Chapter 2

One-sided functions

All physical systems share the property that they do not respond before they are excited. Thus the impulse response of any physical system is a one-sided time function (it vanishes before $t = 0$). In system theory such a filter function is called realizable. In wave propagation this property is associated with causality in that no wave may begin to arrive before it is transmitted. The lag-time point $t = 0$ plays a peculiar and an important role. For this reason, many subtle matters will be much more clearly understood with sampled time than with continuous time. When a filter responds at and after lag time $t = 0$, we will say the filter is realizable or causal. The word *causal* is appropriate in physics where stress may cause (practically) instantaneous strain and vise versa, but one should revert to the more precise words *realizable* or *one-sided* when using filter theory to describe economic or social systems where simultaneity is quite different from cause and effect.

2.1 INVERSE FILTERS

To understand causal filters better, we now take up the task of undoing what a causal filter has done. Consider the output y_t of a filter b_t is known but the input x_t is unknown. See Figure 2.1.

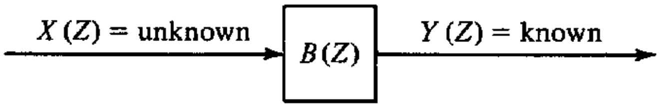
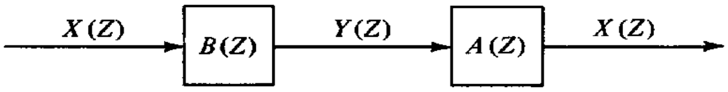


Figure 2.1: Sometimes the input to a filter is unknown. $\boxed{\text{c2-2-1}}$ [NR]

This is the problem that one always has with a transducer/recorder system. For example, the output of a seismometer is a wiggly line on a piece of paper from which the seismologist may wish to determine the displacement, velocity, or acceleration of the ground. To undo the filtering operation of the filter $B(Z)$, we will try to find another filter $A(Z)$ as indicated in Figure 2.2.

Figure 2.2: The filter $A(Z)$ is inverse to the filter $B(Z)$. $\boxed{\text{c2-2-2}}$ [NR]



To solve for the coefficients of the filter $A(Z)$, we merely identify coefficients of powers of Z in $B(Z)A(Z) = 1$. For $B(Z)$, a three-term filter, this is

$$(a_0 + a_1 Z + a_2 Z^2 + a_3 Z^3 + \cdots)\,(b_0 + b_1 Z + b_2 Z^2) \quad = \quad 1 \tag{2.1}$$

The coefficients of $Z^0, Z^1, Z^2, \cdots$ in (2.1) are

$$\begin{aligned}
a_0 b_0 &= 1 && (2.2)\\
a_1 b_0 + a_0 b_1 &= 0 && (2.3)\\
a_2 b_0 + a_1 b_1 + a_0 b_2 &= 0 && (2.4)\\
a_3 b_0 + a_2 b_1 + a_1 b_2 &= 0 && (2.5)\\
a_4 b_0 + a_3 b_1 + a_2 b_2 &= 0 && (2.6)\\
\cdots\cdots\cdots\cdots\cdots\cdots & && \\
a_k b_0 + a_{k-1} b_1 + a_{k-2} b_2 &= 0 && (2.7)
\end{aligned}$$

From (2.2) one may get a_0 from b_0. From (2.3) one may get a_1 from a_0 and the b_k. From (2.4) one may get a_2 from a_1, a_0, and the b_k. Likewise, in the general case a_k may be found from a_{k-1}, a_{k-2}, and the b_k. Specifically, from (2.7) the a_k may be determined recursively by

$$a_k \quad = \quad \frac{-\sum_{i=1}^{2} a_{k-i} b_i}{b_0} \tag{2.8}$$

Consider the example where $B(Z) = 1 - Z/2$; then, by equations like (2.2) to (2.7), by the binomial theorem, by polynomial division, or by Taylor's power series formula we obtain

$$A(Z) \quad = \quad \frac{1}{1 - Z/2} \quad = \quad 1 + \frac{Z}{2} + \frac{Z^2}{4} + \frac{Z^3}{8} + \cdots \tag{2.9}$$

We see that there are an infinite number of filter coefficients but that they drop off rapidly in size so that approximation in a computer presents no problem. The situation is not so rosy with the filter $B(Z) = 1 - 2Z$. Here we obtain

$$A(Z) \quad = \quad \frac{1}{1 - 2Z} \quad = \quad 1 + 2Z + 4Z^2 + 8Z^3 + 16Z^4 + 32Z^5 + \cdots \tag{2.10}$$

The coefficients of the series increase without bound. The outputs of the filter $A(Z)$ depend infinitely strongly on inputs of the infinitely distant past. [Recall that the present output of $A(Z)$ is a_0 times the present input x_1 plus a_1 times the previous input x_{t-1}, etc., so a_n represents memory on n time units earlier.] The implication of this is that some filters $B(Z)$ will not have useful finite approximate inverses $A(Z)$ determined from (2.2) to (2.8). We now seek ways to identify the good filters from the bad ones. With a two-pulse filter, the criterion is merely that the first pulse in $B(Z)$ be larger than the second. A more mathematical description of the state of affairs results from solving for the roots of $B(Z)$, that is, find the values of Z_0 for which $B(Z_0) = 0$. For example $1 - Z/2$ we find $Z_0 = 2$. For the example $1 - 2Z$, we find $Z_0 = \frac{1}{2}$. The general case for wavelets with complex coefficients is that, if the solution value Z_0 of $B(Z_0) = 0$ lies inside the unit circle in the complex plane, then $1/B(Z)$ will have coefficients which blow up; and if the root lies outside the unit circle, then the inverse $1/B(Z)$ will be bounded.

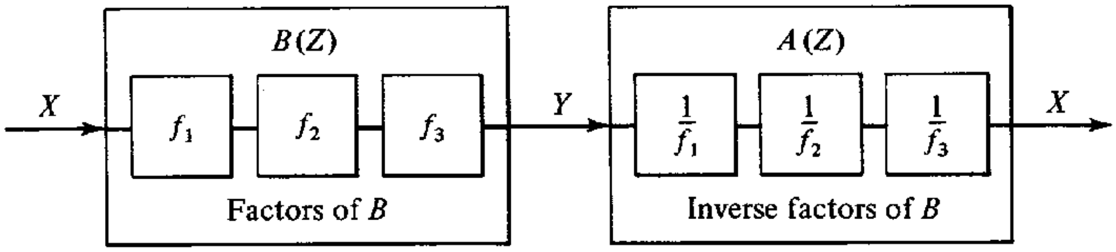


Figure 2.3: Factoring the polynomial $B(Z)$ breaks the filter into many two-term filters. Each one should have a bounded inverse. c2-2-3 [NR]

Recalling earlier discussion that a polynomial $B(Z)$ of degree N may be factored into N subsystems and that the ordering of subsystems is unimportant (see Figure 2.3), we suspect that if any of the N roots of $B(Z)$ lies inside the unit circle we may have difficulty with $A(Z)$. Actual proof of this suspicion relies on a theorem from complex-variable theory about absolutely convergent series. The theorem is that the product of absolutely convergent series is convergent, and conversely the product of any convergent series with a divergent series is divergent. Another proof may be based upon the fact that a power series for $1/B(Z)$ converges in a circle about the origin with a radius from the origin out to he first pole [the zero of $B(Z)$ of smallest magnitude]. Convergence of $A(Z)$ on the unit circle means, in terms of filters, that the coefficients of $A(Z)$ are decreasing. Thus, if all the zeros of $B(Z)$ are outside the unit circle, we will get a convergent filter from (2.8).

Can anything at all be done if there is one root or more inside the circle? An answer is suggested by the example

$$\frac{1}{1-2Z} \quad = \quad \frac{1}{2Z}\frac{1}{1-1/2Z} \quad = \quad -\frac{1}{2Z}\left[1+\frac{1}{2Z}+\frac{1}{(2Z)^2}+\cdots\right] \tag{2.11}$$

Equation (2.11) is a series expansion in $1/Z$, that is, a Taylor series about infinity. It converges from $Z = \infty$ all the way in to a circle of radius 1/2. This means that the inverse converges on the unit circle where it must, if the coefficients are to be bounded. In terms of filters it means that the inverse filter must be one of those filters which responds to future inputs and hence is not physically realizable but may be used in computer simulation.

In the general case, then, one must factor $B(Z)$ into two parts: $B(Z) = B_{out}(Z)B_{in}(Z)$ where B_{out} contains roots outside the unit circle and B_{in} contains the roots inside. Then the inverse of B_{out} is expressed as a Taylor series about the origin and the inverse of B_{in} is expressed as a Taylor series about infinity. The final expression for $1/B(Z)$ is called a Laurent expansion for $1/B(Z)$, and it converges on a ring surrounding the unit circle. Cases with zeros exactly on the unit circle present special problems. Sometimes you can argue yourself out of the difficulty but at other times roots on or even near the circle may mean that a certain computing scheme won't work out well in practice.

Finally, let us consider a mechanical interpretation. The stress (pressure) in a material may be represented by x_t, and the strain (volume charge) may be represented by y_t. The following two statements are equivalent; that is, in some situations they are both true, and in other situations they are both false:

STATEMENT A The stress in a material may be expressed as a linear combination of present and past strains. Likewise, the strain may be deduced from present and past stresses.

STATEMENT B The filter which relates stress to strain and vice versa has all poles and zeros outside the unit circle.

EXERCISES:

1 Find the filter which is inverse to $(2-5Z+2Z^2)$. You may just drop higher-order powers of Z, but an exact expression for the coefficients of any power of Z is preferred. (Partial fractions is a useful, though not necessary, technique.) Sketch the impulse response.

2 Show that multiplication by $(1 - Z)$ in discretized time is analogous to time differentiation in continuous time. Show that dividing by $(1 - Z)$ is analogous to integration. What are the limits on the integral?

3 Describe a general method for determining $A(Z)$ and $B(Z)$ from a Taylor series of $B(Z)/A(Z) = C_0 + C_1 Z + C_2 Z^2 + \cdots + C_\infty Z^\infty$ where $B(Z)$ and $A(Z)$ are polynomials of unknown degree n and m, respectively. Work out the case $C(Z) = \frac{1}{2} - \frac{3}{4}Z - \frac{3}{8}Z^2 - \frac{3}{16}Z^3 - \frac{3}{32}Z^4 - \cdots$. Don't try this problem unless you are quite familiar with determinants. [HINT: Identify coefficients of $B(Z) = A(Z)C(Z)$.]

2.2 MINIMUM PHASE

In Sec. 2-1 we learned that knowledge of convergence of the Taylor series of $1/B(Z)$ on $|Z| = 1$ is equivalent to knowledge that $B(Z)$ has no roots inside the unit circle. Now we will see that these conditions are also equivalent to a certain behavior of the phase of $B(Z)$ on the unit circle.

Let us consider the phase shift of the two-term filter

$$\begin{aligned} B &= 1 - Z/Z_0 \quad (Z_0 = \rho e^{i\omega_o}) \\ &= 1 - \rho^{-1} e^{i(\omega - \omega_o)} \\ &= 1 - \rho^{-1}\cos(\omega - \omega_o) - i\rho^{-1}\sin(\omega - \omega_o) \end{aligned}$$

By definition, phase is the arctangent of the ratio of the imaginary part to the real part.

A graph of phase as a function of frequency looks radically different for $\rho < 1$ than for $\rho > 1$. See Figure 2.4 for the case $\rho > 1$.

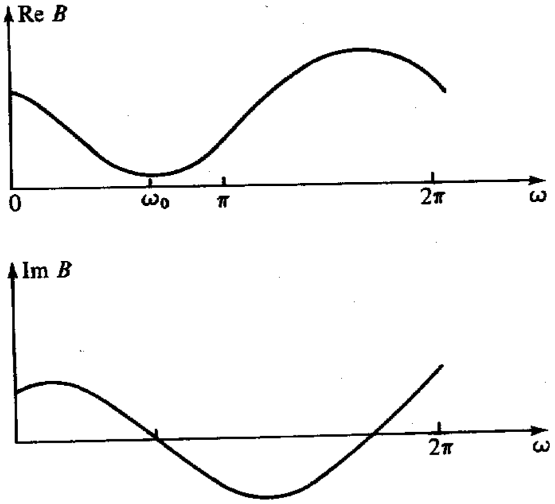


Figure 2.4: Real and imaginary parts of the Z transform $1 - Z/(1.25e^{i2\pi/3})$. c2-2-4 [NR]

The phase is the arctangent of ImB/ReB. The easiest way to keep track of the phase is in the complex B plane. This is shown in Figure 2.5.

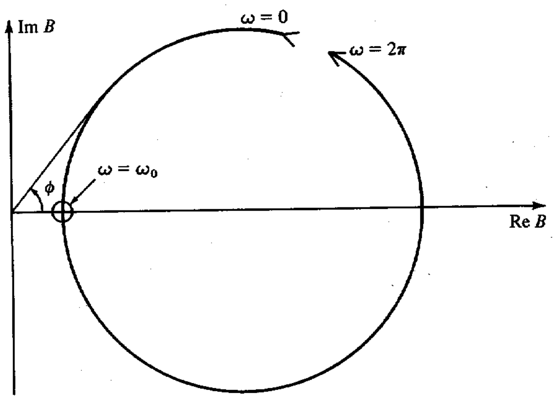


Figure 2.5: Phase of the two-term filter of Figure 2.4. c2-2-5 [NR]

Thus phase as a function of frequency is shown in Figure 2.6. Notice that the phase ϕ at $\omega = 0$ is the same as the phase at $\omega = 2\pi$. This follows because the real and imaginary parts are periodic with 2π. The situation will be different when there

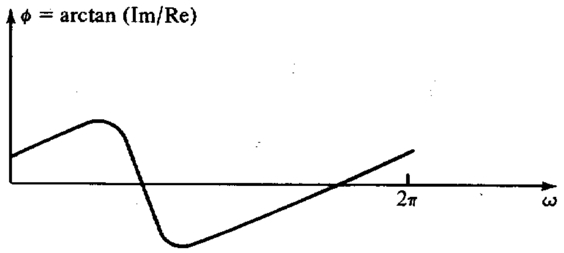


Figure 2.6: The phase of a two-term minimum-phase filter. c2-2-6 [NR]

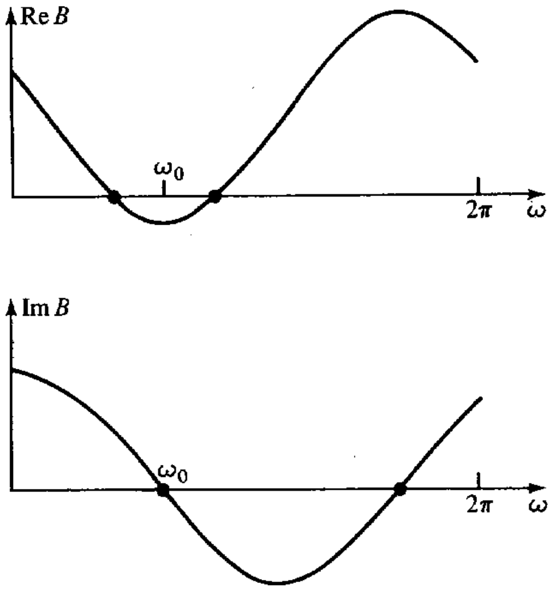


Figure 2.7: Real and imaginary parts of the two-term nonminimum-phase filter, $1 - 1.25Ze^{-i2\pi/3}$. c2-2-7 [NR]

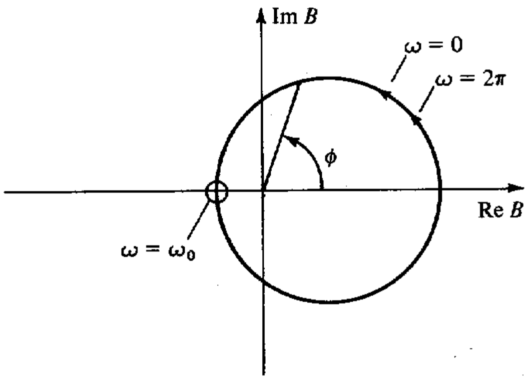


Figure 2.8: Phase in complex plane. c2-2-8 [NR]

is a zero inside the unit circle; that is, $\rho < 1$. The real and imaginary parts are shown in Figure 2.7 and the complex plane in Figure 2.8.

The phase ϕ increases by 2π as ω goes from zero to 2π because the circular path surrounds the origin. The phase curve is shown in Figure 2.9. The case $\rho > 1$ where $\phi(\omega) = \phi(\omega + 2\pi)$ has come to be called *minimum phase* or *minimum delay.*

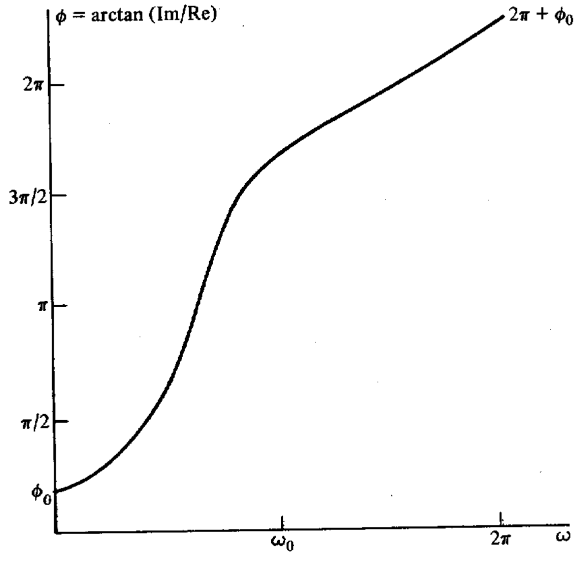


Figure 2.9: The phase of a two-term nonminimum-phase filter. $\boxed{\text{c2-2-9}}$ [NR]

Now we are ready to consider a complicated filter like

$$B(Z) \quad = \quad \frac{(Z - c_1)(Z - c_2) \cdots}{(Z - a_1)(Z - a_2) \cdots} \tag{2.12}$$

By the rules of complex-number multiplication the phase of $B(Z)$ is the sum of the phases in the numerator minus the sum of the phases in the denominator. Since we are discussing realizable filters the denominator factors must all be minimum phase, and so the denominator phase curve is a sum of curves like Figure 2.6. The numerator factors may or may not be minimum phase. Thus the numerator phase curve is a sum of curves like either Figure 2.6 or Figure 2.9. If any factors at all are like Figure 2.9, then the total phase will resemble Figure 2.9 in that the phase at $\omega = 2\pi$ will be greater than the phase at $\omega = 0$. Then the filter will be nonminimum phase.

2.3 FILTERS IN PARALLEL

We have seen that in a cascade of filters the filter polynomials are multiplied together. One might conceive of adding two polynomials $A(Z)$ and $G(Z)$ when they correspond to filters which operate in parallel. See Figure 2.10.

When filters operate in parallel their Z transforms add together. We have seen that a cascade of filters is minimum phase if, and only if, each element of the product

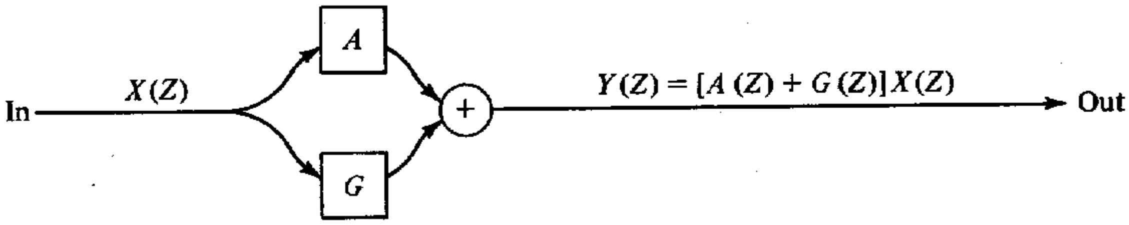


Figure 2.10: Filters operating in parallel. c2-2-10 [NR]

is minimum phase. Now we will see a sufficient (but not necessary) condition that the sum $A(Z) + G(Z)$ be minimum phase. First of all, let us assume that $A(Z)$ is minimum phase. Then we may write

$$A(Z) + G(Z) \quad = \quad A(Z)\left[1 + \frac{G(Z)}{A(Z)}\right] \tag{2.13}$$

The question whether $A(Z)+G(Z)$ is minimum phase is now reduced to determining whether $A(Z)$ and $1+G(Z)/A(Z)$ are both minimum phase. We have assumed that $A(Z)$ is minimum phase. Before we ask whether $1 + G(Z)/A(Z)$ is minimum phase we need to be sure that it is causal. Since $1/A(Z)$ is expandable in positive powers of Z only, then $G(Z)/A(Z)$ is also causal. We will next see that a sufficient condition for $1 + G(Z)/A(Z)$ to be minimum phase is that the spectrum of A exceeds that of G at all frequencies. In other words, for any real ω, $\mid A \mid > \mid G \mid$. Thus, if we plot the curve of $G(Z)/A(Z)$ in the complex plane, for real $0 \leq \omega \leq 2\pi$ it lies everywhere inside the unit circle. Now if we add unity —getting $1 + G(Z)/A(Z)$, the curve will always have a positive real part. See Figure 2.11.

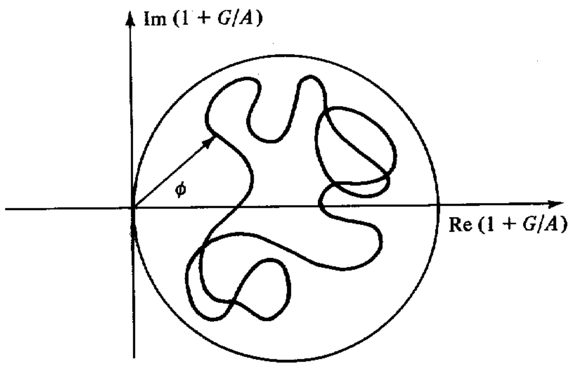


Figure 2.11: Phase of a positive real function lies between $\pm\pi/2$. c2-2-11 [NR]

Since the curve cannot enclose the origin, the phase must be that of a minimum-phase function. In words, "You can add garbage to a minimum-phase wavelet if you do not add too much." This somewhat abstract theorem has an immediate physical consequence. Suppose a wave characterized by a minimum phase $A(Z)$ is emitted from a source and detected at a receiver some time later. At a still later time an echo bounces off a nearby object and is also detected at the receiver. The receiver sees the signal $Y(Z) = A(Z) + Z^n \alpha A(Z)$ where n measures the delay from the first arrival to the echo and α represents the amplitude attenuation of the echo. To see that $Y(Z)$ is minimum phase, we note that the magnitude of Z^n is unity and that the reflection coefficient α must be less than unity (to avoid perpetual motion) so that $Z^n \alpha A(Z)$ takes the role of $G(Z)$. Thus a minimum-phase wave along with its echo is minimum phase. We will later consider wave propagation situations with echoes of the echoes ad infinitum.

EXERCISES:

1 Find two nonminimum-phase wavelets whose sum is minimum phase.

2 Let $A(Z)$ be a minimum-phase polynomial of degree N. Let $A'(Z) = Z^N \bar{A}(1/Z)$. Locate in the complex Z plane the roots of $A'(Z)$. $A'(Z)$ is called *maximum phase*. [HINT: Work the simple case $A(Z) = a_0 + a_1 Z$ first.]

3 Suppose $A(Z)$ is maximum phase and that the degree of $G(Z)$ is less than or equal to the degree of $A(Z)$. Assume $\mid A \mid > \mid G \mid$. Show that $A(Z) + G(Z)$ is maximum phase.

4 Let $A(Z)$ be minimum phase. Where are the roots of $A(Z) + cZ^N \bar{A}(1/Z)$ in the three cases $\mid c \mid < 1, \mid c \mid > 1, \mid c \mid = 1$? (HINT: The roots of a polynomial are continuous functions of the polynomial coefficients.)

2.4 POSITIVE REAL FUNCTIONS

Two similar types of functions called *admittance functions* $Y(Z)$ and *impedance functions* $I(Z)$ occur in many physical problems. In electronics, they are ratios of current to voltage and of voltage to current; in acoustics, impedance is the ratio of pressure to velocity. When the appropriate electrical network or acoustical region contains no sources of energy, then these ratios have the positive real property. To see this in a mechanical example, we may imagine applying a known force $F(Z)$ and observing the resulting velocity $V(Z)$. In filter theory, it is like considering that $F(Z)$ is input to a filter $Y(Z)$ giving output $V(Z)$. We have

$$V(Z) \quad = \quad Y(Z)F(Z) \tag{2.14}$$

The filter $Y(Z)$ is obviously causal. Since we believe we can do it the other way around, that is, prescribe the velocity and observe the force, there must exist a convergent causal $I(Z)$ such that

$$F(Z) \quad = \quad I(Z)V(Z) \tag{2.15}$$

Since Y and I are inverses of one another and since they are both presumed bounded and causal, then they both must be minimum phase.

First, before we consider any physics, note that if the complex number $a+ib$ has a positive real part a, then the real part of $(a+ib)^{-1}$ namely $a/(a^2+b^2)$ is also positive. Taking $a+ib$ to represent a value of $Y(Z)$ or $I(Z)$ on the unit circle, we see the obvious fact that if either Y or I has the positive real property, then the other does, too.

Power dissipated is the product of force times velocity, that is

$$\text{Power} \quad = \quad \cdots + f_0 v_0 + f_1 v_1 + f_2 v_2 + \cdots \tag{2.16}$$

This may be expressed in terms of Z transforms as

$$\begin{aligned} \text{Power} &= \frac{1}{2} \text{ coeff of } Z^0 \text{ of } V\left(\frac{1}{Z}\right) F(Z) + F\left(\frac{1}{Z}\right) V(Z) \\ &= \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi} \left[V\left(\frac{1}{Z}\right) F(Z) + F\left(\frac{1}{Z}\right) V(Z)\right] d\omega \end{aligned} \tag{2.17}$$

Using (2.14) to eliminate $V(Z)$ we get

$$\text{Power} \quad = \quad \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi} F\left(\frac{1}{Z}\right)\left[Y\left(\frac{1}{Z}\right) + Y(Z)\right] F(Z)\, d\omega \tag{2.18}$$

We note that $Y(Z)+Y(1/Z)$ looks superficially like a spectrum because the coefficient of Z^k equals that of Z^{-k}, which shows the symmetry of an autocorrelation function. Defining

$$R(Z) \quad = \quad Y(Z) + Y\left(\frac{1}{Z}\right) \tag{2.19}$$

(2.17) becomes

$$\text{Power} \quad = \quad \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi} R(Z) F\left(\frac{1}{Z}\right) F(Z)\, d\omega \tag{2.20}$$

The integrand is the product of the arbitrary positive input force spectrum and $R(Z)$. If the power dissipation is expected to be positive at all frequencies (for all $\bar{F}F$), then obviously $R(Z)$ must be positive at all frequencies; thus R is indeed a spectrum. Since we have now discovered that $Y(Z)$ and $Y(1/Z)$ must be positive for all frequencies, we have discovered that $Y(Z)$ is not an arbitrary minimum-phase filter. The real part of both $Y(Z)$ and $Y(1/Z)$ is

$$\text{Re}[Y(Z)] \quad = \quad \text{Re}\left[Y\left(\frac{1}{Z}\right)\right] \quad = \quad y_o + y_1 \cos\,\omega + y_2 \cos\, 2\omega + \cdots \tag{2.21}$$

Since the real part of the sum must be positive, then obviously the real part of each of the equal parts be positive.

Now if the material or mechanism being studied is passive (contains no energy sources) then we must have positive dissipation over a time gate from minus infinity up to any time t. Let us find an expression for dissipation in such a time gate. For simplicity take both the force and velocity vanishing before $t = 0$. Let the end of the time gate include the point $t = 2$ but not $t = 3$.

Define

$$f'_t \quad = \quad \begin{cases} f_t & t \le 2 \\ 0 & t > 2 \end{cases} \tag{2.22}$$

To find the work done over all time we may integrate (2.20) over all frequencies. To find the work done in the selected gate we may replace F by F' and integrate over all frequencies, namely

$$W_2 \quad = \quad \frac{1}{2}\frac{1}{2\pi}\int_{-\pi}^{+\pi} F'\left(\frac{1}{Z}\right) R(Z)F'(Z)\, d\omega \tag{2.23}$$

As we have seen, this integral merely selects the coefficient of Z^0 of the integrand. Let us work this out. First, collect coefficients of powers of Z in $R(Z)F'(Z)$. We have

$$\begin{array}{lcl} Z^0 & : & r_o f'_0 + r_{-1} f'_1 + r_{-2} f'_2 \\ Z^1 & : & r_1 f'_0 + r_0 f'_1 + r_{-1} f'_2 \\ Z^2 & : & r_2 f'_0 + r_1 f'_1 + r_0 f'_2 \end{array}$$

To obtain the coefficient of Z^0 in $F'(1/Z)[R(Z)F'(Z)]$ we must multiply the top row above by f'_0, the second row by f'_1 and the third row by f'_2. The result can be arranged in a very orderly fashion by

$$\begin{aligned} W_2 \quad &= \quad \frac{1}{2}[f_0 \ f_1 \ f_2] \begin{bmatrix} r_0 & r_{-1} & r_{-2} \\ r_1 & r_0 & r_{-1} \\ r_2 & r_1 & r_0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \\ &= \quad \frac{1}{2}[f_0 \ f_1 \ f_2] \begin{bmatrix} 2y_0 & y_1 & y_2 \\ y_1 & 2y_0 & y_1 \\ y_2 & y_1 & 2y_0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \end{aligned} \tag{2.24}$$

Not only must the 3 x 3 quadratic form (2.24) be positive (i.e., $W_2 \ge 0$ for arbitrary f_t) but all $t \times t$ similar quadratic forms W_t must be positive.

In conclusion, the positive real property in the frequency domain means that $Y(Z) + Y(1/Z)$ is positive for any real ω and the positive real property in the time domain means that all $t \times t$ matrices like that of (2.24) are positive definite. Figure 2.12 summarizes the function types which we have considered.

EXERCISES:

1 In mechanics we have force and velocity of a free unit mass related by $dv/dt = f$ or $v = \int_{-\infty}^{t} f dt$. Compute the power dissipated as a function of frequency if integration is approximated by convolution with $(.5, 1., 1., 1., ...)$. [HINT: Expand $(1 + Z)/2(1 - Z)$ in positive powers of Z.]

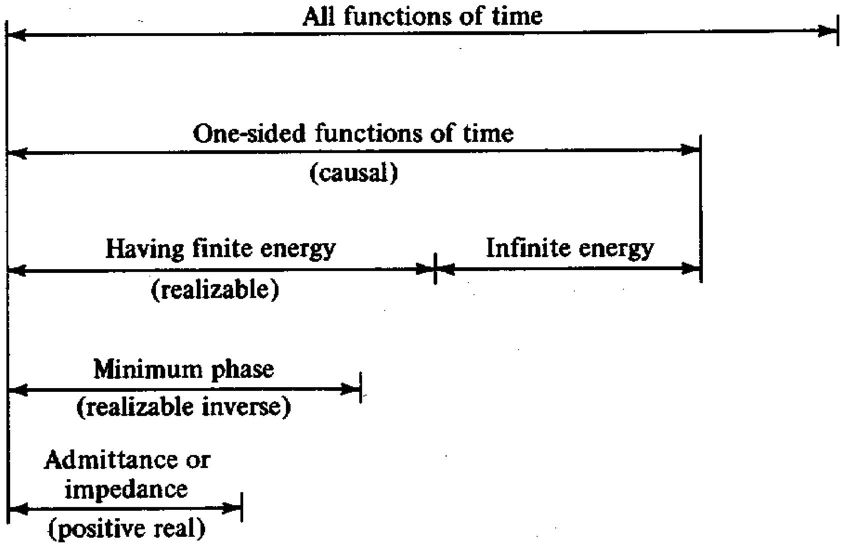


Figure 2.12: Important classes of time functions. c2-2-12 [NR]

2 Construct an example of a simple function which is minimum phase but not positive real.

2.5 NARROW-BAND FILTERS

Filters are often used to modify the spectrum of given data. With input $X(Z)$, filters $B(Z)$, and output $Y(Z)$ we have $Y(Z) = B(Z)X(Z)$ and the Fourier conjugate $\bar{Y}(1/Z) = \bar{B}(1/Z)\bar{X}(1/Z)$. Multiplying these two relations together we get

$$\overline{Y}Y \quad = \quad (\bar{B}B)(\bar{X}X) \tag{2.25}$$

which says that the spectrum of the input times the spectrum of the filter equals the spectrum of the output. Filters are often characterized by the shape of their spectra. Some examples are shown in Figure (2.13).

We will have frequent occasion to deal with sinusoidal time functions. A simple way to represent a sinusoid by Z transforms is

$$\frac{1}{1 - Ze^{i\omega_0}} \quad = \quad 1 + Ze^{i\omega_0} + Z^2 e^{i2\omega_0} + \cdots \tag{2.26}$$

The time function associated with this Z transform is $e^{i\omega_0 t}$, but it is "turned on" at $t = 0$. Actually, the left-hand side of (2.26) contains a pole exactly on the unit circle, so that the series sits on the borderline between convergence and divergence. This can cause paradoxical situations [you could expand (2.26) so that the sinusoid turns

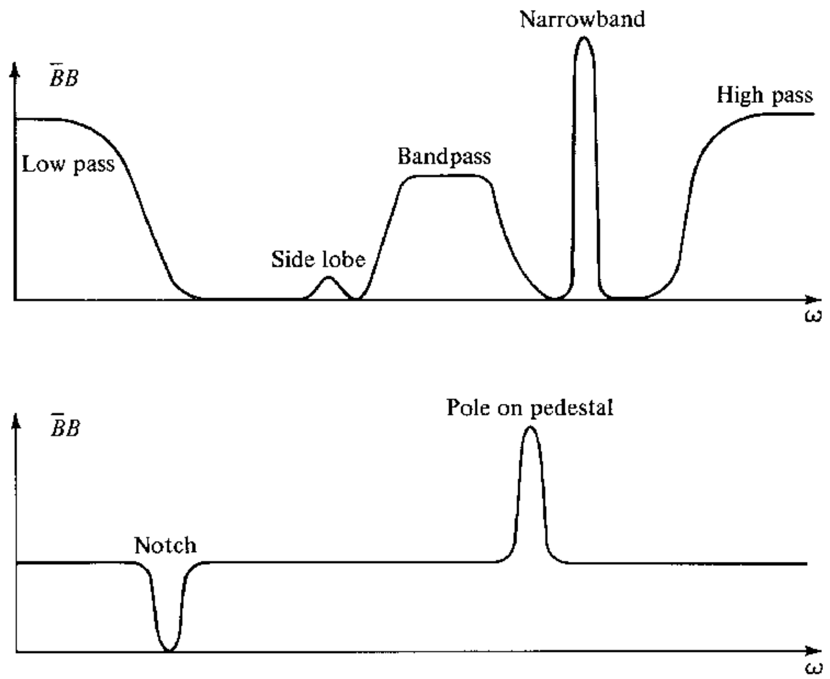


Figure 2.13: Spectra of various filters. $\boxed{\text{c2-2-13}}$ [NR]

off at $t = 0$] which we will avoid by pushing the pole from the unit circle to a small distance ε outside the unit circle. Let $Z_0 = (1+\varepsilon)e^{i\omega_0}$. Then define

$$\begin{aligned} B(Z) &= \frac{1}{A(Z)} \quad = \quad \frac{1}{1 - Z/Z_0} \\ &= 1 + \frac{Z}{Z_0} + \left(\frac{Z}{Z_0}\right)^2 + \cdots \end{aligned} \tag{2.27}$$

The time function corresponding to $B(Z)$ is zero before $t = 0$ and is $e^{-i\omega_0 t}/(1+\varepsilon)^t$ after $t = 0$. It is a sinusoidal function which decreases gradually with time according to $(1+\varepsilon)^{-t}$. The coefficients are shown in Figure 2.14.

It is intuitively obvious, although we will prove it later, that convolution with the coefficients of (2.27), which are sketched in Figure 2.14, is a narrow-banded filtering operation. If the pole is chosen very close to the unit circle, the filter bandpass becomes narrower and the coefficients of $B(Z)$ drop off more and more slowly. To actually perform the convolution it is necessary to truncate, that is, to drop powers of Z beyond a certain practical limit. It turns out that there is a very much cheaper method of narrow-band filtering than convolution with the coefficients of $B(Z)$. This method is polynomial division by $A(Z)$. We have for the output $Y(Z)$

$$Y(Z) \quad = \quad B(Z)X(Z) \tag{2.28}$$

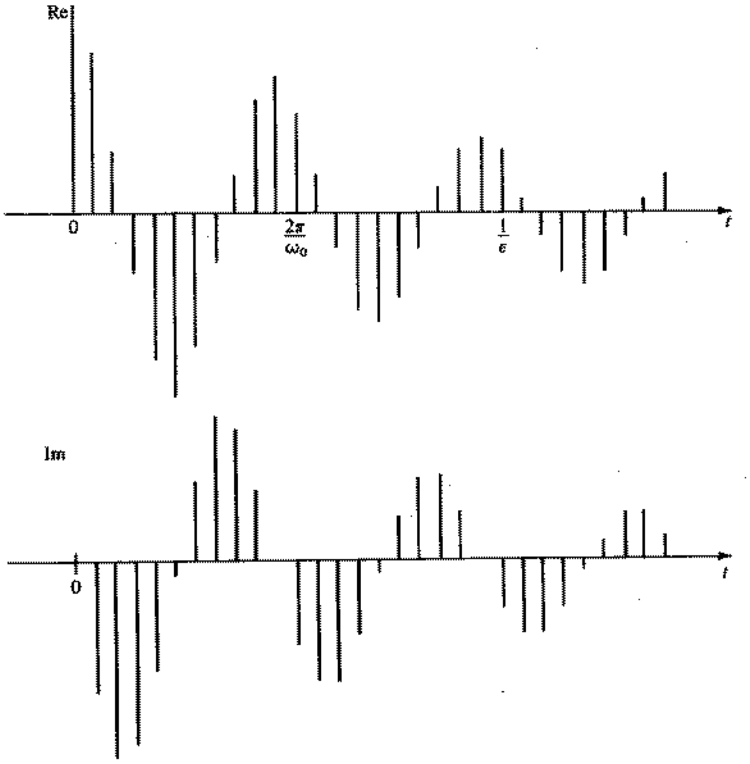


Figure 2.14: The time function associated with a simple pole just outside the unit circle at $Z_0 = 1.1e^{i\pi/5}$. $\boxed{\text{c2-2-14}}$ [NR]

$$Y(Z) \quad = \quad \frac{X(Z)}{A(Z)} \tag{2.29}$$

Multiply both sides of (2.29) by $A(Z)$

$$Y(Z)A(Z) \quad = \quad X(Z) \tag{2.30}$$

For definiteness, let us suppose the x_t and y_t vanish before $t = 0$. Now identify coefficients of successive powers of Z. We get

$$\begin{aligned} y_0a_0 &= x_0 \\ y_1a_0 + y_0a_1 &= x_1 \\ y_2a_0 + y_1a_1 + y_0a_2 &= x_2 \\ y_3a_0 + y_2a_1 + y_1a_2 + y_0a_3 &= x_3 \\ y_4a_0 + y_3a_1 + y_2a_2 + y_1a_3 &= x_4 \\ &\cdots \end{aligned} \tag{2.31}$$

A general equation is

$$y_ka_0 + \sum_{i=1}^{\infty} y_{k-i}a_i = x_k \tag{2.32}$$

Solving for y_k we get

$$y_k \quad = \quad \frac{x_k - \sum\limits_{i=1}^{\infty} y_{k-i}a_i}{a_0} \tag{2.33}$$

Equation (2.33) may be used to solve for y_k once $y_{k-1}, y_{k-2}, \cdots$ are known. Thus the solution is recursive, and it will not diverge if the a_i are coefficients of a minimum-phase polynomial. In practice the infinite limit on the sum is truncated whenever you run out of coefficients of either $A(Z)$ or $Y(Z)$. For the example we have been considering, $B(Z) = 1/A(Z) = 1/(1 - Z/Z_0)$, there will be only one term in the sum. Filtering in this way is called *feedback filtering*, and for narrowband filtering it will be vastly more economical than filtering by convolution, since there are much fewer coefficients in $A(Z)$ than $B(Z) = 1/A(Z)$. Finally, let us examine the spectrum of $B(Z)$. We have

$$\begin{aligned} A(Z) &= 1 - Z/Z_0 \\ &= 1 - \frac{e^{i\omega}}{(1+\varepsilon)e^{i\omega_0}} \\ &= 1 - \frac{e^{i(\omega-\omega_0)}}{(1+\varepsilon)} \end{aligned}$$

and

$$\bar{A}\left(\frac{1}{Z}\right) = 1 - \frac{e^{-i(\omega-\omega_0)}}{1+\varepsilon}$$

so

$$\begin{aligned} \bar{A}\left(\frac{1}{Z}\right) A(Z) &= \left(1 - \frac{e^{-i(\omega-\omega_0)}}{1+\varepsilon}\right)\left(1 - \frac{e^{i(\omega-\omega_0)}}{1+\varepsilon}\right) \\ &= 1 + \frac{1}{(1+\varepsilon)^2} - \frac{1}{1+\varepsilon}\left(e^{-i(\omega-\omega_0)} + e^{i(\omega-\omega_0)}\right) \\ &= 1 + \frac{1}{(1+\varepsilon)^2} - \frac{2\cos(\omega-\omega_0)}{1+\varepsilon} \\ &= 1 + \frac{1}{(1+\varepsilon)^2} - \frac{2}{1+\varepsilon} + \frac{2}{1+\varepsilon}[1 - \cos(\omega-\omega_0)] \\ &= \left(1 - \frac{1}{1+\varepsilon}\right)^2 + \frac{4}{1+\varepsilon}\sin^2\frac{\omega-\omega_0}{2} \\ \bar{B}\left(\frac{1}{Z}\right) B(Z) &= \frac{(1+\varepsilon)^2}{\varepsilon^2 + 4(1+\varepsilon)\sin^2\left(\frac{\omega-\omega_0}{2}\right)} \end{aligned} \tag{2.34}$$

To a good approximation this function may be thought of as $1/[\varepsilon^2 + (\omega - \omega_0)^2]$. A plot of (2.34) is shown in Figure 2.15.

Now it should be apparant why this is called a narrowband filter. It amplifies a vary narrow band of frequencies and attenuates all others. The frequency window of this filter is said to be $\Delta\omega \approx 2\varepsilon$ in width. The time window is $\Delta t = 1/\varepsilon$, the damping time constant of the dampend sinusoid b_t.

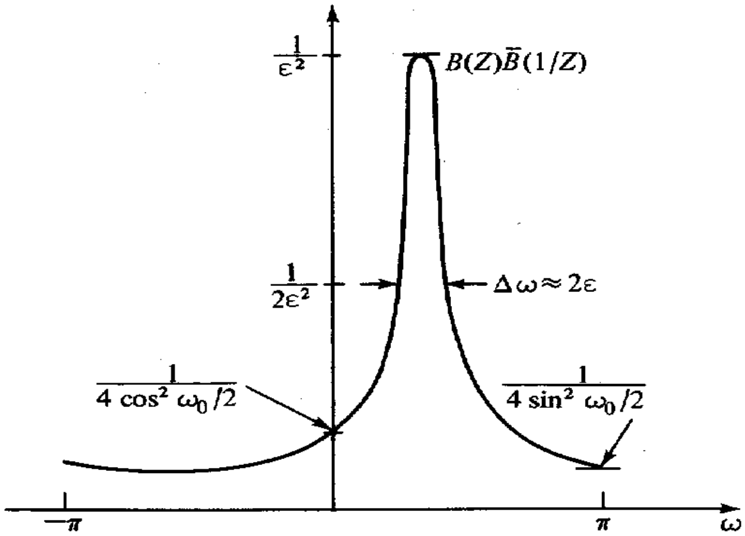


Figure 2.15: Spectrum associated with a single pole at $Z_0 = (1 + \varepsilon)e^{i\omega_0}$. $\boxed{\text{c2-2-15}}$ [NR]

One practical disadvantage of the filter under discussion is that although its input may be a real time series its output will be a complex time series. For many applications a filter with real coefficients may be preferred.

One approach is to follow the filter $[1, e^{i\omega_0}/(1+\varepsilon)]$ by the time-domain, complex conjugate filter $[1, e^{-i\omega_0}/(1+\varepsilon)]$. The composit time-domain operator is now $[1, (2\cos\omega_0)/(1+\varepsilon), 1/(1+\varepsilon)^2]$ which is real. [Note that the complex conjugate in the frequency domain is $\bar{B}(1/Z)$ but in the time domain it is $\bar{B}(Z) = \bar{b}_0 + \bar{b}_1 Z + \cdots$]. The composite filter may be denoted by $B(Z)\bar{B}(Z)$. The spectrum of this filter is $[B(Z)\bar{B}(1/Z)][\bar{B}(Z)B(1/Z)]$. One may quickly verify that the spectrum of $\bar{B}(Z)$ is like that of $B(Z)$, but the peak is at $-\omega_0$ instead of $+\omega_0$. Thus, the composite spectrum is the product of Figure 2.15 with itself reversed along the frequency axis. This is shown in Figure 2.16.

EXERCISES:

1 A simple feedback operation is $y_t = (1-\varepsilon)y_{t-1} + x_t$. This operation is called leaky integration. Give a closed form expression for the output y_t if x_t is an impulse. What is the decay time τ of your solution (the time it takes for y_t to drop to $e^{-1}y_0$)? For small ε, say $= 0.1$, $.001$, or 0.0001, what is τ?

2 How far from the unit circle are the poles of $1/(1 - .1Z + .9Z^2)$? What is the decay time of the filter and its resonant frequency?

3 Find a three-term real feedback filter to pass 59-61 Hz on data which are sampled at 500 points/sec. Where are the poles? What is the decay time of the filter?

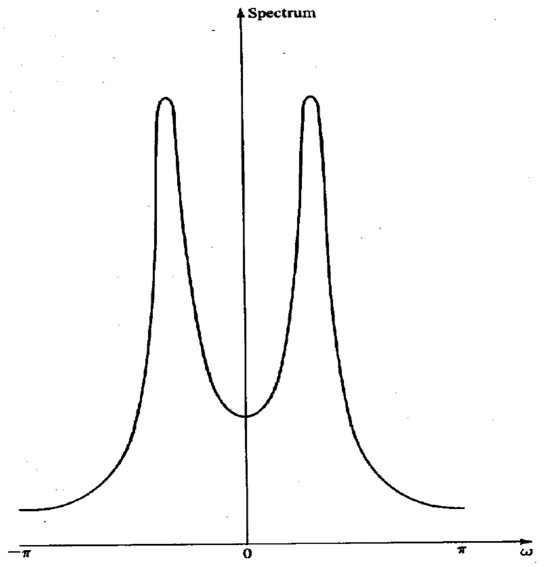


Figure 2.16: Spectrum of a two-pole filter where one pole is like Figure 2.15 and the other is at the conjugate position. $\boxed{\text{c2-2-16}}$ [NR]

2.6 ALL-PASS FILTERS

In this section we consider filters with constant unit spectra, that is, $B(Z)\bar{B}(1/Z) = 1$. In other words, in the frequency domain $B(Z)$ takes the form $e^{i\phi(\omega)}$ where ϕ is real and is called the *phase shift*. Clearly $B\bar{B} = 1$ for all real ϕ. It is an easy matter to construct a filter with any desired phase shift; one merely Fourier transforms $e^{i\phi(\omega)}$ into the time domain. If $\phi(\omega)$ is arbitrary, the resulting time function is likely to be two-sided. Since we are interested in physical processes which are causal, we may wonder what class of functions $\phi(\omega)$ corresponds to one-sided time functions. The easiest way to proceed is to begin with a simple case of a single-pole, single-zero all-pass filter. Then more elaborate all-pass filters can be made up by cascading these simple filters. Consider the filter

$$P(Z) \quad = \quad \frac{Z - 1/(\bar{Z})_0}{1 - Z/Z_0} \tag{2.35}$$

Note that this is a simple case of functions of the form $Z^N \bar{A}(1/Z)/A(Z)$, where $A(Z)$ is a polynomial of degree N or less. Now observe that the spectrum of the filter p_t is indeed a frequency-independent constant. The spectrum is

$$\bar{P}\left(\frac{1}{Z}\right) P(Z) \quad = \quad \frac{1/Z - 1/Z_0}{1 - 1/(Z\bar{Z}_0)} \frac{Z - 1/\bar{Z}_0}{Z - 1/Z_0} \tag{2.36}$$

Multiply top and bottom on the left by Z. We now have

$$\bar{P}\left(\frac{1}{Z}\right) P(Z) \quad = \quad \frac{1 - Z/Z_0}{Z - 1/\bar{Z}_0} \frac{Z - 1/\bar{Z}_0}{1 - Z/Z_0} \quad = \quad 1 \tag{2.37}$$

It is easy to show that $\bar{P}(1/Z)P(Z) = 1$ for the general form $P(Z) = Z^N \bar{A}(1/Z)/A(Z)$. If Z_0 is chosen outside the unit circle, then the denominator of (2.35) can be expanded in positive powers of Z and the expansion in convergent on the unit circle. This means that causality is equivalent to Z_0 outside the unit circle. Setting the numerator of $P(Z)$ equal to zero, we discover that the zero $Z = 1/\bar{Z}_0$ is then inside the unit circle. The situation is depicted in Figure 2.17. To see that the pole and zero are on the same radius line, express Z_0 in polar form $r_0 e^{i\phi_0}$.

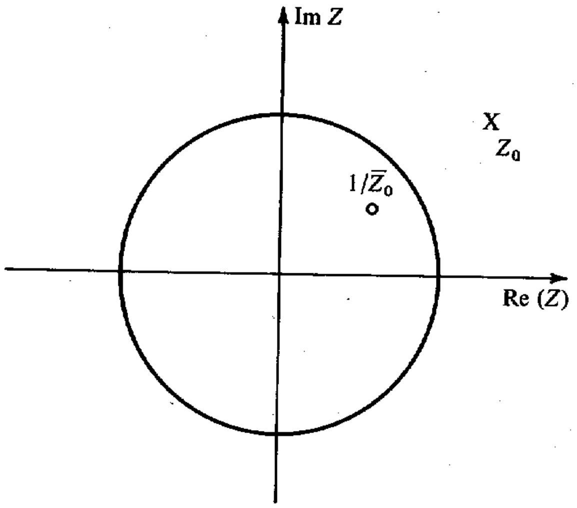


Figure 2.17: The pole of the all-phase filter lies outside the unit circle and the zero is inside. They lie on the same radius line. $\boxed{\text{c2-2-17}}$ [NR]

From Section 2.2 (on minimum phase) we see that the numerator of P is not minimum phase and its phase is augmented by 2π as ω goes from 0 to 2π. Thus the average delay $d\phi/d\omega$ is positive. Not only is the average positive but, in fact, the group delay turns out to be positive in every frequency. To see this, first note that

$$\begin{aligned} Z &= e^{i\omega} \\ \frac{dZ}{d\omega} &= ie^{i\omega} = iZ \\ \frac{d}{d\omega} &= \frac{dZ}{d\omega}\frac{d}{dZ} = iZ\frac{d}{dZ} \end{aligned} \tag{2.38}$$

The phase of the all-pass filter (or any complex number) may be written as

$$\phi \quad = \quad \ln P(Z) \tag{2.39}$$

Since $|P| = 1$ the real part of the log vanishes; and so, for the all-pass filter (only) we may specialize (2.39) to

$$\begin{aligned} \phi &= \frac{1}{i}\ln P(Z) \quad = \quad \frac{1}{i}\ln\frac{Z - 1/\bar{Z}_0}{1 - Z/Z_0} \\ &= \frac{1}{i}\left[\ln\left(Z - \frac{1}{\bar{Z}_0}\right) - \ln\left(1 - \frac{Z}{Z_0}\right)\right] \end{aligned} \tag{2.40}$$

Using (2.38) the group delay is now found to be

$$\begin{aligned}
\tau_g &= \frac{d\phi}{d\omega} \quad = \quad iZ\frac{d\phi}{dZ} \quad = \quad Z\left(\frac{1}{Z - 1/\bar{Z}_0} + \frac{1/Z_0}{1 - Z/Z_0}\right) \\
&= \frac{1}{1 - 1/\bar{Z}_0 Z} + \frac{Z/Z_0}{1 - Z/Z_0} \\
&= \frac{1 - Z/Z_0 + (1 - 1/\bar{Z}_0 Z)(Z/Z_0)}{(1 - 1/\bar{Z}_0 Z)(1 - Z/Z_0)} \quad = \quad \frac{1 - 1/Z_0 \bar{Z}_0}{(1 - 1/\bar{Z}_0 Z)(1 - Z/Z_0)} \qquad (2.41)
\end{aligned}$$

The numerator of (2.41) is a positive real number (since $|Z_0| > 1$), and the denominator is of the form $\bar{A}(1/Z)A(Z)$, which is a spectrum and also positive. Thus we have shown that the group delay of this causal all-pass filter is always positive.

Now if we take a filter and follow it will an all-pass filter, the phases add and the group delay of the composite filter must necessarily be greater than the group delay of original filter. By the same reasoning the minimum-phase filter must have less group delay than any other filter with the same spectrum.

In summary, a single-pole, single-zero all-pass filter passes all frequency components with constant gain and a phase shift which may be adjusted by the placement of a pole. Taking Z_0 near the unit circle causes most of the phase shift to be concentrated near the frequency where the pole is located. Taking the pole further away causes the delay to be spread over more frequencies. Complicated phase shifts or group delays may be built up by cascading several single-pole filters.

EXERCISES:

1 An example of an all-pass filter is the time function $p_t = (\frac{1}{2}, -\frac{3}{4}, -\frac{3}{8}, -\frac{3}{16}, \cdots)$. Calculate a few lags of its autocorrelation by summing some infinite series.

2 Sketch the amplitude, phase, and group delay of the all-pass filter $(1-\bar{Z}_0 Z)/(Z_0 - Z)$ where $Z_0 = (1+\varepsilon)e^{i\omega_0}$ and ε is small. Indicate important parameters on the curve.

3 Show that the coefficients of an all-pass, phase-shifting filter made by cascading $(1 - \bar{Z}_0 Z)/(Z_0 - Z)$ with $(1 - Z_0 Z)/(\bar{Z}_0 - Z)$ are real.

4 A continuous time function is the impulse response of a continuous-time, all-pass filter. Describe the function in both time domain and frequency domain. Interchange the words *time* and *frequency* in your description of the function. What is a physical example of such a function? What happens to the statement: "The group delay of an all-pass filter is positive."?

5 A graph of the group delay $\tau_g(\omega)$ in equation (2.41) shows τ_g to be positive for all ω. What is the area under τ_g in the range $0 < \omega < 2\pi$. (HINT: This is a trick question you can solve in your head.)

2.7 NOTCH FILTER AND POLE ON PEDESTAL

In some applications it is desired to reject a very narrow frequency band leaving the rest of the spectrum little changed. The most common example is 60-Hz noise from power lines. Such a filter can easily be made with a slight variation on the all-pass filter. In the all-pass filter the pole and zero have an equal (logarithmic) relative distance from the unit circle. All we need to do is to put the zero closer to the circle. In fact, there is no reason why we should not put the zero right on the circle. Then the frequency at which the zero is located is exactly canceled from the spectrum of input data. If the undesired frequency need not be completely rejected, then the zero can be left just inside or outside the circle. As the zero is moved farther away from the circle, the notch becomes less deep until finally the zero is farther from the circle than the pole and the notch has become a hump. The resulting filter which will be called *pole on pedestal* is in many respects like the narrowband filter discussed earlier. Some of these filters are illustrated in Figures 2.18 and 2.19. The difference between the pole-on-pedestal and the narrowband filters is in the asymptotic behavior away from ω_0. The former is flat, while the latter continues to decay with increasing $\mid \omega - \omega_0 \mid$. This makes the pole on pedestal more convenient for creating complicated filter shapes by cascades of single-pole filters.

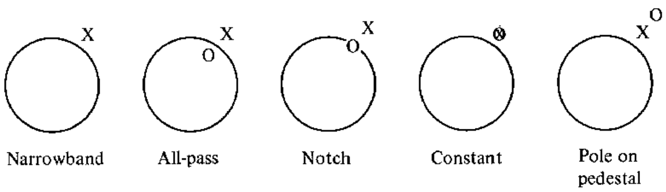


Figure 2.18: Pole and zero locations for some simple filters. Circles are unit circles in the Z plane. Poles are marked by X and *zeros* by 0. $\boxed{\text{c2-2-18}}$ [NR]

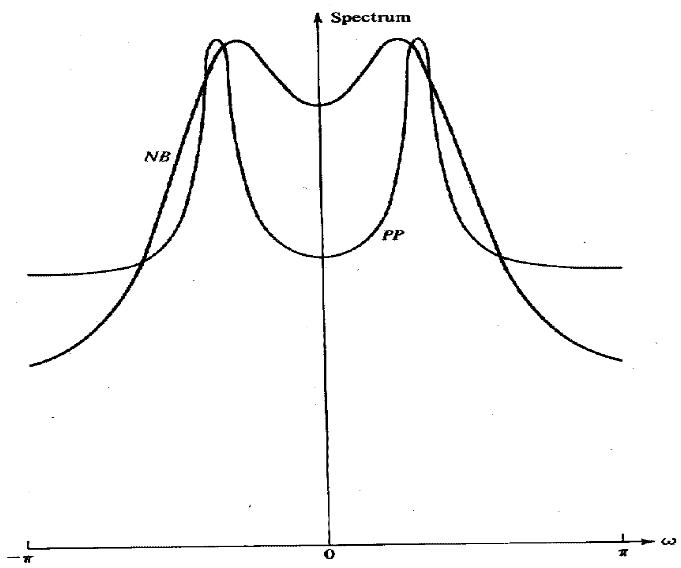


Figure 2.19: Amplitude vs. frequency for narrowband filter (NB) and pole-on-pedestal filter (PP). Each has one pole at $Z_0 = 1.2e^{i\pi/3}$. A second pole at $Z_0 = 1.2e^{-i\pi/3}$ enables the filters to be real in the time domain. $\boxed{\text{c2-2-19}}$ [NR]

Narrowband filters and sharp cutoff filters should be used with caution. An ever-present penalty for such filters is that they do not decay rapidly in time. Although this may not present problems in some applications, it will do so in others. Obviously, if the data collection duration is shorter or comparable to the impulse response of the narrowband filter, then the transient effects of starting up the experiment will not have time to die out. Likewise, the notch should not be too narrow in a 60-Hz rejection filter. Even a bandpass filter (easier to implement with fast Fourier transform than with a few poles) has a certain decay rate in the time domain which may be too slow for some experiments. In radar and in reflection seismology the importance of a signal is not related to its strength. Late-arriving echoes may be very weak, but they contain information not found in earlier echoes. If too sharp a frequency characteristic is used, then filter resonance from early strong arrivals may not have decayed sufficiently by the time that the weak late echoes arrive.

EXERCISES:

1 Consider a symmetric (nonrealizable) filter which passes all frequencies less than ω_0 with unit gain. Frequencies above ω_0 are completely attenuated. What is the rate of decay of amplitude with time for this filter?

2 Waves spreading from a point source decay in energy as the area on a sphere. The amplitude decays as the square root of energy. This implies a certain decay in time. The time-decay rate is the same if the waves reflect from planar interfaces. To what power of time t do the signal amplitudes decay? For waves backscattered to the source from point reflectors, energy decays as distance to the minus fourth power. What is the associated decay with time?

3 Discuss the use of the filter of Exercise 1 on the data of Exercise 2.

4 Design a single-pole, single-zero notch filter to reject 59 to 61 Hz on data which are sampled at 500 points per second.

2.8 THE BILINEAR TRANSFORM

Z transforms and Fourier transforms are related by the relations $Z = e^{i\omega}$ and $i\omega = \ln Z$. A problem with these relations is that simple ratios of polynomials in Z do not translate to ratios of polynomials in ω and vice versa. The approximation

$$-i\hat{\omega} \quad = \quad 2\frac{1-Z}{1+Z} \tag{2.42}$$

is easily solved for Z as

$$Z \quad = \quad \frac{1+\hat{\omega}/2}{1-\hat{\omega}/2} \tag{2.43}$$

These approximations are often useful. They are truncations of the exact power series expansions

$$-i\omega \quad = \quad -\ln e^{i\omega} \quad = \quad -\ln Z \quad = \quad 2\left[\frac{1-Z}{1+Z} + \frac{1}{3}\frac{(1-Z)^3}{(1+Z)^3} + \frac{1}{5}\cdots\right] \tag{2.44}$$

and

$$Z \quad = \quad e^{i\omega} \quad = \quad \frac{e^{i\omega/2}}{e^{-i\omega/2}} \quad = \quad \frac{1 + i\omega/2 + (i\omega/2)^2/2! + \cdots}{1 - i\omega/2 + (i\omega/2)^2/2! + \cdots} \tag{2.45}$$

For a Z transform $B(Z)$ to be minimum phase, any root Z_0 of $0 = B(Z_0)$ should be outside the unit circle. Since $Z_0 = \exp\{i[\mathrm{Re}(\omega_0) + i\,\mathrm{Im}(\omega_0)]\}$ and $\mid Z_0 \mid = e^{-\mathrm{Im}(\omega_0)}$, it means that for a minimum phase $\mathrm{Im}\,(\omega_0)$ should be negative. (In other words, ω_0 is in the lower half-plane.) Thus it may be said that $Z = e^{i\omega}$ maps the exterior of the unit circle to the lower half-plane. By inspection of Figures 2.20 and 2.21, it is found that the bilinear approximation (2.42) or (2.43) also maps the exterior of the unit circle into the lower half-plane.

	Z	$\omega = 2\pi n - i \ln Z$	$\hat{\omega} = 2i\dfrac{1-Z}{1+Z}$
A	1	$2\pi n + 0$	0
B	i	$2\pi n + \pi/2$	2
C	-1	$2\pi n + \pi$	$\pm\infty$
D	$-i$	$2\pi n - \pi/2$	-2
E	$\frac{1}{2}$	$2\pi n + .693i$	$i\frac{2}{3}$
F	2	$2\pi n - .693i$	$-i\frac{2}{3}$

Figure 2.20: Some typical points in the Z-plane, the ω-plane, and the $\hat{\omega}$-plane. $\boxed{\text{c2-2-20}}$ [NR]

Thus, although the bilinear approximation is an approximation, it turns out to exactly preserve the minimum-phase property. This is very fortunate because if a stable differential equation is converted to a difference equation via (2.42), the resulting difference equation will be stable. (Many cases may be found where the approximation of a time derivative by multiplication with $1 - Z$ would convert a stable differential equation into an unstable difference equation.)

A handy way to remember (2.42) is that $-i\omega$ corresponds to time differentiation of a Fourier transform and $(1 - Z)$ is the first differencing operator. The $(1 + Z)$ in the denominator gets things "centered" at $Z^{1/2}$

To see that the bilinear approximation is a low-frequency approximation, multiply top and bottom of (2.42) by $Z^{-1/2}$

$$\begin{aligned} -i\hat{\omega} &= 2\,\frac{Z^{-1/2} - Z^{1/2}}{Z^{-1/2} + Z^{1/2}} \\ &= -2i\,\frac{\sin\,\omega/2}{\cos\,\omega/2} \\ \hat{\omega} &= 2\,\tan\,\omega/2 \end{aligned} \tag{2.46}$$

Equation (2.46) implicitly refers to a sampling rate of one sample per second. Taking an arbitrary sampling rate Δt, the approximation (2.46) becomes

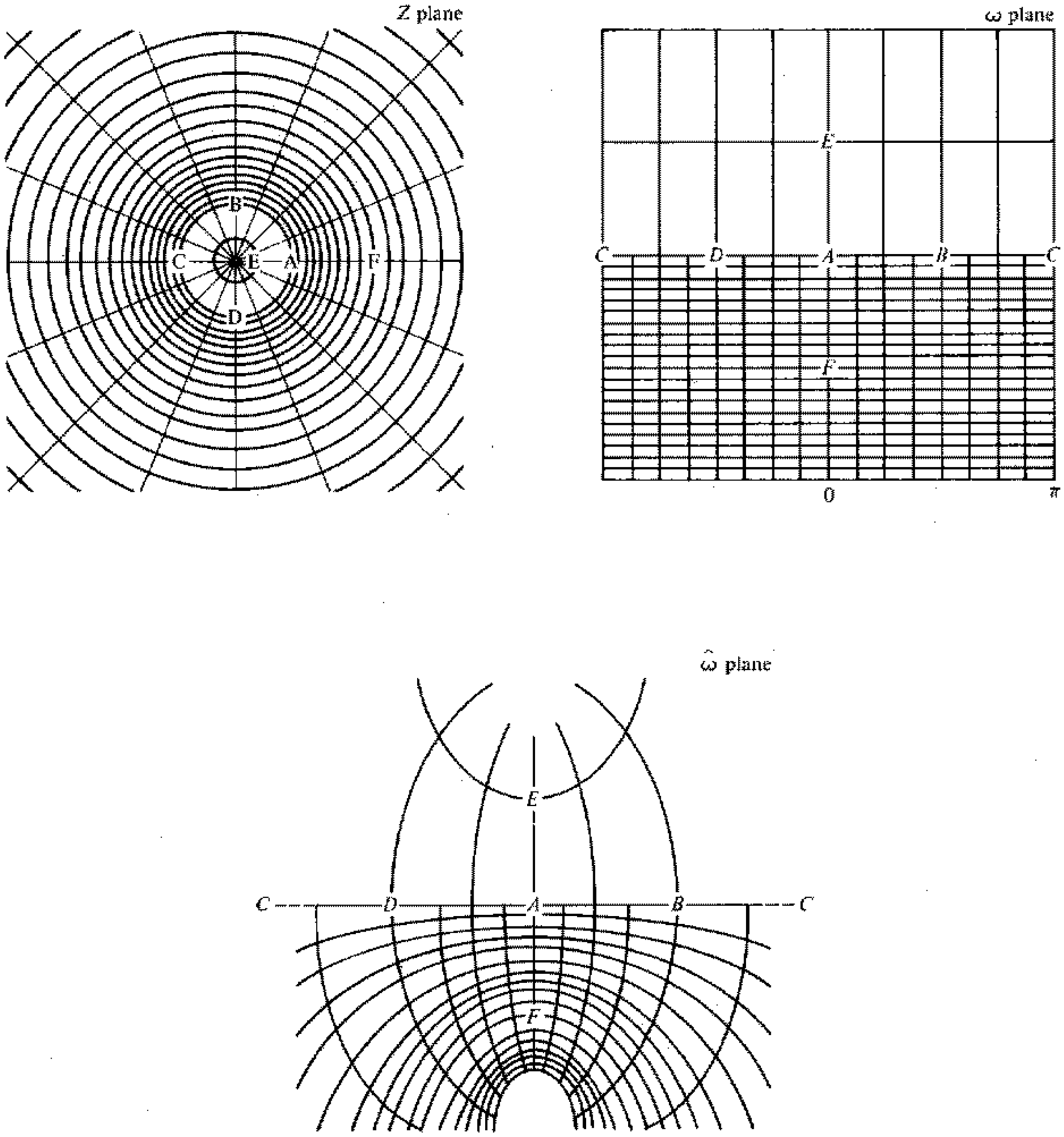


Figure 2.21: The points of Figure 2.20 displayed in the Z plane, the ω plane, and the $\hat{\omega}$-plane. $\boxed{\text{c2-2-21}}$ [NR]

$$\omega \Delta t \quad \approx \quad 2 \tan \omega \Delta t/2 \tag{2.47}$$

This approximation is plotted in Figure 2.22. Clearly, the error can be made as small as one wishes merely by sampling often enough; that is, taking Δt small enough.

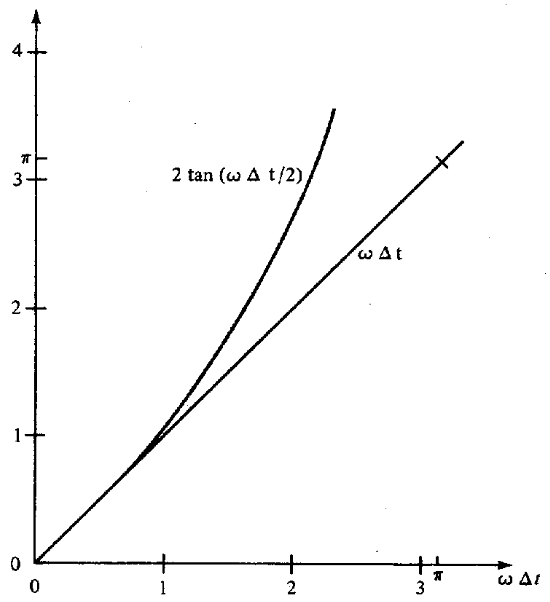


Figure 2.22: The accuracy of the bilinear transformation approximation. $\boxed{\text{c2-2-22}}$ [NR]

From Figure 2.22 we see that the error will be only a few percent if we choose Δt small enough so that $\omega_{\max}\Delta t \leq 1$. Readers familiar with the folding theorem will recall that it gives the less severe restraint $\omega_{\max}\Delta t < \pi$. Clearly, the folding theorem is too generous for applications involving the bilinear transform.

Now, by way of example, let us take up the case of a pole $1/-i\omega$ at zero frequency. This is integration. For reasons which will presently be clear, we will consider the slightly different pole

$$P \quad = \quad \frac{1}{-i\omega + \varepsilon} \tag{2.48}$$

where ε is small. Inserting the bilinear transform, we get

$$\begin{aligned} P \quad &= \quad \frac{1}{2[(1-Z)/(1+Z)] + \varepsilon} \quad = \quad \frac{0.5(1+Z)}{1 - Z + \varepsilon[(1+Z)/2]} \\ &= \quad \frac{0.5(1+Z)}{(1+\varepsilon/2) - Z(1-\varepsilon/2)} \end{aligned} \tag{2.49}$$

By inspection of (2.49) we see that the time-domain function is real, and as ε goes to zero it takes the form (.5, 1, 1, 1, ...). (Taking ε positive forces the step to go out into positive time, whereas ε negative would cause the step to rise at negative time.) The properties of this function are summarized in Figure 2.23.

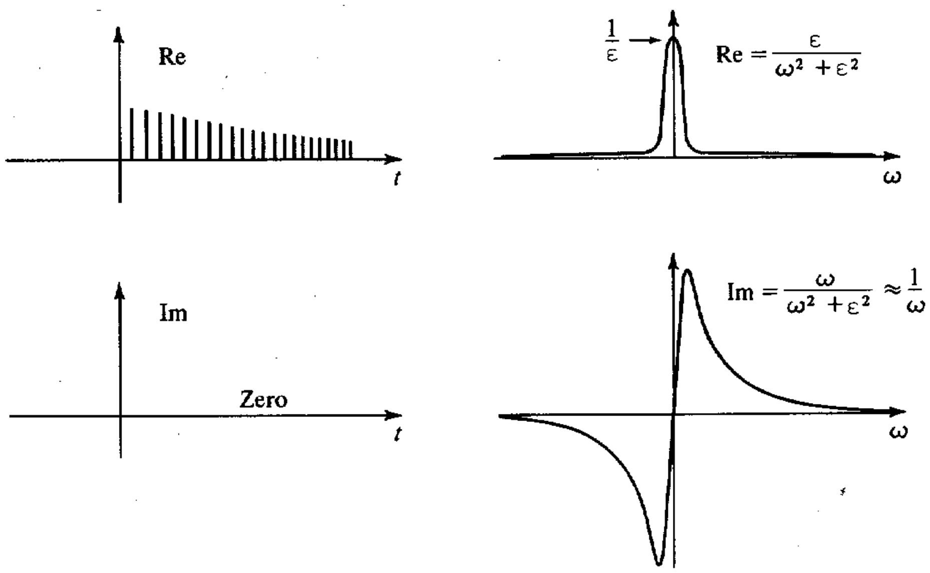


Figure 2.23: Properties of the integration operator. $\boxed{\text{c2-2-23}}$ [NR]

EXERCISES:

1 In the solution to diffusion problems, the factor $F(\omega) = 1/(-i\omega)^{1/2}$ often arises as a multiplier. To see the equivalent convolution operation, find a causal, sampled-time representation f_t of $F(\omega)$ by identification of powers of Z in

$$(f_0 + f_1 Z + f_2 Z^2 + \cdots)^2 \quad = \quad 1/(-i\omega) \quad \simeq \quad \frac{1}{2}(1+Z)/(1-Z) \tag{2.50}$$

Solve numerically for f_0 through f_7.