Invitation to Futterman inversion

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ABSTRACT

A constant Q earth model attenuates amplitude inversely with the number of wavelengths propagated, so the attenuation factor is $e^{-|\omega|(z/v)/Q}$. We call an impulse response in this model a Futterman wavelet. A collection of Futterman wavelets $\mathbf{F}$ for all depths is a seismogram modeling operator. Applying $\mathbf{F}^T$ to the data converts a collection of Futterman responses to a collection of symmetric autocorrelations. In this way it recovers event arrival time while (unfortunately) squaring the frequency response. We build a unitary operator that compensates Futterman phase response without changing the data spectrum. We build a quasi-analytic inverse that considers data precision while attempting to restore pulses that are late-arriving hence high frequency weak.

INTRODUCTION

Rocks are heterogeneous at all scales and they absorb seismic energy. A rule-of-thumb valid over many decades in frequency is that amplitude is reduced inversely with the number of wavelengths. Define travel time distance as distance over velocity $\tau = z/v$. Amplitude A follows a rule like $dA/d\tau = -|\omega|A$ which has the solution $A = e^{-|\omega|\tau}$. More correctly, a material quality property Q is needed so the spectrum is depleted by the amount $e^{-|\omega|\tau/Q}$. (Other authors may have slightly different definitions of Q.) This frequency function is plotted in the top line of Figure 1.

It is widely believed that material velocity has a hard limit, viscosity slows waves, never speeding them. Thus the change of an impulse after propagation must be spreading only in the direction of further delay. Moving with the wave, it fattens out only behind. The mathematical problem of finding such a function (called a causal function) given only a spectrum (such as $e^{-|\omega|\tau/Q}$) is subtle and was first solved only in the 20th century (most of the math we use being older). The causal response shape with spectrum $e^{-|\omega|\tau/Q}$ we call the Futterman wavelet. Along with working code, it may be found in GIEE. This waveform stretches in a self-similar manner with τ and inversely with Q. It is shown in Figure 1.

Use of the Futterman wavelet is not quite the usual process of filtering. It is not time-invariant as filtering normally is although the variation is slow. Deconvolution is the process of estimating a time-invariant waveform (called the "shot waveform") in field data. That time-invariant waveform is also present, but its estimation might be confounded [We don't know yet.] by the strong presence of the Futterman time-variant wavelet. Here I build tools for dealing with the Futterman wavelet.

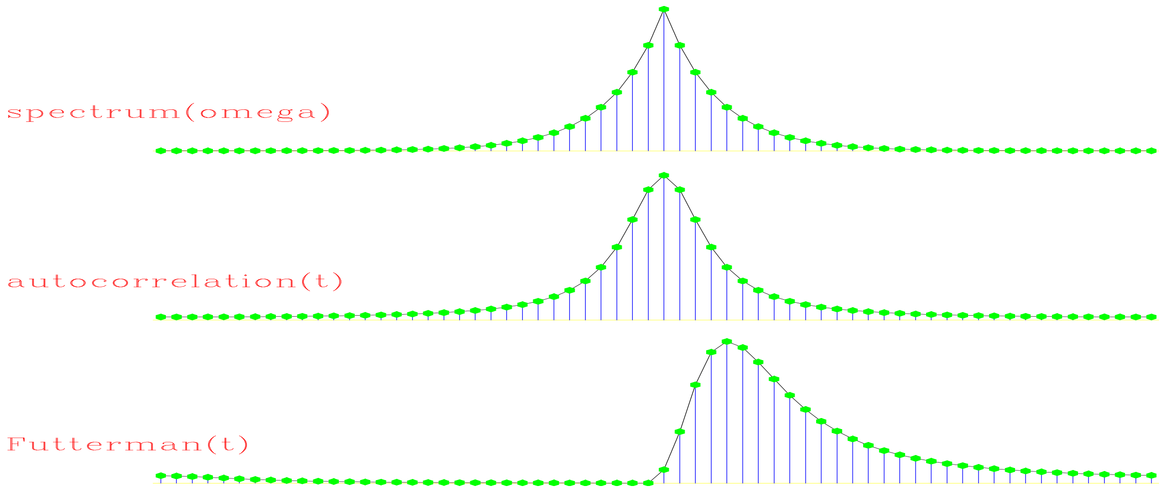

Figure 1: Autocorrelate the bottom signal to get the middle whose FT is the top. Spectral factorization (Kolmogoroff) works the other way, from top to bottom. [from GIEE page 102]

THE FUTTERMAN WAVELET

The middle function in Figure 1 is the autocorrelation that gives the constant-Q spectrum $e^{-|\omega|\tau/Q}$ (top). The third is the spectral factorization that is the Futterman wavelet. An impulse entering an absorptive medium comes out with this shape. It is causal. It begins off with a strong upward curvature and ends out with a broad and gentle downward sweep. A long wavelength (low frequency) cannot be packed in a small space, so we may say it is spread throughout the wavelet. The high frequencies live near the strong curvature at the beginning.

There is no physics in this analysis, only mathematics. A physical system could easily cause the factored wave to be more spread out (effectively by an additional all-pass filter), but physics cannot make it more compact because a long wavelength cannot be compacted into a small space. Physical systems might exist where viscosity adds to wave speed, but evidently they are rare, because I am not aware of them.

FUTTERMAN VERSUS RICKER

Water surface reflection at the gun and then again at the hydrophone convert what should be a single impulse to something more like $(1, -2, 1)$ or $(-1, +2, -1)$ or likewise with more blanks between the ones and twos. Hence Ricker proposed the second derivative of a Gaussian as being an all-purpose estimate of the marine seismic wave source. That led me to the attitude that the second derivative of any "blob" would be a likely candidate for an all-purpose estimate of a seismic wavelet. Figure 2 shows that the Futterman blob contradicts this idea. Actually, any nonsymmetric blob may have this effect.

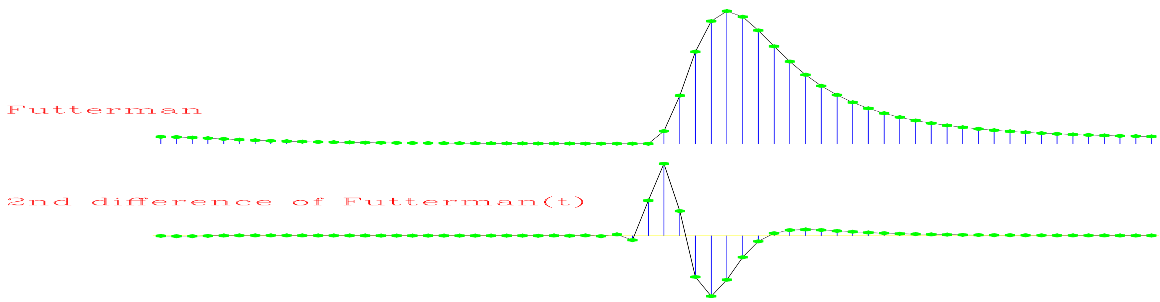

Figure 2: The Futterman wavelet and its second finite difference. This explains why the water bottom could seem a Ricker wavelet while the top of salt would seem a doublet. That third lobe is still there, but it's mighty small. And, the first two look about the same size.

FUTTERMAN BASIS FUNCTIONS

Figure 3 shows the matrix operator that transforms a reflectivity model $\mathbf{m}$ vector as a function of traveltime τ to data $\mathbf{d}$ as a funtion of time t. In each column hangs a Futterman wavelet. In the absence of absorption, it would be an identity matrix. Here a pulse in model space throws out a Futterman wavelet in data space. At later times the wavelets become emergent, thus the bright zone in Figure 3 appears slightly below the main diagonal though on the main diagonal it exists but is minuscle.

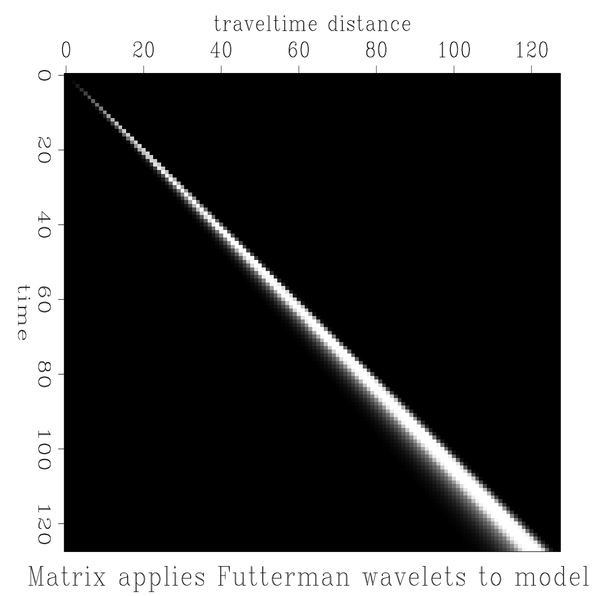

Figure 3: This Futterman matrix operator $\mathbf{F}$ converts a model space (function of traveltime depth) to a data space (function of time) $\mathbf{d} = \mathbf{Fm}$. For display columns are multiplied by t, the one-dimensional Kjartanson display gain.

Kjartansson gain

Setting the damping function $e^{-|\omega|\tau/Q}$ to any constant, say e^{-1} gives the trajectory $|\omega| = Q/\tau$. Kjartansson interpreted $|\omega|$ as a bandwidth dropping off as Q/τ. Thus, a pulse of infinite bandwidth drops off in strength as $1/\tau$ motivating data gaining proportional to time t with 1-D modeling (and t^2 on field data). I put $\sqrt{t}$ in both modeling and adjoint operators. This combination keeps amplitudes roughly constant in time. In Figure 3 I scaled the earth response by t, Kjartansson's one dimensional gain.

Futterman adjoint

Let $\mathbf{F}$ be a matrix similar to an identity matrix, but let the diagonal ones be replaced by constant Q earth responses from increasing depths. This matrix is shown graphically in Figure 3. Let $\mathbf{m}$ be an earth model with several widely spaced layers. Synthetic data is $\mathbf{d} = \mathbf{F}\mathbf{m}$. The synthetic data shown in Figure 4 is $\mathbf{d} = \mathbf{F}\mathbf{m}$ with episodic Futterman arrivals.

Before attempting to create $\mathbf{F}^{-1}$ we try out $\mathbf{F}^T$ and discover earth impulses are replaced by the autocorrelations of the depth dependent Futtermans. Application of the adjoint Futterman filter gives us an estimated model $\hat{\mathbf{m}} = \mathbf{F}^T\mathbf{d} = \mathbf{F}^T(\mathbf{F}\mathbf{m}) = (\mathbf{F}^T\mathbf{F})\mathbf{m}$. For any filter matrix $\mathbf{F}$ the matrix $\mathbf{F}^T\mathbf{F}$ has autocorrelations on the main diagonal. Thus impulsive signals in $\mathbf{m}$ become shaped into autocorrelations functions in $\hat{\mathbf{m}} = \mathbf{F}^T\mathbf{d}$ as is shown in Figure 4. In conclusion, peaks in the autocorrelations now time align with peaks in the underlying model. Hooray! Unfortunately, we now have squared the absorption spectrum—not ideal for data viewing.

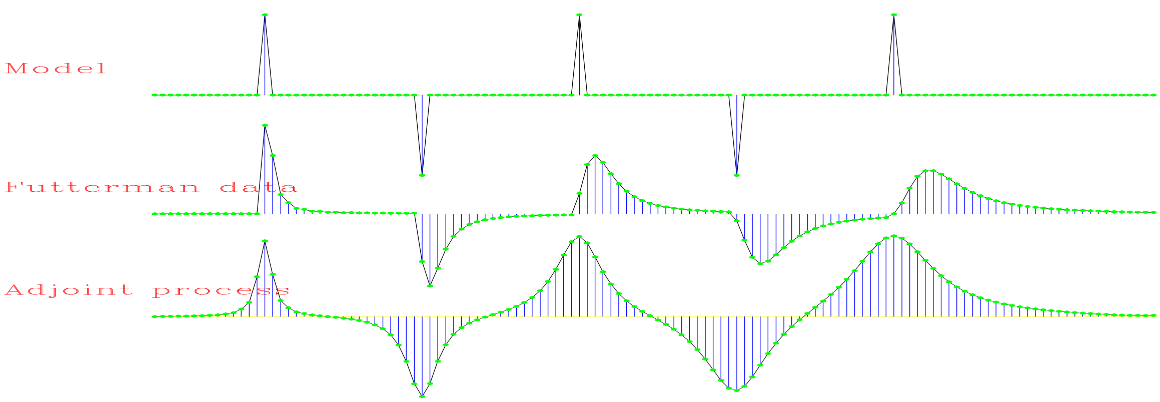

Figure 4: (top) A model space $\mathbf{m}$ with separated pulses in it. (next) A data space $\mathbf{d} = \mathbf{F}\mathbf{m}$. (next) A model estimate $\hat{\mathbf{m}}$ found by filtering the data with $\mathbf{F}^T$, so $\hat{\mathbf{m}} = \mathbf{F}^T\mathbf{d} = \mathbf{F}^T(\mathbf{F}\mathbf{m}) = (\mathbf{F}^T\mathbf{F})\mathbf{m}$. Matrix $\mathbf{F}^T\mathbf{F}$ has autocorrelations on its main diagonal. The autocorrelation peaks precisely align with model spikes.

MAKING ADJOINTS AND INVERSES

Given $\mathbf{F}$, it is easy to operate with adjoint $\mathbf{F}^T$, but there is another route that builds the operator $\mathbf{F}^T$ directly as well as enlightening us towards the building of operators that are suitable approximations to $(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T$ and better.

Delightful in practice will be a phase correction operator, one with the phase of $\mathbf{F}^T$ but without its ugly squaring the frequency spectrum. Imagine this: $(\mathbf{F}^T\mathbf{F})^{-1/2}\mathbf{F}^T$. Then we'll want something like $(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T$ but giving us control in the ugly struggle to recover frequencies that have been scattered, absorbed, or filtered away. Next we see how to achieve both.

Reach into the Kolmogoroff algorithm. Its final step is to exponentiate a complex something. After that for each frequency we have a complex number $|r|e^{i\phi} = e^{\ln|r|}e^{i\phi} = e^{\ln|r|+i\phi}$. In the Kolmogoroff program find that thing about to be exponentiated, $\ln|r| + i\phi$, and set its real part to zero. After exponentiating you will have a phase-only filter, a unitary operator. What phase do we want? We want the phase of the adjoint operator. Thinking about $e^{i\omega t}$ you understand that changing the sign of i is exactly like changing the sign of time t, like changing convolution to correlation. So, we need to change the polarity of the imaginary part of the complex number Kolmogoroff is about to exponentiate. The result is shown in Figure 5.

Inverse Futterman operator

First thoughts for the inverse Futterman operator tend to traditional approaches such as $\mathbf{F}^{-1} \approx (\mathbf{F}^T\mathbf{F} + \epsilon\mathbf{I})^{-1}\mathbf{F}^T$. But, another approach has something better than ϵ. We wish to specify an inverse filter $1/(|r|e^{i\phi}) = (1/|r|)e^{-i\phi} = e^{-\ln|r|-i\phi}$ given that we have its logarithm. To do this we need to take the negative of the real part and the negative of the imaginary part. That's theory. In practice there is more.

If nature is diminishing signals with $e^{-|\omega|\tau/Q}$, our inverse will be growing them back by $e^{+|\omega|\tau/Q}$. To add (or FT) numbers in single precision we cannot allow their range to exceed 10^6 or small numbers are the same as zeros. With field data we should likely keep the range under 1000. So, we need to scan the real parts of the logarithms about to be exponentiated and be sure their range does not exceed ln(1000). Let s be the smallest (most negative) value among the $\ln|r_i|$, namely $s = \min_i(\ln|r_i|)$. Then reassign the logarithms to limit their range, $\ln|r_i| \leftarrow \min(\ln|r_i|,\ s + \ln(1000))$.

Figure 5 shows that this pseudoinverse recovers the early two impulses very well, after which thresholding increasingly broadens impulses. Nature had taken energy away proportional to exponential $-|\omega|\tau$ and now we are boosting it back proportional to exponential $+|\omega|\tau$, so the thresholding takes effect at larger τ. I had anticipated setting the threshold at ln(1000) but used ln(40) instead, the likely reason being not data precision, but data truncation and wraparound in this tiny test case. Notice noise near the time origin. Recall the Futterman operator itself along with its adjoint required a gain of one power of t to bring later signals up to the same scale as the model. Now we see the converse, late arrivals when given more bandwidth will grow, so we compensate by scaling downward with $1/t$. A side effect is divergence at $t = 0$ boosting any noise there. Noise distributed along the time axis may also result from the sharp corner in the thresholding function of frequency. We should smooth it.

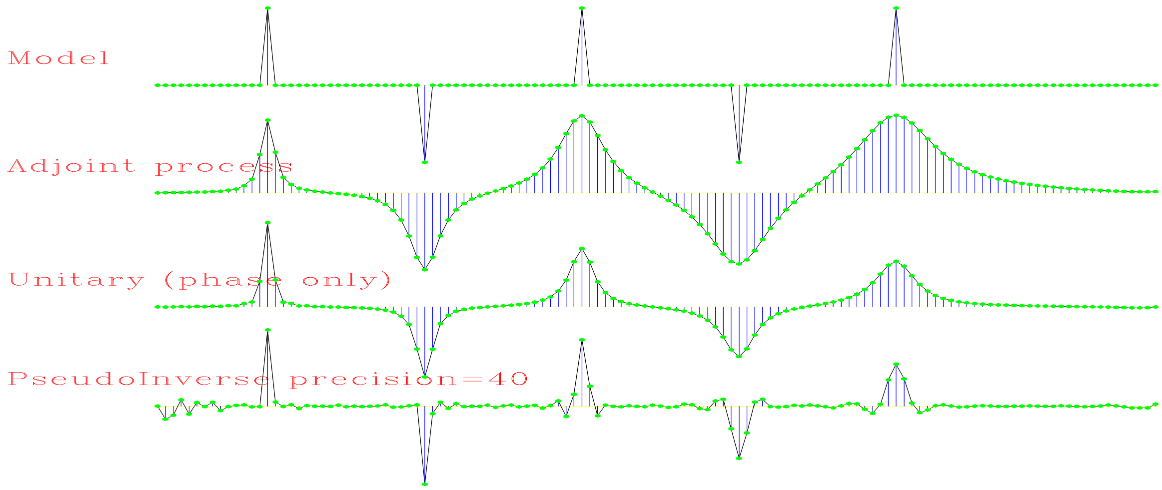

Figure 5: Successive approximations to the inverse Futterman operation applied to a spike model. The adjoint repairs the Futterman phase but squares the amplitude spectrum. The unitary repairs the phase without changing the data spectrum. The pseudoinverse attempts to recover the original spikes, failing on the later ones because specified precision does not permit restoring high frequencies at late time.

CONCLUSIONS AND FUTURE WORK

Do we have sufficient motivation to continue work with Futterman? Well, the time shifts implied by its phase are certainly nontrivial and surely affect velocity estimation. Very likely such shifts are on the same order as those accompanying anisotropy.

The most valuable direct result of this study is the unitary-adjoint Q correction (UAQC) process. (Elsewhere, it may be known as Futterman phase correction.) Secondly, this new inverse Q process may sound too bold to be practically useful, but having its precision cutoff turns it into what could become a practical workhorse. It's the natural lever for inching up high frequencies at late times.

But, even for constant Q I haven't yet come to definitive statements about Q estimation. That's embarrassing! Embarrassing again that Q estimation by spectral ratios may be the most practical method for estimating Q despite it not seemingly related to "inverse theory."

I have long felt that seismograms receive their essential color from the source waveform. But, now we see the source waveform is actually quite broad band and that the essential color in a seismogram arises from Q. I don't know why it took me until this old age to recognize that the Futterman functions are the natural basis functions for all seismograms.

I want seismic polarity revealing geology, not geophysical effects. My dream is to process seismic data such that correct polarity is readily apparent, meaning that effects of offset, Q, and shot waveform including ghost effects have been suitably removed.

When I was working on shot waveform, I always suspected, but could not confirm, that the greatest confounding aspect was divergence correction. Then I had not realized the powerful effect of Q. I'm wishing we had done Q correction before beginning with shot waveform estimation. I should go back and repeat the old shot-waveform studies doing UAQC first, even better, alternate between UAQC and shot waveform estimation.

The last big issue is offset. My earlier shot waveform work did not adequately incorporate offset, and it needs to. Luckily, Q fits pretty easily with offset. In a companion paper we open the door to nonstationary PEFs. Should they be helping deal with offset? The bubble effect is offset independent while the ghosts depend on offset in a fairly straightforward manner. At zero offset I separated them on on the basis of phase lag where ghosts where earlier than 60 ms and bubbles later. What is the next step?

BECOMING A COAUTHOR

Any of these should do it.

1. Assume NMO stretch is caused by shot waveform. Say how to suppress it.

2. Assume NMO stretch is caused by Futterman. Say how to suppress it.

3. How to better estimate shot waveforms by understanding the effect of Q?

4. How do Q and shot waveform affect various migration and velocity algorithms?

5. Wanting to routinely use the unitary operator on field data, how do we default the parameter Q?

6. What is the derivative of a seismogram with respect to Q?

7. Try estimating $Q(\tau)$ on field data.

8. Reorganize code here to be easily accessible with SEPLIB main program and linkable operators. Include zero padding. Verify the dot product test.

9. My code filters by matrix multiplication. But the matrix is really very sparse. How do we recode to take advantage of sparsity?

10. What parameters and defaults should SEP's production anti-Futterman code have for nonzero offset on land data? on marine data?

11. Put some sparse and semi-sparse models into Futterman and test inversion, and test prediction error, and test missing data estimation.

12. Our present basis functions for seismograms are impulses at all lags. But, we are oversampled at late times. Were we to switch to the model space of Futterman filters we would not need so many late time basis functions. Suppose a sparseness optimization program were able to turn off unneeded basis functions at late time, would the remaining ones correlate with nearby seismograms illuminating event slopes? or would we see only noise?

13. The shot waveform operator $\mathbf{S}$ is a time-invariant convolution matrix clustered fairly near the main diagonal (because the shot energy is within a few dozen millisec.) The Futterman operator $\mathbf{F}$ is also a filter matrix but it changes slowly down the main diagonal. These matrices approximately commute, but thinking of their non-commutivity (perhaps pedantically) one should come before the other. Describe migration or velocity or tomography environments in which you know which filter comes first?