Hubbert math

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ABSTRACT

M. King Hubbert fits the growth and decay of petroleum production using the logistic function. The concepts may be expressed as four different equations. The four are stated here, then derived from one of them, thus showing they are equivalent. Hubbert's fitting function presumes the initial climb rate matches the ultimate decline rate. Most applications (including his) are better fit to distinguishing the two rates.

INTRODUCTION

Over the long haul, populations grow and decay. To describe the growth and decay of civilization's dependence on nuclear and fossil fuels, M. King Hubbert chose an equation that describes many natural processes. Introduce bacteria to food and their population will grow exponentially until there no longer is food. As we catch all the fish in the lake our daily catch will be proportional to the number of remaining fish. Hubbert's presumption that the two rates are the same led him to presume in 1956 that worldwide oil production would be peaking about about 2008. It did not.

Hubbert's math has four different expressions which we examine before showing they are mathematically equivalent.

Lastly, we recognize that most applications may be better fit by not requiring the growth rate to match the decline rate. Hubbert's expression is easily modified for the two different rates, suitable for two different mechanisms. Unfortunately, observing the asymptotic upslope tells little about the asymptotic downslope.

THE FOUR FORMS OF HUBBERT'S EQUATION

We define:

- t is time in years

- $Q(t)$ is cumulative production in billion barrels at year t.

- Q_∞ is the ultimate recoverable resource.

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[1]2008 paper updated in 2020

- $P(t) = dQ/dt$ is production in billion barrels/year at year t.

- τ is the year at which production peaks.

- ω is an inverse decay time (imaginary frequency).

The Hubbert's equation can be expressed in four forms. First, the differential form

$$\frac{dQ}{dt} = P = \omega \, Q \left(1 - \frac{Q}{Q_\infty}\right) \tag{1}$$

This equation is non-linear in Q but it reduces to familiar linear equations near the beginning and $Q \approx 0$ and near the end at $Q \approx Q_\infty$. As production begins and Q/Q_∞ is small, equation (1) reduces to $dQ/dt = \omega Q$ which displays exponential growth at a rate ω. As production ends near $Q \approx Q_\infty$ the non-linear equation reduces to exponential decay. To prove this fact change variables from Q to q where $Q = Q_\infty - q$ and evaluate the result at small q. Dividing equation (1) by Q we get the second form of the Hubbert equation sometimes called the *Hubbert Linearization*.

$$\frac{P}{Q} = \omega \, \left(1 - \frac{Q}{Q_\infty}\right) \tag{2}$$

The important thing about this equation is that it is linear in the two variables Q and P/Q. If you have historical measurements of P_i and Q_i, you can plot these points in the $(Q, P/Q)$-plane and hope for them to reasonably fit a straight line. Fitting the best line to the scattered points we can read the axis intercepts. At $Q = 0$ with equation (2) we can read off the value of the growth/decay parameter $\omega = (P/Q)_{\text{intercept}}$. For world oil, according to Deffeyes it is 5.3 percent/year. At the other intercept, $P/Q = 0$ we must have $Q = Q_\infty$. Again, according to Deffeyes, Q_∞ is two trillion barrels.

The third form of Hubbert's equation is the one best known. It looks like a Gaussian, but it isn't. (A Gaussian decays much faster.) The current production $P = dQ/dt$ is

$$P(t) = Q_\infty \, \omega \, \frac{1}{\left(e^{-(\omega/2)(\tau-t)} + e^{(\omega/2)(\tau-t)}\right)^2} \tag{3}$$

This is the equation of a blob, also known as "Hubbert's pimple", symmetric about the point $t = \tau$. Asymptotically it decreases (or increases) exponentially towards its maximum The function resembles a gaussian but exponential decay is much weaker than gaussian decay. Exponential growth is common in ecological systems which may also decay exponentially as resources are depleted or predator numbers grow exponentially.

All that remains is to figure out τ. The Hubbert curve is symmetrical and reaches its maximum when half the oil is gone. That happens when $Q = Q_\infty/2$. In the case of USA production which has passed its peak we can find the year that Q reached that value (about 1973). There is some debate about what year world production peaks, but general agreement is that it is about now (2008). Under Hubbert assumptions the decline curve is a mirror of the rise curve. That means we start down gently over the next decade, but about 25 years from now we hit the inflection point and see a 5 percent/year decline every year thereafter.

In real life there is no reason for the decay rate to match the growth rate. The decay could be faster because of horizontal drilling. The decay could be slower because we tax to conserve or successfully invest in technologies. As liquid oil depletes, society is switching to mining tar sands.

The Hubbert equation, in all its forms, follows as a consequence of the definition of the "logistic" function $Q(t)$. It ranges from 0 in the past to Q_∞ in the future.

$$Q(t) = \frac{Q_\infty}{1 + e^{\omega(\tau - t)}} \tag{4}$$

VERIFICATION THE FOUR FORMS ARE EQUIVALENT

If you buy the idea that your data scatter in $(Q_i, P_i/Q_i)$-space is a straight line, then you have bought equation (2). If you buy any one of equations (1),(2),(3), or (4), then you have bought them all because they are mathematically equivalent. Starting from the definition (4) using the rule from calculus that $d(1/v)/dt = -(dv/dt)/v^2$ yields equation (3).

$$\frac{dQ}{dt} = P(t) = Q_\infty \, \omega \, \frac{e^{\omega(\tau - t)}}{(1 + e^{\omega(\tau - t)})^2} \tag{5}$$

$$P(t) = Q_\infty \, \omega \, \frac{1}{(e^{-(\omega/2)(\tau - t)} + e^{(\omega/2)(\tau - t)})^2} \tag{6}$$

which is equation (3).

Equation (4) allows us to eliminate the denominator in equation (5) getting equation (2)

$$P/Q = (Q/Q_\infty) \, \omega \, e^{\omega(\tau - t)} \tag{7}$$

$$P/Q = (Q/Q_\infty) \, \omega \, ((1 + e^{\omega(\tau - t)}) - 1) \tag{8}$$

$$P/Q = (Q/Q_\infty) \, \omega \, (Q_\infty/Q - 1) \tag{9}$$

$$P/Q = \omega \, (1 - Q/Q_\infty) \tag{10}$$

which is equation (2). Multiplying both sides by Q gives equation (1).

DISTINGUISHING FALL FROM RISE

We grab equation (6), neglect $Q_\infty \omega$ and redefine the omegas to distinguish them.

$$P(t) \;=\; \frac{1}{\left(e^{\omega_+(t-\tau)} + e^{\omega_-(\tau-t)}\right)^2} \tag{11}$$

(The rise and fall rates are not Gaussian because $(e^{-\omega t})^2$ is $e^{-2\omega t}$ not $e^{-\omega t^2}$.)

A reader planning to model the coronavirus pandemic might adapt these ideas to a form where the pandemic remains endemic, namely that the prevalence does not return to zero, examples in Figures 1 and 2.

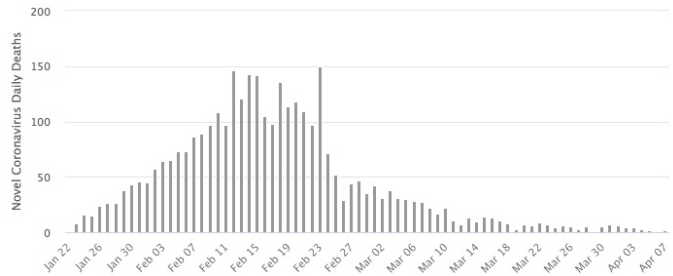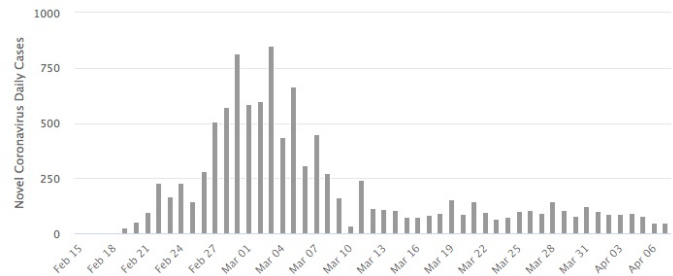Figure 1: Apparent pandemic example with China data

Figure 2: Apparent endemic example with Korean data

REFERENCE

1. http://www.hubbertpeak.com/hubbert/1956/1956.pdf contains Hubbert(1956, M.King Hubbert, Nuclear energy and fossil fuels, Publication 95, Shell Oil Company.

2. Kenneth S. Deffeyes, 2006, Beyond Oil: The view from Hubbert's Peak: Hill and Wang ISBN 0-8090-2957-X.