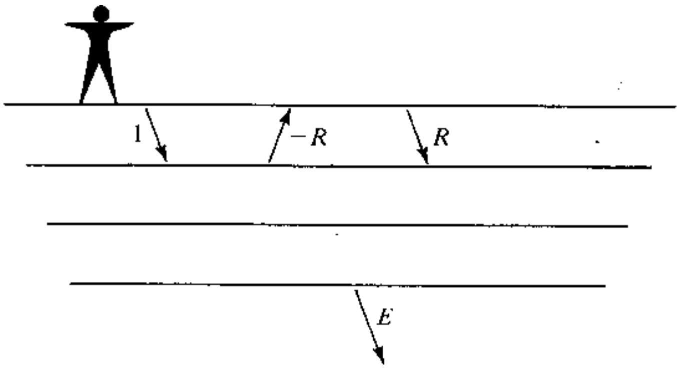

Figure 8.7: Basic reflection seismology geometry. The person initiates an impulse going downward. The earth sends back $-R(Z)$ to the surface. Since the surface is perfectly reflective, the surface sends $R(Z)$ back into the earth. Escaping from the bottom of the layers is a wave $E(Z)$ which is heading toward the other side of the earth. c8-8-7 [NR]

This energy flux theorem leads quickly to some sweeping statements about the waveforms scattered from layered structures. Figure 8.7 shows the basic geometry of reflection seismology. Applying the energy flux theorem to this geometry we may say that the energy flux in the top layer equals that in the lower halfspace so

$$Y_1 \left\{ R\left(\frac{1}{Z}\right) R(Z) - \left[1 + R\left(\frac{1}{Z}\right)\right] [1 + R(Z)] \right\} \quad = \quad -Y_k E\left(\frac{1}{Z}\right) E(Z)$$

or

$$1 + R\left(\frac{1}{Z}\right) + R(Z) \quad = \quad \frac{Y_k}{Y_1} E\left(\frac{1}{Z}\right) E(Z) \tag{8.19}$$

This very remarkable result says that if we were to observe the escaping wave $E(Z)$, we could by autocorrelation construct the waveform seen at the surface. We will later see that $E(Z)$ is minimum-phase so that E could be constructed from R by spectral factorization.

Now let us turn our attention to the earthquake seismology geometry depicted in Figure 8.8. Applying the energy flux theorem to this geometry we obtain

$$Y_1 \left[X\left(\frac{1}{Z}\right) X(Z) - X\left(\frac{1}{Z}\right) X(Z) \right] \quad = \quad Y_k \left[1 - P(Z) P\left(\frac{1}{Z}\right) \right]$$

or

$$1 \quad = \quad P(Z) P\left(\frac{1}{Z}\right) \tag{8.20}$$

The interpretation of the result is that the backscattered waveform $P(Z)$ has the form of an all-pass filter. This result may have been anticipated on physical grounds since all the energy which is incident is ultimately reflected without attenuation; thus the only thing which can happen is that there will be frequency-dependent delay.

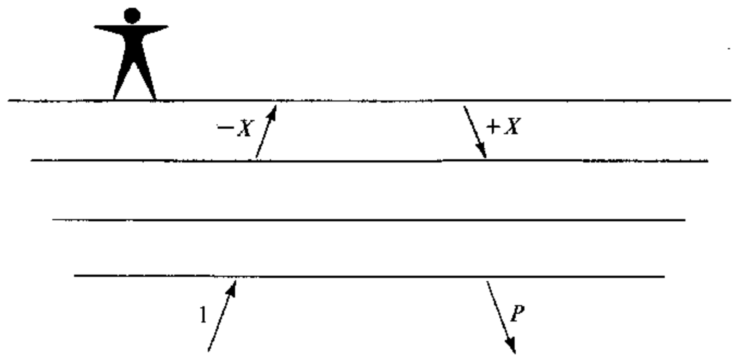

Figure 8.8: Earthquake seismology geometry. An impulse 1 is incident from below. The waveform $X(Z)$ is incident upon the free surface and is reflected back down. The waveform $P(Z)$ scatters back into earth. $\boxed{\text{c8-8-8}}$ [NR]

Finally, we will derive a theorem which relates energy flux to impedance and admittance functions (these functions have Fourier transforms with a positive real part). Suppose that a downgoing wave $D(Z)$ is stronger than an upgoing wave $U(Z)$ at all frequencies, i.e.

$$D\left(\frac{1}{Z}\right) D(Z) - U\left(\frac{1}{Z}\right) U(Z) \quad > \quad 0 \qquad \text{on } |Z| = 1 \tag{8.21}$$

(Note that this does not imply $|d_t| > |u_t|$.) We will abbreviate (8.21) by

$$\bar{D}D - \bar{U}U \quad > \quad 0 \tag{8.22}$$

From (8.21) or (8.22) we will deduce that $(D-U)/(D+U)$ has a Fourier transform with a positive real part. We have

$$\begin{aligned} 2\,\mathrm{Re}\,\frac{D-U}{D+U} &= \frac{D-U}{D+U} + \frac{\bar{D}-\bar{U}}{\bar{D}+\bar{U}} = \frac{(D-U)(\bar{D}+\bar{U}) + (\bar{D}-\bar{U})(D+U)}{(D+U)(\bar{D}+\bar{U})} \\ &= \frac{2(\bar{D}D - \bar{U}U)}{(D+U)(\bar{D}+\bar{U})} \end{aligned} \tag{8.23}$$

The numerator of (8.23) is positive by hypothesis (8.22) and the denominator of (8.23) is positive, since it is the spectrum of the time function $d_t + u_t$ and any spectrum is always positive. Thus $(D-U)/(D+U)$ is called "positive real." The acoustical interpretation of $(D-U)/(D+U)$ is that $(D-U)$ represents the vertical component of material velocity and $(D+U)$ represents the material pressure.

8.3 GETTING THE WAVES FROM THE REFLECTION COEFFICIENTS

A layered material may be specified by giving the reflection coefficient at each interface. Alternate descriptions are to give any one of the scattered waves $R(Z)$, $E(Z)$, $X(Z)$, or $P(Z)$. Our ultimate objective is to get such a good grip on the algebra of this kind of problem that we will be able to start with any descriptor of the layers and from it deduce all the other descriptors.

An important result of the last section was the development of a "layer matrix" (8.14) that is, a matrix which can be used to extrapolate waves observed in one layer to the waves observed in the next layer. This process may be continued indefinitely. To see how to extrapolate from layer 1 to layer 3 substitute (8.14) with $k = 1$ into (8.14) with $k = 2$, obtaining

$$\begin{aligned} \begin{bmatrix} U \\ D \end{bmatrix}_3 &= \frac{1}{\sqrt{Z}t_2}\begin{bmatrix} 1 & Zc_2 \\ c_2 & Z \end{bmatrix} \frac{1}{\sqrt{Z}t_1}\begin{bmatrix} 1 & Zc_1 \\ c_1 & Z \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1 \\ &= \frac{1}{\sqrt{Z}^2 t_1 t_2}\begin{bmatrix} 1+Zc_1c_2 & Zc_1+Z^2c_2 \\ c_2+Zc_1 & Zc_1c_2+Z^2 \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix}_1 \end{aligned} \tag{8.24}$$

Inspection of this example suggests the general form for a product of k layer matrices

$$\frac{1}{\sqrt{Z}^k \prod_{j=1}^k t_j}\begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \tag{8.25}$$

Now let us verify that (8.25) is indeed the general form. We assume (8.25) is correct for $k-1$; then we multiply (8.25) by another layer matrix and see if the product retains the same form with $k-1$ increased to k. The product is

$$\begin{aligned} &\frac{1}{\sqrt{Z}t_k}\begin{bmatrix} 1 & c_kZ \\ c_k & Z \end{bmatrix}\frac{1}{\sqrt{Z}^{k-1}\Pi_{j=1}^{k-1}t_j}\begin{bmatrix} F(Z) & Z^{k-1}G\left(\frac{1}{Z}\right) \\ G(Z) & Z^{k-1}F\left(\frac{1}{Z}\right) \end{bmatrix} = \\ &= \frac{1}{\sqrt{Z}^k \Pi_1^k t_j}\left[\begin{array}{c:c} F(Z)+c_kZG(Z) & Z^{k-1}G\left(\frac{1}{Z}\right)+c_kZ^kF\left(\frac{1}{Z}\right) \\ \hdashline c_kF(Z)+ZG(Z) & c_kZ^{k-1}G\left(\frac{1}{Z}\right)+Z^kF\left(\frac{1}{Z}\right) \end{array}\right] \end{aligned} \tag{8.26}$$

By inspecting the product we see that the scaling factor is of the same form with $k-1$ changed to k. Also the 22 matrix element can be obtained from the 11 element by replacing Z with $1/Z$ and multiplying by Z^k. Likewise, the 21 element is obtained from the 12 element; thus (8.25) does indeed represent a general form. The polynomials $F(Z)$ and $G(Z)$ of order k are built up, starting from $F_1 = 1$ and $G_1 = c_1$, in the following way [from the first column of the right-hand side of (8.26)]:

$$F_k(Z) = F_{k-1}(Z) + c_k Z G_{k-1}(Z) \tag{8.27}$$
$$G_k(Z) = c_k F_{k-1}(Z) + Z G_{k-1}(Z) \tag{8.28}$$

By inspecting (8.27) and (8.28) we can see some of the details of F and G. From (8.27) we see that the lead coefficient f_0 of $F(Z)$ does not change with k. It is always $(f_0)_k = 1$. Knowing this from (8.28) we see that $(g_0)_k = c_k$. Also with knowledge that $F(Z)$ and $G(Z)$ are of the same degree in Z, we see that (8.28) implies that the highest coefficient of $G(Z)$, say $(g_k)_k$ does not change with k and therefore it equals the starting value of c_1. Finally, with this knowledge and (8.27) we deduce that the highest coefficient in $F(Z)$ will always be $c_1 c_k$. Thus, for $k > 1$

$$F_k(Z) = 1 + f_1 Z + f_2 Z^2 + \cdots + c_1 c_k Z^{k-1} \tag{8.29}$$
$$G_k(Z) = c_k + g_1 Z + g_2 Z^2 + \cdots + c_1 Z^{k-1} \tag{8.30}$$

It may be noted in (8.29) and (8.30) and proved from the recurrence relations (8.27) and (8.28) that the coefficients of F contain even powers of c and that G contains odd powers of c. This means that if all c change sign, G will change sign but F is unchanged.

The polynomials $F(Z)$ and $G(Z)$ are not independent and a surprising energy-flux-like relationship exists between them. By substitution from (8.27) and (8.28) one may directly verify that

$$\left[F(Z)F\left(\frac{1}{Z}\right) - G(Z)G\left(\frac{1}{Z}\right)\right]_k = (1-c_k^2)\left[F(Z)F\left(\frac{1}{Z}\right) - G(Z)G\left(\frac{1}{Z}\right)\right]_{k-1} \tag{8.31}$$

Since $F_1(Z) = 1$ and $G_1(Z) = c_1$ we have by iterative application of (8.31) that

$$\left[F(Z)F\left(\frac{1}{Z}\right) - G(Z)G\left(\frac{1}{Z}\right)\right]_k = \prod_1^k (1 - {c_k}^2) = \prod_1^k t't \tag{8.32}$$

Equation (8.32) is a surprising equation because on the left-hand side we have two spectra, the spectrum of f_t and the spectrum of g_t, but the right-hand side is a positive, frequency-independent constant. Since the spectrum of f_t is thus greater than the spectrum of g_t, we may apply the theorem of adding garbage to a minimum-phase wavelet to deduce from (8.27) and from knowledge that $|c_k| < 1$ that $F_k(Z)$ is minimum-phase if $F_{k-1}(Z)$ is minimum-phase. Since $F_1(Z) = 1$ is minimum-phase, we see that all $F_k(Z)$ are minimum-phase. Since $F(Z)$ is minimum-phase, then $F(Z)$ may be calculated from its spectrum $F(Z)F(1/Z)$ or the spectrum of g_t (along with the single number $\Pi t't$). However, we cannot get G from F. Before continuing our algebraic discussion we take up an example.

Let a stack of layers be sandwiched in between two halfspaces Figure 8.9. An impulse is incident from below. The backscattered wave is called $C(Z)$ and the transmitted wave is called $T(Z)$.

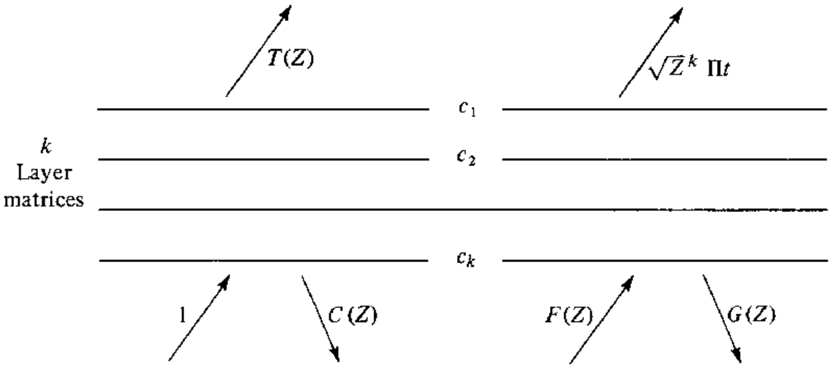

Figure 8.9: Waves incident, reflected, and transmitted from a stack of layers between two half-spaces. $\boxed{\text{c8-8-9}}$ [NR]

Mathematically, we describe the situation with the equations

$$\begin{bmatrix} 1 \\ C(Z) \end{bmatrix} \quad = \quad \frac{1}{\sqrt{Z}^k \Pi t} \begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \begin{bmatrix} T(Z) \\ 0 \end{bmatrix} \tag{8.33}$$

We may solve the first of (8.33) for the transmitted wave $T(Z)$

$$T(Z) \quad = \quad \frac{\sqrt{Z}^k \Pi t}{F(Z)} \tag{8.34}$$

and introduce the result back into the second of (8.33) to obtain the backscattered wave

$$C(Z) \quad = \quad \frac{G(Z)T(Z)}{\sqrt{Z}^k \Pi t} \quad = \quad \frac{G(Z)}{F(Z)} \tag{8.35}$$

The mathematical fact that $F(Z)$ is minimum-phase corresponds to the physical fact that the $C(Z)$ and $T(Z)$ have finite energy; therefore the denominators of (8.34) and (8.35) cannot have zeros inside the unit circle. Since we know that the backscattered wave $C(Z)$ contains less energy than the incident wave by reference to (8.23) we know that a positive real function is given by

$$\frac{1 - C(Z)}{1 + C(Z)} \quad = \quad \frac{1 - G(Z)/F(Z)}{1 + G(Z)/F(Z)} \tag{8.36}$$

Now let us see how to reconstruct the reflection coefficients c_j from the observed scattered wave $C(Z)$. Referring to Figure 8.9 we have

$$\begin{bmatrix} 1 \\ C(Z) \end{bmatrix} \quad = \quad \frac{1}{t_k \sqrt{Z}} \begin{bmatrix} 1 & c_k Z \\ c_k & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \tag{8.37}$$

The first coefficient of $C(Z)$ is c_k [this is physically obvious but may also be seen from (8.30)]. Thus the layer matrix in (8.37) is known. Multiplying (8.37) through by the inverse of the layer matrix we will have obtained $U_{k-1}(Z)$ and $D_{k-1}(Z)$. The next reflection coefficient c_{k-1} is obviously d_0/u_0. Thus we may proceed until all the c_k are determined.

Next let us reconsider the reflection seismology geometry. We have

$$\begin{bmatrix} 0 \\ E(Z) \end{bmatrix} = \frac{1}{\sqrt{Z}^k \Pi t} \begin{bmatrix} F(Z) & Z^k G\left(\frac{1}{Z}\right) \\ G(Z) & Z^k F\left(\frac{1}{Z}\right) \end{bmatrix} \begin{bmatrix} -R \\ 1+R \end{bmatrix} \tag{8.38}$$

From the first equation we may solve for $R(Z)$

$$R(Z) = +\frac{Z^k G(1/Z)}{F(Z) - Z^k G(1/Z)} \tag{8.39}$$

The denominator occurs so often that we give it the name $A(Z)$

$$A_k(Z) = F_k(Z) - Z^k G_k\left(\frac{1}{Z}\right) \tag{8.40}$$

$A(Z)$, like $F(Z)$, is minimum-phase. The second of (8.38) gives the escaping wave as

$$\begin{aligned} E(Z) &= \frac{Z^k F(1/Z) + [-G(Z) + Z^k F(1/Z)]R}{\sqrt{Z}^k \Pi t} \\ &= \frac{Z^k F(1/Z)[F(Z) - Z^k G(1/Z)] + [-G(Z) + Z^k F(1/Z)] Z^k G(1/Z)}{A(Z)\sqrt{Z}^k \Pi t} \end{aligned}$$

simplifying with (8.32) we get

$$E(Z) = \frac{\sqrt{Z}^k \Pi t'}{A(Z)} \tag{8.41}$$

Let us confirm[1] that the definition (8.40) of $A_k(Z)$ is consistent with our chapter 3 definition of the Levinson recursion (3.18). First, rewrite (8.40) in various ways

$$A_k(Z) = F_k(Z) - Z^k G_k\left(\frac{1}{Z}\right) \tag{8.42}$$

$$A_{k-1}(Z) = F_{k-1}(Z) - Z^{k-1} G_{k-1}\left(\frac{1}{Z}\right) \tag{8.43}$$

$$A_{k-1}\left(\frac{1}{Z}\right) = F_{k-1}\left(\frac{1}{Z}\right) - Z^{1-k} G_{k-1}(Z) \tag{8.44}$$

$$Z^k A_{k-1}\left(\frac{1}{Z}\right) = Z^k F_{k-1}\left(\frac{1}{Z}\right) - Z G_{k-1}(Z) \tag{8.45}$$

[1]This confirmation is accidentally missing from the 1976 version of this book.

Now insert the recurrence relations (8.27) and (8.28) into the definition (8.42) getting

$$A_k(Z) = F_{k-1}(Z) + c_k Z G_{k-1}(Z) - Z^k \left[c_k F_{k-1}\left(\frac{1}{Z}\right) + Z^{-1} G_{k-1}\left(\frac{1}{Z}\right)\right] \quad (8.46)$$

$$A_k(Z) = \left[F_{k-1}(Z) - Z^{k-1} G_{k-1}\left(\frac{1}{Z}\right)\right] - c_k \left[Z^k F_{k-1}\left(\frac{1}{Z}\right) - Z G_{k-1}(Z)\right] \quad (8.47)$$

$$A_k(Z) = A_{k-1}(Z) - c_k Z^k A_{k-1}\left(\frac{1}{Z}\right) \quad (8.48)$$

which is the Levinson recursion as asserted. The recursion begins from $A_0 = 1$.

The positive real function is

$$\frac{D-U}{D+U} = \frac{1+R-(-R)}{1+R-R} = 1 + 2R(Z) \quad (8.49)$$

$$= \frac{\text{Vertical velocity} = 1 + 2R.}{\text{Pressure} = 1}$$

As mentioned earlier, if the equations are interpreted in terms of acoustics, then $Y(D-U)/(D+U)$ is interpreted as vertical velocity divided by pressure. It is called the *admittance* which is the inverse of the impedance.

We have now completed the task of solving for the waves given the reflection coefficients. In the subsequent section we attach the inverse problems of getting the reflection coefficients from knowledge of various waves.

EXERCISES:

1 In Figure 8.9 let $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, and $c_3 = \frac{1}{3}$. What are the polynomial ratios $T(Z)$ and $C(Z)$?

2 For a simple interface, we had the simple relations $t = 1 + c$, $t' = 1 + c'$, and $c = -c'$. What sort of analogous relations can you find for the generalized interface of Figure 8.9? [For example, show $1 - T(Z)T'(1/Z) = C(Z)C(1/Z)$ which is analogous to $1 - tt' = c^2$.]

3 Show that $T(Z)$ and $T'(Z)$ are the same waveforms within a scale factor. Deduce that many different stacks of layers may have the same $T(Z)$.

4 Let an impulse be incident on a stack of layers and let a wave $C(Z)$ be reflected. What is the reflection coefficient at the first layer encountered? What would be the reflected wave as a function of C for a situation which differs from the above by the removal of the first reflector?

5 Consider the earth to be modeled by layers over a halfspace. Let an impulse be incident from below (Figure 8.10). Given $F(Z)$ and $G(Z)$, elements of the product of the layer matrices, solve for X and for P. Check your answer by showing that $P(Z)\bar{P}(1/Z) = 1$. How is X related to E? This relation illustrates the principle of reciprocity which says source and receiver may be interchanged.

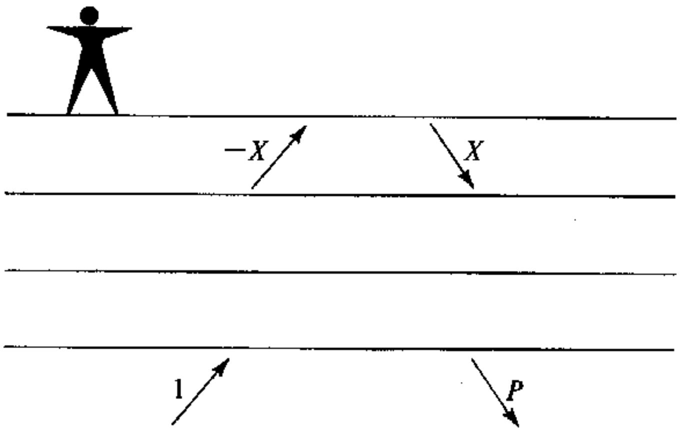

Figure 8.10: Earthquake geometry. c8-E8-3-5 [NR]

6 Show that $1+R(1/Z)+R(Z) = (\text{scale factor})X(Z)X(1/Z)$, which shows that one may autocorrelate the transmission seismogram to get the reflection seismogram.

7 Refer to Figure 8.11. Calculate R' from R.

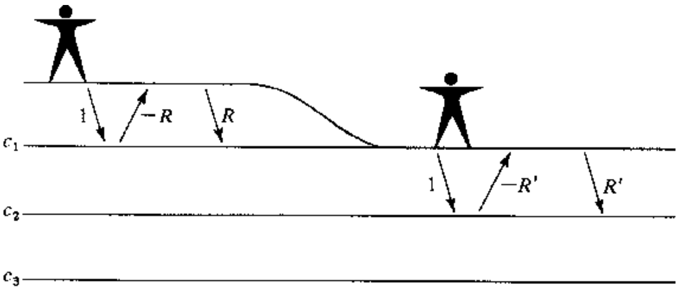

Figure 8.11: Stripping off the surface layer. c8-E8-3-7 [NR]

8.4 GETTING THE REFLECTION COEFFICIENTS FROM THE WAVES

The best starting point for inverse problems is the Kunetz equation (8.19).

$$1 + R\left(\frac{1}{Z}\right) + R(Z) \quad = \quad \frac{Y_k}{Y_1} E\left(\frac{1}{Z}\right) E(Z) \tag{8.50}$$

We need also the expression for the escaping wave (8.41)

$$E(Z) \quad = \quad \frac{\sqrt{Z}^k \Pi t'}{A(Z)} \tag{8.51}$$

We also need to recall that $Y_k/Y_1 = \Pi t/t'$. With this (8.50) becomes

$$1 + R(Z) + R\left(\frac{1}{Z}\right) \quad = \quad \frac{\Pi t' t}{A(Z)A(1/Z)} \tag{8.52}$$

Multiplying through by $A(Z)$ we get

$$\left[1 + R(Z) + R\left(\frac{1}{Z}\right)\right] A(Z) \quad = \quad \frac{\Pi t' t}{A(1/Z)} \tag{8.53}$$

Since $A(Z)$ is minimum-phase, $A(Z)$ may be written as $1/B(Z)$ or $A(1/Z) = 1/B(1/Z)$. Thus (8.53) becomes

$$[1 + R(Z) + R(1/Z)]A(Z) \quad = \quad (\Pi t' t)\left(1 + \frac{b_1}{Z} + \frac{b_2}{Z^2} + \cdots\right) \tag{8.54}$$

Identifying coefficients of zero and positive powers of Z as simultaneous equations, we get a set of equations which for a three-layer model looks like ($r_0 = 1$).

$$\begin{bmatrix} r_0 & r_1 & r_2 & r_3 \\ r_1 & r_0 & r_1 & r_2 \\ r_2 & r_1 & r_0 & r_1 \\ r_3 & r_2 & r_1 & r_0 \\ r_4 & r_3 & r_2 & r_1 \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ -c_3 \end{bmatrix} = \begin{bmatrix} \Pi t' t \\ 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \end{bmatrix} \tag{8.55}$$

In (8.55) we see our old friend the Toeplitz matrix. It used to work for factoring spectra and predicting time series. Notice that $-c_3$ has been inserted in (8.55) as the highest coefficient of $A(Z)$. This is justified by reference back to the definition of $A(Z)$ in terms of $F(Z)$ and $G(Z)$ which were in turn defined from the c_k. It is by reexamining the Toeplitz simultaneous equations (8.55) and the Levinson method of solution (3.18) that we will learn how to compute the reflection coefficients from the waves.

The first four equations in (8.55) would normally be thought of as follows: Given the first three reflected pulses r_1, r_2, and r_3 we may solve the equations for A, incidentally getting the reflection coefficient c_3. Knowing A, the 5th equation in (8.55) may be used to compute r_4. If the model were truly a three-layer model, it would come out right; if not, the discrepancy would be indicative of another reflector c_4 which could be found by expanding equation (8.55) from 4th order to 5th order. In summary, given the reflected pulses r_k, the Levinson recursion successively turns out the reflection coefficients c_k.

Now suppose we begin by observation of the escaping wave $E(Z)$. One way to determine the reflection coefficients would be to form $1 + R(Z) + R(1/Z)$ by the autocorrelation of $E(Z)$; then, the Levinson recursion could be used to solve for the reflection coefficients. The only disadvantage of this method is that $E(Z)$ contains an infinite number of coefficients so that in practice some truncation must be done. The truncation is avoided by an alternative method. Given $E(Z)$ polynomial division will find $A(Z)$. The heart of the Levinson recursion is the building up of $A(Z)$ by $A_k(Z) = A_{k-1}(Z) - c_k Z^k A_{k-1}(1/Z)$. In particular, from (3.20) we have

$$\begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_3 = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{bmatrix}_2 - c_3 \begin{bmatrix} 0 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_2 \tag{8.56}$$

which shows how to get $A_3(Z)$ from $A_2(Z)$ and c_3. To do it backwards, we see first that c_3 is $-a_3$. Then write (8.56) upside-down

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 0 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_2 - c_3 \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{bmatrix}_2 \tag{8.57}$$

Next multiply (8.56) by $1/(1 - {c_3}^2)$ and add the product to (8.57) multiplied by $c_3/(1 - {c_3}^2)$. Notice that the upside-down vectors on the right-hand side cancel, leaving

$$\frac{1}{1 - {c_3}^2} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_3 + \frac{c_3}{1 - {c_3}^2} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ 1 \end{bmatrix}_3 = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{bmatrix}_2 \tag{8.58}$$

Equation (8.58) is the desired result which shows how to reduce $A_{k+1}(Z)$ to $A_k(Z)$ while learning c_{k+1}. A program to continue this process is

```
      COMPLEX A,C,AL,BE,TOP,CONJG
      C(1)=-1.; R(1)=1.; A(1)=1.; V(1)=1.
300   DO 310 I=1,N
310   C(I)=A(I)
      DO 330 K=1,N
      J=N-K+2
      AL=1./(1.-C(J)*CONJG(C(J)))
      BE=C(J)*AL
      JH=(J+1)/2
      DO 320 I=1,JH
      TOP=AL*C(I)-BE*CONJG(C(J-I+1))
      C(J-I+1)=AL*C(J-I+1)-BE*CONJG(C(I))
320   C(I)=TOP
330   C(J)=-BE/AL
```

A program to compute reflection coefficients c_k from the prediction-error filter $A(Z)$. The complex arithmetic is optional.

An inverse program to get R and A from c is

```
      COMPLEX C,R,A,BOT,CONJG
      C(1)=-1.; R(1)=1.; A(1)=1.; V(1)=1.
100   DO 120 J=2,N
      A(J)=0.
      R(J)=C(J)*V(J-1)
      DO 110 I=2,J
110   R(J)=R(J)-A(I)*R(J-I+1)
      JH=(J+1)/2
      DO 120 I=1,JH
      BOT=A(J-I+1)-C(J)*CONJG(A(I))
      A(I)=A(I)-C(J)*CONJG(A(J-I+1))
120   A(J-I+1)=BOT
```

This program computes both R and A from c.

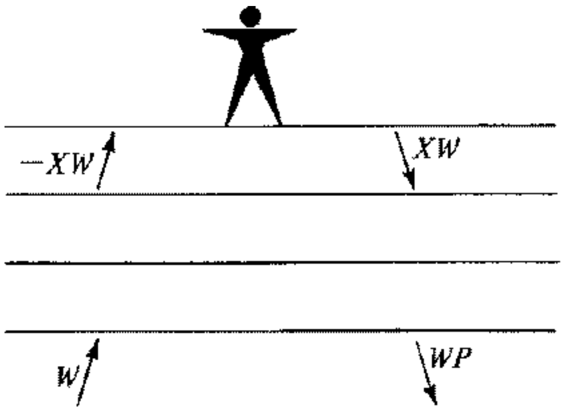

Figure 8.12: Earthquake seismogram geometry with white light incident from below. In the top layer, the sum of the waves vanishes representing zero pressure at the free surface. The difference of up- and down- is the observed vertical component of velocity. $\boxed{\text{c8-8-12}}$ [NR]

Finally, let us see how to do a problem where there are random sources. Figure 8.12 shows the "earthquake geometry." However, in order to introduce a statistical element, the pulse incident from below has been convolved with a white light series w_t of random numbers. Consequently, all the waves internal to Figure 8.12 are given by the convolution of w_t with the corresponding wave in the impulse incident model. Now suppose we are given the top-layer waves $D = 1 - U = XW$ and wish to consider downward continuation. We have the layer matrix

$$\begin{bmatrix} U \\ D \end{bmatrix}_{k+1} = \frac{1}{\sqrt{Z}t_k} \begin{bmatrix} 1 & cZ \\ c & Z \end{bmatrix} \begin{bmatrix} U \\ D \end{bmatrix}_k \tag{8.59}$$

which can be re-written as

$$\begin{bmatrix} -U \\ D \end{bmatrix}_{k+1} = \frac{1}{\sqrt{Z}t_k} \begin{bmatrix} 1 & -cZ \\ -c & Z \end{bmatrix} \begin{bmatrix} -U \\ D \end{bmatrix}_k \tag{8.60}$$

The Burg prediction-error scheme can be written in the form

$$\begin{bmatrix} e^+ \\ e^- \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & -c_k \\ -c_k & 1 \end{bmatrix} \begin{bmatrix} e^+ \\ e^- \end{bmatrix}_k \tag{8.61}$$

which makes it equivalent within a scale factor to downward continuing surface waveforms. The remaining question is whether Burg's estimate of the reflection coefficient, namely,

$$\hat{c}_k \quad = \quad \frac{2\sum_t e_k^+ e_k^-}{\sum_t e_k^+ e_k^+ + e_k^- e_k^-} \tag{8.62}$$

turns out to estimate the reflection coefficient c_k in the physical model. To see how Burg's $\hat{c}_k$ is related to the c_k arising in the Levinson recursion, we define $\mathbf{f}^+$ and $\mathbf{f}^-$ for $k = 2$ as

$$\begin{bmatrix} \\ \\ \mathbf{f}^+ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \\ & x_2 & x_1 \\ & & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \\ \\ \mathbf{f}^- \\ \\ \\ \end{bmatrix} = \begin{bmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \\ & x_2 & x_1 \\ & & x_2 \end{bmatrix} \begin{bmatrix} 0 \\ a_1 \\ 1 \end{bmatrix} \tag{8.63}$$

Next from the dot product

$$\begin{aligned} (\mathbf{f}^-)^T\mathbf{f}^+ &= [0 \quad a_1 \quad 1] \begin{bmatrix} x_0 & x_1 & x_2 & & \\ & x_0 & x_1 & x_2 & \\ & & x_0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_0 & & \\ x_1 & x_0 & \\ x_2 & x_1 & x_0 \\ & x_2 & x_1 \\ & & x_2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ 0 \end{bmatrix} \\ &= [0 \quad a_1 \quad 1] \begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ 0 \end{bmatrix} \end{aligned} \tag{8.64}$$

Now utilize the fact that $(1, a_1)$ satisfies the 2×2 system. Following the Levinson recursion (8.64) can be written as

$$(\mathbf{f}^-)^T\mathbf{f}^+ \quad = \quad [0 \quad a_1 \quad 1] \begin{bmatrix} v \\ 0 \\ e \end{bmatrix} \quad = \quad e \tag{8.65}$$

Likewise we can deduce that $\mathbf{f}^+ \cdot \mathbf{f}^+ = \mathbf{f}^- \cdot \mathbf{f}^- = v$. Thus, the Levinson calculation of the reflection coefficient can be written as

$$c \quad = \quad \frac{2(\mathbf{f}^+ \cdot \mathbf{f}^-)}{(\mathbf{f}^+ \cdot \mathbf{f}^+) + (\mathbf{f}^- \cdot \mathbf{f}^-)} \quad = \quad \frac{2e}{2v} \tag{8.66}$$

The Burg treatment differs from the Levinson treatment in that Burg omits end-effect terms on (8.63). Instead of (8.63) he has

$$\begin{bmatrix} e^+ \end{bmatrix} = \begin{bmatrix} x_1 & x_0 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e^- \end{bmatrix} = \begin{bmatrix} x_1 & x_0 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} a_1 \\ 1 \end{bmatrix} \tag{8.67}$$

For a sufficiently long data sequence the Burg method and the Levinson technique thus become indistinguishable. For a data sample of finite duration we must make a choice. The Levinson technique with (8.63) is equivalent to assuming the data sample vanishes off the ends of the interval in which it is observed. In most applications this is untrue, and so the Burg technique is usually preferable.

EXERCISES:

1 An impulse and the first part of a reflection seismogram, that is, $1 + 2R(Z)$ is $1 + 2(Z/4 + Z^2/16 + Z^3/4 + \cdots)$. What are the first three reflection coefficients? Assuming there are no more reflectors what is the next point in the reflection seismogram?

2 A seismogram $X(Z) = 1/(1 - .1Z + .9Z^2)$ is observed at the surface of some layers over a halfspace. Sketch the time function and indicate its resonance frequency and decay time. Find the reflection coefficients if $X(Z)$ is due to an impulsive source of unknown magnitude in the halfspace below the layers.

3 A source $b_0 + b_1 Z$ deep in the halfspace produces a seismogram $B(Z)X(Z) = 1 - Z + Z^2/2 - Z^3/2 + Z^4/4 - Z^5/4 + Z^6/8 - Z^7/8 + \cdots$. What are the layered structure and the source time function?