Chapter 1

Transforms

The first step in data analysis is to learn how to represent and manipulate waveforms in a digital computer. Time and space are ordinarily regarded as continuous, but for purposes of computer analysis we must discretize them. This discretizing is also called digitizing or sampling. Discretizing continuous functions may at first be regarded as an evil that is necessary only because our data are not always known analytic functions. However, after gaining some experience with sampled functions, one realizes that many mathematical concepts are easier with sampled time than with continuous time. For example, in this chapter, the concept of the Z transform is introduced and is shown to be equivalent to the Fourier transform. The Z transform is readily understood on a basis of elementary algebra, whereas the Fourier transform requires substantial experience in calculus.

1.1 SAMPLED DATA AND Z TRANSFORMS

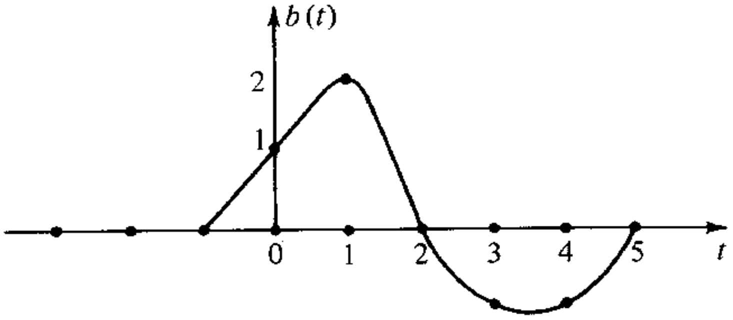

Figure 1.1: A continuous time function sampled at uniform time intervals. c1-1-1 [NR]

Consider the time function in Figure 1.1. To analyze such an observed time function in a computer it is necessary to approximate it in some way by a list of numbers. The usual way to do this is to evaluate or observe $b(t)$ at a uniform spacing of points in time. For this example, such a digital approximation to the continuous function could be denoted by the vector

$$b_t \quad = \quad (\ldots\ 0,\ 0,\ 1,\ 2,\ 0,\ -1,\ -1,\ 0,\ 0,\ \ldots) \tag{1.1}$$

Of course if time points were taken more closely together we would have a more accurate approximation. Besides a vector, a function can be represented as a polynomial where the *coefficients* of the polynomial represent the values of $b(t)$ at successive time points. In this example, we have

$$B(Z) \quad = \quad 1 + 2Z + 0Z^2 - Z^3 - Z^4 \tag{1.2}$$

This polynomial is called a Z transform. What is the meaning of Z in this polynomial? The meaning is not that Z should take on some numerical value; the meaning of Z is that it is the unit delay operator. For example the coefficients of $ZB(Z) = Z + 2Z^2 - Z^4 - Z^5$ are plotted in Figure 1.2. It is the same waveform as in Figure 1.1, but it has been delayed.

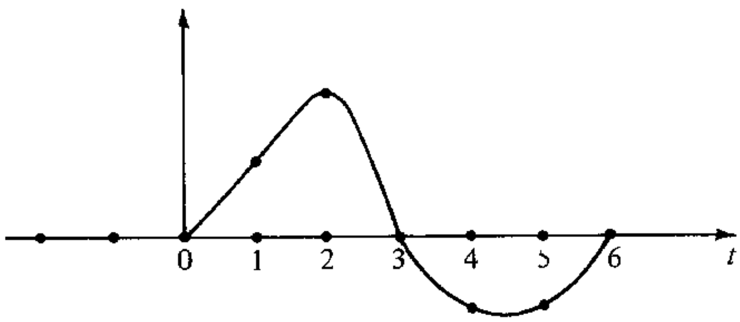

Figure 1.2: Coefficients of $ZB(Z)$ are shifted version of the coefficients of $B(Z)$ $\boxed{\text{c1-1-2}}$ [NR]

We see that the time function b_t is delayed n time units when $B(Z)$ is multiplied by Z^n. The delay operator Z is very important in analyzing waves simply because waves take a certain amount of time to get from place to place.

Another value of the delay operator is that it may be used to build up more complicated time functions from simpler ones. Suppose $b(t)$ represents the acoustic pressure function or the seismogram observed after a distant explosion. Then $b(t)$ is called the *impulse response.* If another explosion occurs at $t = 10$ time units after the first, we expect the pressure function $y(t)$ depicted in Figure 1.3.

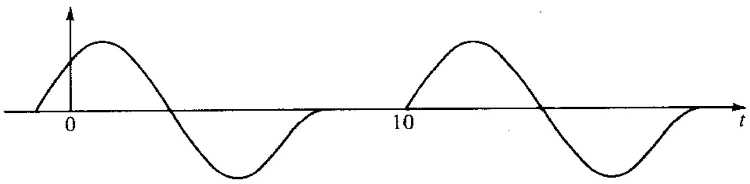

Figure 1.3: Response to two explosions. $\boxed{\text{c1-1-3}}$ [NR]

In terms of Z transforms this would be expressed as $Y(Z) = B(Z) + Z^{10}B(Z)$. If the first explosion were followed by an implosion of half strength, we would have $B(Z) - \frac{1}{2}Z^{10}B(Z)$. If pulses overlap one another in time [as would be the case if $B(Z)$

was of degree greater than 10], the waveforms would just add together in the region of overlap. The supposition that they just add together without any interaction is called the *linearity assumption*. This linearity assumption is very often true in practical cases. In seismology we find that—although the earth is a very heterogeneous conglomerations of rocks of different shapes and types—when seismic waves (of usual amplitude) travel through the earth, they do not interfere with one another. They satisfy linear superposition. The plague of nonlinearity arises from large amplitude disturbances. Nonlinearity does not arise from geometrical complications.

Now suppose there was an explosion at $t = 0$, a half-strength implosion at $t = 1$, and another, quarter-strength explosion at $t = 3$. This sequence of events determines a "source" time series, $x_t = (1, -\frac{1}{2}, 0, \frac{1}{4})$. The Z transform of the source is $X(Z) = 1 - \frac{1}{2}Z + \frac{1}{4}Z^3$. The observed y_t for this sequence of explosions and implosions through the seismometer has a Z transform $Y(Z)$ given by

$$\begin{aligned} Y(Z) &= B(Z) - \frac{Z}{2}\,B(Z) + \frac{Z^3}{4}\,B(Z) \\ &= \left(1 - \frac{Z}{2} + \frac{Z^3}{4}\right)\,B(Z) \\ &= X(Z)\,B(Z) \end{aligned} \tag{1.3}$$

The last equation illustrates the underlying basis of linear-system theory that the output $Y(Z)$ can be expressed as the input $X(Z)$ times the impulse response $B(Z)$.

There are many examples of linear systems. A wide class of electronic circuits is comprised of linear systems. Complicated linear systems are formed by taking the output of one system and plugging it into the input of another. Suppose we have two linear systems characterized by $B(Z)$ and $C(Z)$, respectively. Then the question arises whether the two combined systems of Figure 1.4 are equivalent. The use of Z transforms makes it obvious that these two systems are equivalent since products of polynomials commute, i.e.,

$$\begin{aligned} Y_1(Z) &= [X(Z)B(Z)]\,C(Z) = XBC \\ Y_2(Z) &= [X(Z)C(Z)]\,B(Z) = XCB = XBC \end{aligned} \tag{1.4}$$

Consider a system with an impulse response $B(Z) = 2 - Z - Z^2$. This polynomial can be factored into $2 - Z - Z^2 = (2 + Z)\,(1 - Z)$, and so we have the three equivalent systems in Figure 1.5. Since any polynomial can be factored, any impulse response can be simulated by a cascade of two-term filters (impulse responses whose Z transforms are linear in Z).

What do we actually do in a computer when we multiply two Z transforms together? The filter $2 + Z$ would be represented in a computer by the storage in memory of the coefficients $(2, 1)$. Likewise, for $1 - Z$ the numbers $(1, -1)$ are stored. The polynomial multiplication program should take these inputs and produce the

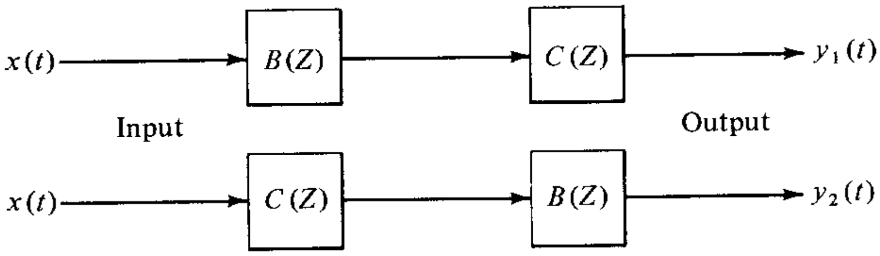

Figure 1.4: Two equivalent filtering systems. $\boxed{\text{c1-1-4}}$ [NR]

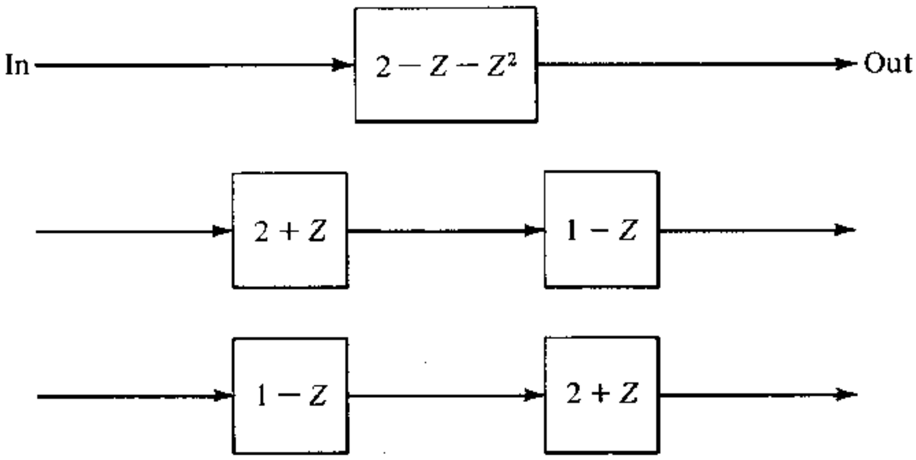

Figure 1.5: Three equivalent filtering systems. $\boxed{\text{c1-1-5}}$ [NR]

sequence $(2, -1, -1)$. Let us see how the computation proceeds in a general case, say

$$X(Z)\,B(Z) = Y(Z) \tag{1.5}$$

$$(x_0 + x_1 Z + x_2 Z + \cdots)\,(b_0 + b_1 Z + b_2 Z) = (y_0 + y_1 Z + y_2 Z^2 + \cdots) \tag{1.6}$$

Identifying coefficients of successive powers of Z, we get

$$\begin{aligned} y_0 &= x_0 b_0 \\ y_1 &= x_1 b_0 + x_0 b_1 \\ y_2 &= x_2 b_0 + x_1 b_1 + x_0 b_2 \\ y_3 &= x_3 b_0 + x_2 b_1 + x_1 b_2 \\ y_4 &= x_4 b_0 + x_3 b_1 + x_2 b_2 \end{aligned} \tag{1.7}$$

$$y_k = \sum_{i=0}^{2} x_{k-i} b_i \tag{1.8}$$

Equation (1.8) is called a *convolution equation.* Thus, we may say that the product of two polynomials is another polynomial whose coefficients are found by convolution. A simple Fortran computer program which does convolution, including end effects on both ends, is this:

```
DIMENSION X(LX), B(LB), Y(LY)
        DO 10 IY=1,LY
10      Y(IY) = 0.
        DO 20 IX=1,NX
        DO 20 IB=1,NB
        IY = IX+IB-1
20      Y(IY) = Y(IY) + X(IX)*B(IB)
```

The reader should notice that $X(Z)$ and $Y(Z)$ need not strictly be polynomials; they may contain both positive and negative powers of Z; that is

$$X(Z) = \cdots \frac{x_{-2}}{Z^2} + \frac{x_{-1}}{Z} + x_0 + x_1 Z + \cdots \tag{1.9}$$

$$Y(Z) = \cdots \frac{y_{-2}}{Z^2} + \frac{y_{-1}}{Z} + y_0 + y_1 Z + \cdots \tag{1.10}$$

The effect of using negative powers of Z in $X(Z)$ and $Y(Z)$ is merely to indicate that data are defined before $t = 0$. The effect of using negative powers of Z in the filter is quite different. Inspection of (1.8) shows that the output y_k which occurs at time k is a linear combination of current and previous inputs; that is, $(x_i,\ i \leq k)$. If the filter $B(Z)$ had included a term like b_{-1}/Z, then the output y_k at time k would be a linear combination of current and previous inputs and x_{k+1}, an input which really has not arrived at time k. Such a filter is called a *nonrealizable* filter because it could not operate in the real world where nothing can respond now to an excitation which has not yet occurred. However, nonrealizable filters are occasionally useful in computer simulations where all of the data are prerecorded.

EXERCISES:

1 Let $B(Z) = 1 + Z + Z^2 + Z^3 + Z^4$. Graph the coefficients of $B(Z)$ as a function of the powers of Z. Graph the coefficients of $[B(Z)]^2$.

2 If $x_t = \cos\omega_0 t$, where t takes on integral values, $b_t = (b_0, b_1)$, and $Y(Z) = X(Z)B(Z)$, what are A and B in $y_t = A\cos\omega_0 t + B\sin\omega_0 t$?

3 Deduce that, if $x_t = \cos\omega_0 t$ and $b_t = (b_0,\, b_1,\, \ldots,\, b_n)$, then y_t always takes the form $A\cos\omega_0 t + B\sin\omega_0 t$.

1.2 Z-TRANSFORM TO FOURIER TRANSFORM

We have defined the Z transform as

$$B(Z) = \sum_t b_t Z^t \tag{1.11}$$

If we make the substitution $Z = e^{i\omega}$ we have a "Fourier sum"

$$B(Z) \quad = \quad B(e^{i\omega}) \quad = \quad \sum_t b_t e^{i\omega t} \tag{1.12}$$

This is like a Fourier integral, and we could do a limiting operation to make it into an integral. Another point of view is that the Fourier integral

$$B(\omega) \quad = \quad \int_{-\infty}^{+\infty} b(t)\, e^{i\omega t}\, dt \tag{1.13}$$

reduces to the sum (1.12) when $b(t)$ is not a continuous function of time but is defined as

$$b(t) \quad = \quad \sum_k b_k\, \delta(t-k) \tag{1.14}$$

where δ is an impulse function.

In the last section we saw that to multiply two polynomials the coefficients must be convolved. The same process in Fourier transform language is that a product in the frequency domain corresponds to a convolution in the time domain.

Although one thinks of a Fourier transform as an integral which may be difficult or impossible to do, the Z transform is always easy, in fact trivial. To do a Z transform one merely attaches powers of Z to successive data points. When one has $B(Z)$ one can refer to it either as a time function or a frequency function, depending on whether one graphs the polynomial coefficients or if one evaluates and graphs $B(Z = e^{i\omega})$ for various frequencies ω. Notice that as ω goes from zero to 2π, $Z = e^{i\omega} = \cos\omega + i\sin\omega$ migrates once around the unit circle in the counterclockwise direction.

If taking a Z transform amounts to attaching powers of Z to successive points of a time function, then the inverse Z transform must be merely identifying coefficients of various powers of Z with different points in time. How can this simple "identification of coefficients" be the same as the apparently more complicated operation of inverse Fourier integrals? The inverse Fourier integral is

$$b(t) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} B(\omega)\, e^{-i\omega t}\, d\omega \tag{1.15}$$

First notice that the integration of Z^n about the unit circle or $e^{in\omega}$ over $-\pi \leq \omega < +\pi$ gives zero unless $n = 0$ because cosine and sine are oscillatory; that is,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\omega}\, d\omega \quad &= \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos n\omega + i \sin n\omega)\, d\omega \\ &= \quad \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = \text{non-zero integer} \end{cases} \end{aligned} \tag{1.16}$$

In terms of our discretized time functions, the inverse Fourier integral (1.15) is

$$b_t \quad = \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cdots + b_{-1} e^{-i\omega} + b_0 + b_1 e^{+i\omega} + \cdots)\, e^{-i\omega t}\, d\omega \tag{1.17}$$

Of all the terms in the integrand (1.17) we see by (1.16) that only the term with b_t will contribute to the integral; all the rest oscillate and cancel. In other words, it is only the coefficient of Z to the zero power which contributes to the integral, reducing (1.17) to

$$b_t \quad = \quad \frac{1}{2\pi} b_t \int_{-\pi}^{+\pi} d\omega \quad = \quad b_t \tag{1.18}$$

This shows how inverse Fourier transformation is just like identifying coefficients of powers of Z.

In this book and many others, it is common to assume that the time span between data samples $\Delta t = 1$ is unity. To adapt given equations to other values of Δt, one only need replace ω by $\omega \Delta t$; that is,

$$\omega_{\text{book}} \quad = \quad \omega_{\text{book}}\, \Delta t_{\text{book}} \quad = \quad \omega_{\text{true}}\, \Delta t_{\text{true}} \tag{1.19}$$

With Z transforms we have the spectrum given on a range of 2π for ω_{book}. In the limit Δt_{true} goes to zero, ω_{true} has the same infinite limits as the Fourier integral.

When a continuous function is approximated by a sampled function, it is necessary to take the sample spacing Δt_{true} small enough. The basic result of elementary texts is that, if there is no appreciable energy in a Fourier transform for frequencies higher than some frequency $\omega_{\max}$, then there is no appreciable loss of information if the sampled at least two points per wavelength. Figure 1.6 shows how insufficient sampling of a sine wave often causes it to appear as a sine wave of lower frequency.

If a high-frequency sinusoid is sampled insufficiently often, it becomes indistinguishable from a lower-frequency sinusoid. For this reason $\omega_{\max} = \pi/\Delta t$ is said to

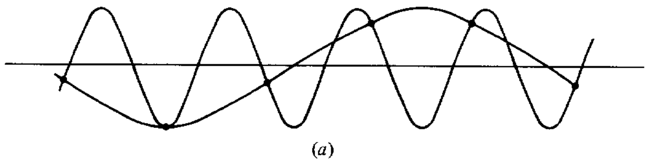

Figure 1.6: Subsampled sinusoid. $\boxed{\text{c1-1-7a}}$ [NR]

be the folding frequency, as higher frequencies are folded down to look like lower frequencies.

In practice, quasi-sinusoidal waves are always sampled more frequently than twice per wavelength.

Next we wish to examine odd/even symmetries to see how they are affected in Fourier transformation. The even part e_t of a time function b_t is defined as

$$e_t = \frac{b_t - b_{-t}}{2} \tag{1.20}$$

The odd part is

$$o_t = \frac{b_t - b_{-t}}{2} \tag{1.21}$$

A function is the sum of its even and odd parts. By adding (1-2-10) and (1-2-11), we get

$$b_t = e_t + o_t \tag{1.22}$$

Consider a simple, real, even time function such as $(b_{-1}, b_0, b_1) = (1, 0, 1)$. Its transform $Z + 1/Z = 2\cos\omega$ is an even function of ω since $\cos\omega = \cos(-\omega)$. Consider the real, odd time function $(b_{-1}, b_0, b_1) = (-1, 0, 1)$. Its transform $Z - 1/Z = 2(\sin\omega)/i$ is imaginary and odd, since $\sin\omega = -\sin(-\omega)$. Likewise, the transform of the imaginary even function $(i, 0, i)$ is the imaginary even function $i\cos\omega$ and the transform of the imaginary odd function $(-i, 0, i)$ is real and odd. Let r and i refer to real and imaginary, e and o refer to even and odd, and lower-case and upper-case refer to time and frequency functions. A summary of the symmetries of Fourier transformation is shown in Figure 1.7.

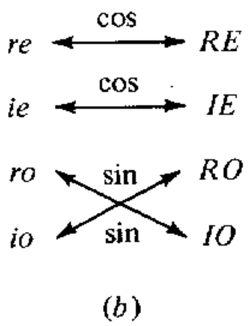

Figure 1.7: Mnemonic table illustrating how even/odd and real/imaginary properties are affected by Fourier transformation. $\boxed{\text{c1-1-7b}}$ [NR]

More elaborate time functions can be made up by adding together the two point functions we have considered. Since sums of even functions are even, and so on, the table of Figure 1.7 applies to all time functions. Note that an arbitrary time function takes the form $b_t = (\mathrm{re} + \mathrm{ro})_t + i(\mathrm{ie} + \mathrm{io})_t$. On transformation of b_t, each of the four individual parts transforms according to the table

EXERCISES:

1 Normally a function is specified entirely in the time domain or entirely in the frequency domain. When one is known, the other is determined by transformation. Now let us give half the information in the time domain by specifying that $b_t = 0$ for $t < 0$, and half in the frequency domain by giving the real part RE + RO in the frequency domain. How can you determine the rest of the function?

1.3 THE FAST FOURIER TRANSFORM

When we write the expression

$$B(Z) \quad = \quad b_0 + b_1 Z + b_2 Z^2 + \cdots \tag{1.23}$$

we have both a time function and its Fourier transform. If we plot the coefficients $(b_0, b_1, \ldots)$, we plot the time function. If we evaluate and plot (1.23) at numerous real ω, then we have plotted the transform. (Note that for real ω, Z is of unit magnitude; i.e., on the unit circle.) Since ω is a continuous variable and everything in a computer is finite, how do we select a finite number of values ω_k for plotting? The usual choice is to take evenly spaced frequencies. The lowest frequency can be zero. [Note $Z(\omega = 0) = e^{io} = 1$.] A frequency as high as $\omega = 2\pi$ [Note $Z(\omega = 2\pi) = e^{i2\pi} = 1$ also] need not be considered, since (1.23) gives the same value for it as for zero frequency. Choosing uniformly spaced frequencies between these limits we have

$$\omega_k \quad = \quad \frac{(0, 1, 2, \ldots, M-1)\, 2\pi}{M} \tag{1.24}$$

where M is some integer. Now let us abbreviate $B(Z(\omega_k))$ as B_k.

For the special case of an N-point time function where $N = 4$, (1.23) may be expressed by the matrix multiplication

$$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad = \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 \\ 1 & W^2 & W^4 & W^6 \\ 1 & W^3 & W^6 & W^9 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1.25}$$

where

$$W \quad = \quad e^{2\pi i/N} \tag{1.26}$$

It is not essential to choose $N = M$ as we have done in (1.25), but it is a convenience. There is no loss of generality because one may always append zeros to a time function before inserting it into (1.25). A convenience of the choice $N = M$ is that the matrix in (1.25) will then be square and there will be an exact inverse. In fact, the inverse to (1.25) may be easily shown to be

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = 1/N \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1/W & 1/W^2 & 1/W^3 \\ 1 & 1/W^2 & 1/W^4 & 1/W^6 \\ 1 & 1/W^3 & 1/W^6 & 1/W^9 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{1.27}$$

Since $1/W$ is the complex conjugate of W, the matrices of (1.25) and (1.27) are just complex conjugates of one another. In fact, one observes no fundamental mathematical difference between time functions and frequency functions. This "duality" would be even more complete if we had used a scale factor of $N^{-1/2}$ in each of (1.25) and (1.27) rather than 1 in (1.25) and N^{-1} in (1.27). Note also that time functions and frequency functions could be interchanged in the mnemonic table describing symmetries. In fact, our earlier observation that the product of two frequency functions amounts to a convolution of two time functions corresponds to the convolution of the corresponding two frequency functions. We will not "provide" this duality as it is standard fare in both mathematics and systems theory books. However we will occasionally call upon the reader to realize that in any theorem the meanings of "time" and "frequency" may be interchanged.

In making a plot of the transform B_k for $(k = 0, 1, \ldots, M-1)$, the frequency axis ranges as $0 \le \omega_k < 2\pi$. It is often more natural to display the interval $-\pi \le \omega < \pi$. Since the transform is periodic with period 2π, values of B_k on the interval $\pi \le \omega < 2\pi$ may simply be moved to the interval $-\pi \le \omega < 0$ for display.

Thus, for $N = 8$ one might plot successively

$$B_4 \quad B_5 \quad B_6 \quad B_7 \quad B_0 \quad B_1 \quad B_2 \quad B_3 \tag{1.28}$$

corresponding to values of ω equal to

$$-\pi, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \tag{1.29}$$

One advantage of this display interval is that for continuous time series which are sampled sufficiently densely in time the transform values B_k get small on both ends. If the time series is real, the real part of B_k has even symmetry about B_0; the imaginary part has odd symmetry about B_0. Then, one need not bother to display half the values. Choice of an odd value of N would enable us to put $\omega = 0$ exactly in the middle of the interval, but the reader will soon see why we stick to an even number of data points.

The matrix times vector operation in (1.25) requires N^2 multiplications and additions. The rest of this section describes a trick method, called the fast Fourier transform, of accomplishing the matrix multiplication in $N \log_2 N$ multiplications and additions. Since, for example, $\log_2 1024$ is 10, this is a tremendous saving in effort.

A basic building block in the fast Fourier transform is called doubling. Given a series $(x_0, x_1, \ldots, x_{N-1})$ and its sampled Fourier transform $(X_0, X_1, \ldots, X_{N-1})$ and another series $(y_0, y_1, \ldots, y_{N-1})$, one finds the transform of the interlaced double-length series

$$z_t \quad = \quad (x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1}) \tag{1.30}$$

The process of doubling is used many times during the process of computing a fast Fourier transform. As the word *doubling* might suggest, it will be convenient to suppose that N is an integer formed by raising 2 to some integer power. Suppose $N = 8 = 2^3$. We begin by dividing our eight-point series of one point each. The Fourier transform of each of the one-point series is just the point. Next, we use doubling four times to get the transforms of the four different two point series (x_0, x_4), (x_1, x_5), (x_2, x_6), and (x_3, x_7). We use doubling twice more to get the transforms of the two different four point series (x_0, x_2, x_4, x_6) and (x_1, x_3, x_5, x_7). Finally, we use doubling once more to get the transform of the original eight-point series $(x_0, x_1, x_2, \ldots, x_7)$.

It remains to look into the details of the doubling process.

Let

$$V \quad = \quad e^{i2\pi/2N} = W^{1/2} \tag{1.31}$$

$$V^N \quad = \quad e^{i\pi} = -1 \tag{1.32}$$

The transforms of two N-point series are by definition

$$X_k \quad = \quad \sum_{j=0}^{N-1} x_j V^{2jk} \quad (k = 0, 1, \ldots, N-1) \tag{1.33}$$

$$Y_k \quad = \quad \sum_{j=0}^{N-1} y_j V^{2jk} \quad (k = 0, 1, \ldots, N-1) \tag{1.34}$$

The transform of the interlaced series $z_j = (x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1})$ is by definition

$$Z_k = \sum_{l=0}^{2N-1} z_l V^{lk} \quad (k = 0, 1, \ldots, 2N-1) \tag{1.35}$$

To make Z_k from X_k and Y_k we require two separate formulas: one for $k = 0$, $1, \ldots, N-1$, and the other for $k = N, N+1, \ldots, 2N-1$.

First

$$Z_k = \sum_{l=0}^{2N-1} z_l V^{lk} \quad (k = 0, 1, \ldots, N-1)$$

We split the sum into two parts, noting that x_j multiplies even powers of V and y_j multiplies odd powers.

$$Z_k \quad = \quad \sum_{j=0}^{N-1} x_j V^{2jk} + V^k \sum_{j=0}^{N=1} y_j V^{2jk} \tag{1.36}$$

$$= \quad X_k + V^k Y_k \tag{1.37}$$

We obtain the last half of the Z_k by

$$Z_k = \sum_{l=0}^{2N-1} z_l V^{lk} \quad (k = N, N+1, \ldots, 2N-1) \tag{1.38}$$

$$= \sum_{l=0}^{2N-1} z_l V^{l(m+N)} \quad (k - N = m = 0, 1, \ldots, N-1) \tag{1.39}$$

$$= \sum_{l=0}^{2N-1} z_l V^{lm} (V^N)^l \tag{1.40}$$

$$= \sum_{l=0}^{2N-1} z_l V^{lm} (-1)^l$$

1.4 Phase delay and group delay

This material was revised and included in my third book, ESA:PVI.

1.5 Correlation and spectra

This material was revised and included in my third book, ESA:PVI.

1.6 Hilbert transform

This material was revised and included in my third book, ESA:PVI.

```
      SUBROUTINE FORK(LX,CX,SIGNI)
C FAST FOURIER                                          2/15/69
C                           LX
C  CX(K) = SQRT(1/LX) SUM (CX(J)*EXP(2*PI*SIGNI*I*(J-1)*(K-1)/LX))
C                           J=1               FOR K=1,2,...,(LX=2**INTEGER)
      COMPLEX CX(LX),CARG,CEXP,CW,CTEMP
      J=1
      SC=SQRT(1./LX)
      DO 30 I=1,LX
      IF(I.GT.J) GO TO 10
      CTEMP=CX(J)*SC
      CX(J)=CX(I)*SC
      CX(I)=CTEMP
 10   M=LX/2
 20   IF(J.LE.M) GO TO 30
      J=J-M
      M=M/2
      IF(M.GE.1) GO TO 20
 30   J=J+M
      L=1
 40   ISTEP=2*L
      DO 50 M=1,L
      CARG=(0.,1.)*(3.14159265*SIGNI*(M-1))/L
      CW=CEXP(CARG)
      DO 50 I=M,LX,ISTEP
      CTEMP=CW*CX(I+L)
      CX(I+L)=CX(I)-CTEMP
 50   CX(I)=CX(I)+CTEMP
      L=ISTEP
      IF(L.LT.LX) GO TO 40
      RETURN
      END
```

Figure 1.8: A program to do fast Fourier transform. Modified from Brenner. Calling this program twice returns the original data. SIGNI should be +1. on one call and −1. on the other. LX must be a power of 2. c1-1-8 [NR]