

Poroelastic Shear Modulus Dependence on
Pore-Fluid Properties
Arising in a Model of Thin Isotropic Layers

James G. Berryman
University of California
Lawrence Livermore National Laboratory
P. O. Box 808 L-200
Livermore, CA 94551-9900

Abstract

Gassmann's fluid substitution formulas for bulk and shear moduli were originally derived for the quasi-static mechanical behavior of fluid saturated rocks. It has been shown recently that it is possible to understand deviations from Gassmann's results at higher frequencies when the rock is heterogeneous, and in particular when the rock heterogeneity anywhere is locally anisotropic. On the other hand, a well-known way of generating anisotropy in the earth is through fine layering. Then, Backus' averaging of the mechanical behavior of the layered isotropic media at the microscopic level produces anisotropic mechanical behavior at the macroscopic level. For our present purposes, the Backus averaging concept can also be applied to fluid-saturated porous media, and thereby permits us to study how deviations from Gassmann's predictions could arise in an elementary fashion. We consider both closed-pore and open-pore boundary conditions between layers within this model in order to study in detail how violations of Gassmann's predictions can arise. After evaluating a number of possibilities, we determine that energy estimates show unambiguously that one of our estimates – $G_{eff}^{(2)} = (C_{11} + C_{33} - 2C_{13} - C_{66})/3$ – is the correct one for our purposes. This choice also possesses the very interesting property that it is one of two sets of choices satisfying a product formula $6K_V G_{eff}^{(1)} = 6K_R G_{eff}^{(2)} = \omega_+ \omega_-$, where ω_{\pm} are eigenvalues of the stiffness matrix for the pertinent quasi-compressional and quasi-shear modes. K_R is the Reuss average for the bulk modulus, which is also the true bulk modulus K for the simple layered system. K_V is the Voigt average. For a polycrystalline system composed of simple layered systems randomly oriented at the microscale, K_V and K_R are the upper and lower bounds respectively on the bulk modulus, and $G_{eff}^{(2)}$ and $G_{eff}^{(1)}$ are the upper and lower bounds respectively on the G_{eff} of interest here. We find that $G_{eff}^{(2)}$ exhibits the expected/desired behavior, being dependent on the fluctuations in the layer shear moduli and also being a monotonically increasing function of Skempton's coefficient B of pore-pressure buildup, which is itself a measure of the pore fluid's ability to stiffen the porous material in compression.

1 Introduction

It has been shown recently (Berryman and Wang, 2001) that local anisotropy in heterogeneous porous media must play a significant role in deviations from the well-known fluid substitution formulas of Gassmann (Gassmann, 1951; Berryman, 1999). In particular, it is not easy to see in an explicit way how fluid dependence of the effective shear modulus G_{eff} of such systems can arise within a Gassmann-like derivation. Locally isotropic materials have been shown to be incapable of producing such results, even though such effects have been observed in experimental data (Berryman *et al.*, 2002a,b). So a simple theoretical means of introducing anisotropy into such a poroelastic system is desirable in order to aid our physical intuition about these problems.

When viewed from a point close to the surface of the Earth, the structure of the Earth is often idealized as being that of a layered (or laminated) medium with essentially homogeneous physical properties within each layer. Such an idealization has a long history and is well represented by famous textbooks such as Ewing *et al.* (1957), Brekhovskikh (1980), and White (1983). The importance of anisotropy due to fine layering (*i.e.*, layer thicknesses small compared to the wavelength of the seismic or other waves used to probe the Earth) has been realized

more recently, but efforts in this area are also well represented in the literature by the work of Postma (1955), Backus (1962), Berryman (1979), Schoenberg and Muir (1987), Anderson (1989), Katsube and Wu (1998), and many others.

In a completely different context, because of the relative ease with which their effective properties may be computed, finely layered composite laminates have been used for theoretical purposes to construct idealized but, in principle, realizable materials to test the optimality of various rigorous bounds on the effective properties of general composites. This line of research includes the work of Tartar (1976), Schulgasser (1977), Tartar (1985), Francfort and Murat (1986), Kohn and Milton (1986), Lurie and Cherkaev (1986), Milton (1986), Avellaneda (1987), Milton (1990), deBotton and Castañeda (1992), and Zhikov *et al.* (1994), among others. Recent books on composites by Cherkaev (2000), Milton (2002), and Torquato (2002) also make frequent use of these ideas.

In this work, we will study some simple means of estimating the effects of fluids on elastic and poroelastic constants and, in particular, we will derive formulas for anisotropic poroelastic media using a straightforward generalization of the method of Backus (1962) for determining the effective constants of a laminated elastic material. There has been some prior work in this area by Norris (1993), Gurevich and Lopatnikov (1995), Gelinsky *et al.* (1998), and others. However, our focus is specific to the issue of shear modulus dependence on pore fluids [see Mavko and Jizba (1991) and Berryman *et al.* (2002b)]. We initially review facts about elastic layered systems and then show that, of all the possible candidates for an effective shear modulus exhibiting mechanical dependence on pore fluids, the evidence shows that one choice is unambiguously preferred. We then use Backus averaging in the layered model to obtain an explicit formula for this shear modulus. The results agree with prior physical arguments indicating that fluid presence stiffens the medium in shear, but the layered material needs substantial inhomogeneity in its shear properties for the effect to be observed. An Appendix provides a simple derivation of some useful product formulas that arise in the analysis.

2 Notation for Elastic Analysis

In tensor notation, the relationship between components of stress σ_{ij} and strain $u_{k,l}$ is given by

$$\sigma_{ij} = C_{ijkl}u_{k,l}, \quad (1)$$

where C_{ijkl} is the stiffness tensor, and repeated indices on the right hand side of (3) are summed. In (1), u_k is the k th Cartesian component of the displacement vector \mathbf{u} , and $u_{k,l} = \partial u_k / \partial x_l$. Whereas for an isotropic elastic medium the stiffness tensor has the form

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (2)$$

depending on only two parameters (the Lamé constants, λ and μ), this tensor can have up to 21 independent constants for general anisotropic elastic media. The stiffness tensor has pairwise symmetry in its indices such that $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$, which will be used later to simplify the resulting equations.

The general equation of motion for elastic wave propagation through an anisotropic medium is given by

$$\rho \ddot{u}_i = \sigma_{ij,j} = C_{ijkl}u_{k,l,j}, \quad (3)$$

where \ddot{u}_i is the second time derivative of the i th Cartesian component of the displacement vector \mathbf{u} and ρ is the density (assumed constant). Equation (3) is a statement that the product of mass times acceleration of a particle is determined by the internal stress force $\sigma_{ij,j}$. For the present purposes, we are more interested in the quasistatic limit of this equation, in which case the left-hand side of (3) vanishes and the equation to be satisfied is just the force equilibrium equation

$$\sigma_{ij,j} = 0. \quad (4)$$

A commonly used simplification of the notation for elastic analysis is given by introducing the strain tensor, where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (5)$$

Then, using one version of the Voigt convention, in which the pairwise symmetries of the stiffness tensor indices are used to reduce the number of indices from 4 to 2 using the rules $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, 23 or $32 \rightarrow 4$, 13 or $31 \rightarrow 5$, and 12 or $21 \rightarrow 6$, we have

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & 2C_{44} & & \\ & & & & 2C_{55} & \\ & & & & & 2C_{66} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \quad (6)$$

Although the Voigt convention introduces no restrictions on the stiffness tensor, we have chosen to limit discussion to the form in (6), which is not completely general. Of the 36 coefficients (of which 21 are generally independent), we choose to treat only those cases for which the 12 coefficients shown (of which nine are generally independent) are nonzero. This form includes all orthorhombic, cubic, hexagonal, and isotropic systems, while excluding triclinic, monoclinic, trigonal, and some tetragonal systems, since each of the latter contains additional off-diagonal constants that may be nonzero. Nevertheless, we will restrict our discussion to (6) or to the still simpler case of transversely isotropic (TI) materials.

For TI materials whose symmetry axis is in the x_3 direction, another common choice of notation is $C_{11} = C_{22} \equiv a$, $C_{12} \equiv b$, $C_{13} = C_{23} \equiv f$, $C_{33} \equiv c$, $C_{44} = C_{55} \equiv l$, and $C_{66} \equiv m$. There is also one further constraint on the constants that $a = b + 2m$, following from rotational symmetry in the x_1x_2 -plane. In such materials, (6) may be replaced by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} a & b & f & & & \\ b & a & f & & & \\ f & f & c & & & \\ & & & 2l & & \\ & & & & 2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}, \quad (7)$$

in which the matrix has the same symmetry as hexagonal systems and of which isotropic symmetry is a special case (having $a = c = \lambda + 2\mu$, $b = f = \lambda$, and $l = m = \mu$).

3 Backus Averaging of Fine Elastic Layers

Backus (1962) presents an elegant method of producing the effective constants for a finely layered medium composed of either isotropic or anisotropic elastic layers. For simplicity, we will assume that the layers are isotropic, in which case the equation relating elastic stresses σ_{ij} to elastic strains e_{ij} is given by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ & \lambda & \lambda + 2\mu & & & \\ & \lambda & \lambda & \lambda + 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \\ & & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \quad (8)$$

The key idea presented by Backus is that these equations can be rearranged into a form where rapidly varying coefficients multiply slowly varying stresses or strains. For simple layering, we know physically (and can easily prove mathematically) that the normal stress and the tangential strains must be continuous at the boundaries between layers. If the layering direction is the z or x_3 direction as is the normal choice in the acoustics and geophysics literature, then σ_{33} , σ_{23} , σ_{31} , e_{11} , e_{22} , and e_{12} are continuous and in fact constant throughout such a laminated material. If the constancy of e_{11} , e_{22} , and e_{12} were not so, the layers would necessarily experience relative slip; while if the constancy of σ_{33} , σ_{23} , and σ_{31} were not so, then there would be force gradients across boundaries necessarily resulting in nonstatic material response to the lack of force equilibrium. By making use of this elegant idea, we arrive at the following equation

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ -e_{33} \\ e_{23} \\ e_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & \frac{2\lambda\mu}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & & & \\ \frac{2\lambda\mu}{\lambda+2\mu} & \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & & & \\ \frac{\lambda}{\lambda+2\mu} & \frac{\lambda}{\lambda+2\mu} & -\frac{1}{\lambda+2\mu} & & & \\ & & & \frac{1}{2\mu} & & \\ & & & & \frac{1}{2\mu} & \\ & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}, \quad (9)$$

which can be averaged essentially by inspection. Equation (9) can be viewed as a Legendre transform of the original equation, to a different set of dependent/independent variables in which both vectors have components with mixed physical significance, some being stresses and some being strains. Otherwise these equations are completely equivalent to the original ones in (8).

Performing the layer average using the symbol $\langle \cdot \rangle$, assuming as mentioned previously that the variation is along the z or x_3 direction, we find, using the notation of (7),

$$\begin{pmatrix} \langle \sigma_{11} \rangle \\ \langle \sigma_{22} \rangle \\ -\langle e_{33} \rangle \\ \langle e_{23} \rangle \\ \langle e_{31} \rangle \\ \langle \sigma_{12} \rangle \end{pmatrix} = \begin{pmatrix} \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle & \left\langle \frac{2\lambda\mu}{\lambda+2\mu} \right\rangle & \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & & & \\ \left\langle \frac{2\lambda\mu}{\lambda+2\mu} \right\rangle & \left\langle \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle & \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & & & \\ \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle & -\left\langle \frac{1}{\lambda+2\mu} \right\rangle & & & \\ & & & \left\langle \frac{1}{2\mu} \right\rangle & & \\ & & & & \left\langle \frac{1}{2\mu} \right\rangle & \\ & & & & & \langle 2\mu \rangle \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ e_{12} \end{pmatrix}$$

where, after some rearrangement of the resulting expressions, we have

$$K = f + \left(\frac{1}{\mu_1^*} + \frac{1}{2\mu_3^*} \right)^{-1}. \quad (18)$$

The new terms in (18) are defined by

$$2\mu_1^* \equiv a + b - 2f \quad \text{and} \quad 2\mu_3^* \equiv c - f. \quad (19)$$

and are measures of shear behavior in the simple layered system. We can write the formula (18) this way, or in another suggestive form

$$\frac{1}{K - f} = \frac{1}{a - m - f} + \frac{1}{c - f}, \quad (20)$$

in anticipation of results concerning various shear modulus measures that will be discussed at length in the next section.

4.2 Effective Bulk Modulus Estimates for Polycrystals

Assuming that the simple layered material we have been studying is present locally at some small (micro-)scale in a heterogeneous (macro-)medium, and also assuming that the axis of symmetry of these local constituents is randomly distributed so the whole composite medium is isotropic, then we have a polycrystalline material. The well-known results of Reuss and Voigt provide simple and useful estimates for moduli of polycrystals [actually lower and upper bounds on the moduli as shown by Hill (1952)].

4.2.1 Reuss average

The Reuss average is obtained by assuming constant stress, which is the same condition we applied already to estimate the bulk modulus K for the simple layered material. The well-known result in terms of compliances ($S = C^{-1}$) is

$$\frac{1}{K_R} = 2S_{11} + 2S_{12} + S_{33} + 4S_{13}. \quad (21)$$

It is straightforward to show that this produces exactly the same result as either (18) or (20). So $K_R = K$, which should be interpreted as meaning that the lower bound on the bulk modulus in the polycrystalline system is equal to the bulk modulus of the simple layered system.

4.2.2 Voigt average

The Voigt average is obtained with a constant strain assumption, and leads directly to the estimate in terms of stiffnesses

$$\begin{aligned} K_V &= \frac{1}{9} (2C_{11} + 2C_{12} + C_{33} + 4C_{13}) \\ &= [9f + 4(a - m - f) + (c - f)] / 9 = [9f + 4\mu_1^* + 2\mu_3^*] / 9, \end{aligned} \quad (22)$$

where the final equality makes use of the definitions from (19). It is well-known that $K_V \geq K_{eff} \geq K_R$ [see Hill (1952)].

For an isotropic system, the bulk modulus $K = \lambda + 2\mu/3$. The results (18) and (22) obtained for K_{eff} suggest that f plays the role of λ and that some combination or combinations of the constants μ_1^* and μ_3^* may play the role of the one nontrivial effective shear modulus G_{eff} for both the simple layered system and for the polycrystalline system. The combinations arising here are

$$G_{KR} \equiv \left[\frac{2}{3} \left(\frac{1}{\mu_1^*} + \frac{1}{2\mu_3^*} \right) \right]^{-1} \quad \text{and} \quad G_{KV} (2\mu_1^* + \mu_3^*) / 3, \quad (23)$$

the harmonic mean and mean, respectively, of μ_1^* and μ_3^* after having accounted for the duplication of μ_1^* in the system. We might anticipate (incorrectly!) that these two estimates of the magnitude the remaining shear response will be, respectively, the lowest and the highest that we will find. However, in fact both these estimates usually take lower values than the ones we study more carefully in the next section.

5 Effective Shear Modulus Estimates for Simple Layers and for Polycrystals

To understand the effective shear modulus G_{eff} and how to estimate it, we need first to introduce some facts about the eigenvalue structure of the elasticity matrices.

5.1 Singular value decomposition

The singular value decomposition (SVD), or equivalently the eigenvalue decomposition in the case of a real symmetric matrix, for (6) is relatively easy to perform. We can immediately write down four eigenvectors:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (24)$$

and their corresponding eigenvalues, respectively $2l$, $2l$, $2m$, and $a - b = 2m$. All four correspond to shear modes of the system. The two remaining eigenvectors must be orthogonal to all four of these and therefore both must have the general form

$$\left(1, 1, \Omega, 0, 0, 0 \right)^T \quad (25)$$

with corresponding eigenvalue

$$\omega = a + b + f\Omega. \quad (26)$$

The remaining condition that determines both Ω and ω is

$$\omega\Omega = 2f + c\Omega, \quad (27)$$

which, after substitution for ω , leads to a quadratic equation having the solutions

$$\Omega_{\pm} = \frac{1}{2} \left(\left[\frac{c-a-b}{f} \right] \pm \sqrt{8 + \left[\frac{c-a-b}{f} \right]^2} \right). \quad (28)$$

Then (26) and (28) imply that

$$\omega_+ \omega_- = (a+b)c - 2f^2 \quad (29)$$

and

$$\omega_+ + \omega_- = a + b + c = 2(a-m) + c, \quad (30)$$

which are two identities that will be used repeatedly later.

The ranges of values for Ω_{\pm} are $0 \leq \Omega_+ \leq \infty$ and, since $\Omega_- = -2/\Omega_+$, $-\infty \leq \Omega_- \leq 0$. The interpretation of the solutions Ω_{\pm} is simple for the isotropic limit where $\Omega_+ = 1$ and $\Omega_- = -2$, corresponding respectively to pure compression and pure shear modes. For all other cases, these two modes have mixed character, indicating that pure compression cannot be excited in the system, and must always be coupled to shear. Some types of pure shear modes can still be excited even in the nonisotropic cases, because the other four eigenvectors in (24) are unaffected by this coupling, and they are all pure shear modes. Pure shear (in strain) and pure compressional (in stress) modes are obtained as linear combinations of these two mixed modes according to

$$\alpha \begin{pmatrix} 1 \\ 1 \\ \Omega_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \Omega_- \\ 0 \\ 0 \\ 0 \end{pmatrix} = (1 + \alpha) \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (31)$$

with $\alpha = -2(\Omega_+ - 1)/[\Omega_+(\Omega_+ + 2)]$ for pure shear, and

$$\begin{pmatrix} 1 \\ 1 \\ \Omega_+ \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ \Omega_- \\ 0 \\ 0 \\ 0 \end{pmatrix} = (1 + \beta) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (32)$$

and with $\beta = \Omega_+(\Omega_+ - 1)/(\Omega_+ + 2)$ for pure compression.

To understand the behavior of Ω_+ in terms of the layer property fluctuations, it is first helpful to note that the pertinent functional $f(x) = \frac{1}{2} [-x + \sqrt{8 + x^2}]$ is easily shown to be a monotonic function of its argument x . So it is sufficient to study the behavior of the argument $x = (a + b - c)/f$.

5.1.1 Exact results in terms of layer elasticity parameters

Combining results from Eqs. (11)–(14), we find after some work on rearranging the terms that

$$\frac{a + b - c}{f} = \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^{-1} \left[\left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle - 6 \left\langle \frac{\Delta\mu}{\lambda + 2\mu} \right\rangle - 8 \left\{ \left\langle \frac{\mu^2}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle^2 \right\} \right], \quad (33)$$

where $\Delta\mu \equiv \mu - \langle \mu \rangle$ is the deviation of the shear modulus from the layer-averaged shear modulus m . Note that the term in curly brackets in (33) is in Cauchy-Schwartz form (*i.e.*, $\langle \alpha^2 \rangle \langle \beta^2 \rangle - \langle \alpha\beta \rangle^2 \geq 0$) and therefore is always non-negative. This term is also effectively quadratic in the deviations of μ from its layer average, and thus is of higher order than the term explicitly involving $\Delta\mu$. This fact, together with the fact that $\langle \Delta\mu/\mu \rangle = 1 - \langle \mu \rangle \langle 1/\mu \rangle \leq 0$, suggests that the dominant corrections to unity (since the leading term is exactly unity) for this expression will be positive if λ and μ are positively correlated throughout all the layers, but the correction could be negative in cases where there is a strong negative correlation between λ and μ . If the fluctuations in shear modulus are very large throughout the layered medium, then the quadratic terms can dominate, in which case the result could be less than unity. Numerical examples developed by applying a code of V. Grechka [used previously in a similar context by Berryman *et al.* (1999)] confirm (and, in fact, motivated) these analytical results.

Our main conclusion is that the shear modulus fluctuations giving rise to the anisotropy due to layering are (as expected) the main source of deviations of (33) from unity. But, there are some other more subtle effects present having to do with the interplay between λ , μ correlations as well as the strength of the μ fluctuations that ultimately determine the magnitude of the deviations of (33) from unity.

5.1.2 A fifth effective shear modulus?

From what has gone before, we know that there are four eigenvalues of the system that are easily identified with effective shear moduli (two are l and two are m). The bulk modulus K of the simple system is well-defined, and the bounds on K_{eff} for polycrystals are quite simple to apply and interpret. But we are still missing an important element of the overall picture of this system, and that is how the remaining degree of freedom is to be interpreted. It seems clear that it should be interpreted as an effective (quasi-)shear mode, since we have already accounted for the bulk mode. It is also clear that analysis of this remaining degree of freedom is not so easy because it is never an eigenfunction of the elasticity/poroelasticity tensor except in the cases that are trivial and therefore of interest to us here only as the isotropic baseline for comparisons.

Although it seems problematic to define a new shear modulus arbitrarily, we will now proceed to enumerate a number of possibilities and then, by a process of elimination, arrive at what appears to be a useful result.

5.2 Bounds for polycrystals

5.2.1 Reuss average for shear

The well-known Reuss result for shear modulus is

$$\begin{aligned} G_R^{-1} &= \frac{1}{15} (8S_{11} - 4S_{12} + 4S_{33} - 8S_{13} + 6S_{44} + 3S_{66}) \\ &= \frac{1}{15} \left(\frac{6}{l} + \frac{6}{m} + \frac{9f + 4(a - m - f) + (c - f)}{(a - m)c - f^2} \right), \end{aligned} \quad (34)$$

but

$$(a - m)c - f^2 = [(a - m - f) + (c - f)] K_R = [\mu_1^* + 2\mu_3^*] K_R, \quad (35)$$

again using the definitions from (19). Combining these results, we have

$$G_R^{-1} = \frac{1}{5} \left(\frac{2}{l} + \frac{2}{m} + [\mu_1^* + 2\mu_3^*]^{-1} \frac{3K_V}{K_R} \right) \quad (36)$$

Since the multiplicity of the shear modulus eigenvalues (l and m) is properly accounted for (2 and 2, respectively, out of 5), this result strongly suggests that one reasonable estimate of the fifth shear modulus for the system is

$$G_{eff}^{(1)} \equiv (\mu_1^* + 2\mu_3^*) \frac{K_R}{3K_V}. \quad (37)$$

5.2.2 Voigt average for shear

The Voigt average for shear modulus is

$$\begin{aligned} G_V &= \frac{1}{15} [2C_{11} - C_{12} + C_{33} - 2C_{13} + 6C_{44} + 3C_{66}] \\ &= \frac{1}{15} [(a - m - f) + (c - f) + 6l + 6m]. \end{aligned} \quad (38)$$

[Note that (38) corrects an error in equation (69) of Berryman and Wang (2001).] This result shows that the combinations $2(a - m - f)$ and $(c - f)$ again play the roles of twice a shear modulus contribution. By analogy to (36) and (37), we could define another effective constant

$$G_{eff}^{(2)} = [(a - m - f) + (c - f)]/3 = (\mu_1^* + 2\mu_3^*)/3. \quad (39)$$

This constant has a sensible dependence on these parameters and is consistent with the rest of our analysis (see *Discussion of G_{eff}* below), and also a somewhat simpler form than (37). Both this constant and $G_{eff}^{(1)}$ reappear in the later analyses. Because these two estimates are related to rigorous bounds, it seems that estimates not lying in the range from $G_{eff}^{(1)}$ to $G_{eff}^{(2)}$ can surely be excluded from consideration.

5.3 Estimates based on matrix invariants

5.3.1 Trace estimates

The trace of the stiffness matrix C is an invariant, and equals the sum of its eigenvalues. Similarly, the trace of the compliance matrix S is also invariant, and equals the sum of its eigenvalues, which are the inverses of the eigenvalues of C . These facts provide two more ways of obtaining estimates of the G_{eff} we seek.

Trace estimate from S

After eliminating the four eigenvalues associated with simple shear, the remainder of the trace of S is just the sum of the inverses of ω_+ and ω_- . To obtain an estimate of G_{eff} , we again make use of the known bulk modulus K and set

$$\frac{1}{3K} + \frac{1}{2G_{eff}^{(3)}} = \frac{1}{\omega_+} + \frac{1}{\omega_-}, \quad (40)$$

which can easily be shown to imply that

$$G_{eff}^{(3)} = (\mu_1^* + 2\mu_3^*) \frac{K_R}{3K_V} = G_{eff}^{(1)}. \quad (41)$$

So this compliance estimate again produces the same result found earlier in (37).

Trace estimate from C

Again eliminating the four eigenvalues associated with simple shear, the remainder of the trace of C is just the sum of ω_+ and ω_- . To obtain another estimate of G_{eff} , we make use of the known bulk modulus K as before and set

$$3K + 2G_{eff}^{(4)} = \omega_+ + \omega_-. \quad (42)$$

After some manipulation, we find

$$2G_{eff}^{(4)} = \frac{2(a - m - f)^2 + (c - f)^2}{(a - m - f) + (c - f)} = 2 \frac{(\mu_1^*)^2 + 2(\mu_3^*)^2}{\mu_1^* + 2\mu_3^*}. \quad (43)$$

So this estimate does not agree with any of the others, but it is nevertheless an interesting new combination of the shear modulus measures $(a - m - f)$ and $(c - f)$.

5.3.2 Determinant estimates

Another invariant of an elasticity matrix is its determinant, which is given by the product of its eigenvalues. Thus,

$$\det C = \omega_+ \omega_- (2l)^2 (2m)^2 \quad \text{and} \quad \det S = [\det C]^{-1}, \quad (44)$$

and so there is only one new estimate available based on this fact. In particular, if we assume that a reasonable estimate of G_{eff} might be obtained from this condition, then we would again

make use of the four known eigenvalues for shear, and the bulk modulus for the simple layered medium. Setting

$$\det C = (3K)(2G_{eff})(2l)^2(2m)^2, \quad (45)$$

comparing to (44), and recalling that $K = K_R$, we immediately find

$$2G_{eff}^{(5)} \equiv \frac{\omega_+\omega_-}{3K} = 2G_{eff}^{(2)}, \quad (46)$$

which is a repeat of an earlier estimate.

It is easy to show that $G_{eff}^{(5)} \leq G_{eff}^{(4)}$, and, since $G_{eff}^{(2)}$ was derived from the Voigt average for shear, now we should be able to exclude $G_{eff}^{(4)}$ safely from further consideration.

5.3.3 Relationship among estimates

The three estimates of G_{eff} found in this subsection can be related, by making use of the fact that $\omega_+ + \omega_- = \omega_+\omega_-(1/\omega_+ + 1/\omega_-)$. The main result is

$$\frac{G_{eff}^{(5)} - G_{eff}^{(3)}}{2G_{eff}^{(3)}} = \frac{G_{eff}^{(4)} - G_{eff}^{(5)}}{3K}, \quad (47)$$

which shows first of all (since $K > 0$ and $G_{eff}^{(3)} > 0$) that if any two of these constants are equal, then they are all equal. Eq. (47) also shows that the value of $G_{eff}^{(5)}$ always lies between the other two estimates, so in general $G_{eff}^{(4)} \geq G_{eff}^{(5)} \geq G_{eff}^{(3)}$, with equality holding only in the case of isotropic composites.

5.4 Energy estimates

The energy W of the elastic system can be written conveniently in either of two ways, based on strain and stress respectively:

$$\begin{aligned} 2W &= a(e_{11}^2 + e_{22}^2) + ce_{33}^2 + 2be_{11}e_{22} + 2fe_{33}(e_{11} + e_{22}) \\ &= S_{11}(\sigma_{11}^2 + \sigma_{22}^2) + S_{33}\sigma_{33}^2 + 2S_{12}\sigma_{11}\sigma_{22} + 2S_{13}\sigma_{33}(\sigma_{11} + \sigma_{22}). \end{aligned} \quad (48)$$

We can use these energy formulas to help decide whether any of the estimates obtained so far are actually fundamental quantities, by which we mean that they actually provide useful measures of the energy stored in the system. For example, it is well-known (as we have already discussed) that, if we set $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$, then

$$W = \sigma^2/2K, \quad (49)$$

where K is given by (20). Thus, even though K is not simply related to the eigenvalues of the system in general, it is still the fundamental measure of compressional energy in the simple layered system.

To check to see if any of the shear constants studied so far play a similar role for shear, we can set $e_{11} = e_{22} = -e_{33}/2 = e/\sqrt{6}$. Then, we find that

$$W = 2[(a - m - f) + (c - f)]e_{11}^2 = G_{eff}^{(2)}e^2. \quad (50)$$

So there is no ambiguity in the result for the shear energy. Clearly, $G_{eff}^{(2)}$ plays the same role for shear energy that K plays for bulk energy in this system, again regardless of the fact that it is not simply related to the eigenvalues.

5.5 Discussion of G_{eff}

Our goal is to obtain some new insight into the effective shear modulus of a poroelastic system in order to understand how the shear and bulk modes become coupled in such systems and thereby violate Gassmann's (1951) results for quasi-static systems having nonzero fluid permeability. The purpose of such an analysis will ultimately be to understand why some laboratory ultrasonics data show that the effective shear modulus of porous saturated and partially saturated rocks/systems has a substantial dependence on saturation when Gassmann's result would appear to deny the possibility of such behavior. It was demonstrated by Berryman and Wang (2001) that such deviations from Gassmann's predictions are expected, but they are surely not universal. For example, local elastic isotropy and spherical pores are two cases in which the shear modulus should remain independent of pore-fluid saturation.

For the transversely isotropic system arising from finely layered isotropic layers, we know that four out of five of the shear modes of the system are always independent of fluid saturation. They depend only on the Reuss and Voigt averages of the shear moduli present in the layered system. The remaining two modes are both of mixed character, not being pure shear or pure compression. So at some intellectual level it is clear that this coupling imposed through the eigenvalues is the reason for the shear wave dependence on fluid saturation. But this statement, although surely correct, is not really helpful in achieving our goal of understanding how the mixing of these effects happens down at the microscale. The best possible way to elucidate this behavior would be to make use of a formula for shear modulus (if one were known), containing the desired effects in it explicitly. But, a *rigorous formula* of this type is probably *not* going to be found. So, the next best option is to have in hand a formula that, although it is known to be approximate, still has the right structure and thereby permits analysis to proceed.

What is the right structure? The appropriate shear strain that contains all the effects of interest clearly is of the form $(1, 1, -2, 0, 0, 0)^T$. If we apply this shear strain to the stiffness matrix, the two distinct stresses generated are proportional to $2(a - m - f)$ [twice] and $(c - f)$ [once]. So the effective shear modulus we seek should depend on these two quantities, each of which acts like 2μ in the limit of an isotropic system. But in the nonisotropic cases of most interest, these combinations both include coupling between λ and μ of the layers through the Backus formulas — coupling that is good for our purposes. Furthermore, since there are two nontrivial constants, it is not obvious what combinations to choose for study. But, the preceding analysis shows that one likely candidate for the effective shear modulus is $G_{eff}^{(1)}$, since it appears naturally in two out of five of the main cases considered above. $G_{eff}^{(2)}$ also appears in two similar cases, as well as in the energy estimate.

The shear moduli $G_{eff}^{(1)}$ and $G_{eff}^{(2)}$ are clearly not eigenvalues; but the most likely candidate eigenvalue ω_- relevant for our study is even more difficult to analyze and interpret (both because

of its eigenvector’s mixed character and because of the complicated formula relating it to the elastic constants) than either $G_{eff}^{(1)}$ or $G_{eff}^{(2)}$. There are other choices that could be made, but we will give preference to $G_{eff}^{(1)}$ and $G_{eff}^{(2)}$ in the following discussion — both for definiteness and because they do seem to be useful constants to quantify and to help focus our attention.

6 Porous Elastic Materials Containing Fluids

Now we want to broaden our outlook and suppose that the materials composing the layers are not homogeneous isotropic *elastic* materials, but rather *poroelastic* materials containing voids or pores. The pores may be air-filled, or alternatively they may be partially or fully saturated with a liquid, some gas, or a fluid mixture. For simplicity, we will suppose here that the pores are either air-filled or they are fully saturated with some other homogeneous fluid. When the porous layers are air-filled, it is generally adequate to assume that the analysis of the preceding section holds, but with the new interpretation that the Lamé parameters are those for the porous elastic medium in the absence of saturating fluids. The resulting effective constants λ_{dr} and μ_{dr} are then said to be those for “dry” — or somewhat more accurately “drained” — conditions. These constants are also sometimes called the “frame” constants, to distinguish them from the constants associated with the solid materials composing the frame, which are often called the “grain” or “mineral” constants.

One simplification that arises immediately is due to the fact, according to a quasi-static analysis of Gassmann (1951), that the presence of pore fluids has no mechanical effect on the layer shear moduli, so $\mu_{dr} = \mu$. There may be other effects on the layer shear moduli due to the presence of pore fluids, such as softening of cementing materials or expansion of interstitial clays, which we will term “chemical” effects to distinguish them from the purely mechanical effects to be considered here. We neglect all such chemical effects in the following analysis. This means that the lamination analysis for the four effective shear moduli (l, l, m, m) associated with eigenvectors (since they are uncoupled from the analysis involving λ) does not change in the presence of fluids. Thus, equations (15) and (16) continue to apply for the poroelastic problem, and we can therefore simplify our system of equations in order to focus on the parts of the analysis that do change in the presence of fluids.

The presence of a saturating pore fluid introduces the possibility of an additional control field and an additional type of strain variable. The pressure p_f in the fluid is the new field parameter that can be controlled. Allowing sufficient time for global pressure equilibration will permit us to consider p_f to be a constant throughout the percolating (connected) pore fluid, while restricting the analysis to quasistatic processes. The change ζ in the amount of fluid mass contained in the pores [see Berryman and Thigpen (1985)] is the new type of strain variable, measuring how much of the original fluid in the pores is squeezed out during the compression of the pore volume while including the effects of compression or expansion of the pore fluid itself due to changes in p_f . It is most convenient to write the resulting equations in terms of compliances rather than stiffnesses, so the basic equation for an individual layer in the stack of

layers to be considered takes the form:

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{12} & -\beta \\ S_{12} & S_{11} & S_{12} & -\beta \\ S_{12} & S_{12} & S_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}. \quad (51)$$

The constants appearing in the matrix on the right hand side will be defined in the following two paragraphs. It is important to write the equations this way rather than using the inverse relation in terms of the stiffnesses, because the compliances S_{ij} appearing in (51) are simply related to the drained constants λ_{dr} and μ_{dr} in the same way they are related in normal elasticity, whereas the individual stiffnesses obtained by inverting the equation in (51) must contain coupling terms through the parameters β and γ that depend on the pore and fluid compliances. Thus, we find easily that

$$S_{11} = \frac{1}{E_{dr}} = \frac{\lambda_{dr} + \mu}{\mu(3\lambda_{dr} + 2\mu)} \quad \text{and} \quad S_{12} = -\frac{\nu_{dr}}{E_{dr}}, \quad (52)$$

where the drained Young's modulus E_{dr} is defined by the second equality of (52) and the drained Poisson's ratio is determined by

$$\nu_{dr} = \frac{\lambda_{dr}}{2(\lambda_{dr} + \mu)}. \quad (53)$$

When the external stress is hydrostatic so $\sigma = \sigma_{11} = \sigma_{22} = \sigma_{33}$, equation (51) telescopes down to

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \frac{1}{K_{dr}} \begin{pmatrix} 1 & -\alpha \\ -\alpha & \alpha/B \end{pmatrix} \begin{pmatrix} \sigma \\ -p_f \end{pmatrix}, \quad (54)$$

where $e = e_{11} + e_{22} + e_{33}$, $K_{dr} = \lambda_{dr} + \frac{2}{3}\mu$ is the drained bulk modulus, $\alpha = 1 - K_{dr}/K_m$ is the Biot-Willis parameter (Biot and Willis, 1957) with K_m being the bulk modulus of the solid minerals present, and Skempton's pore-pressure buildup parameter B (Skempton, 1954) is given by

$$B = \frac{1}{1 + K_p(1/K_f - 1/K_m)}. \quad (55)$$

New parameters appearing in (55) are the bulk modulus of the pore fluid K_f and the pore modulus $K_p^{-1} = \alpha/\phi K_{dr}$ where ϕ is the porosity. The expressions for α and B can be generalized slightly by supposing that the solid frame is composed of more than one constituent, in which case the K_m appearing in the definition of α is replaced by K_s and the K_m appearing explicitly in (55) is replaced by K_ϕ [see Brown and Korrington (1975), Rice and Cleary (1976), Berryman and Milton (1991), Berryman and Wang (1995)]. This is an important additional complication (Berge and Berryman, 1995), but one that we choose not to pursue here.

Comparing (51) and (54), we find easily that

$$\beta = \frac{\alpha}{3K_{dr}} \quad \text{and} \quad \gamma = \frac{\alpha}{BK_{dr}}. \quad (56)$$

All the constants are defined now in terms of “easily” measureable quantities.

When the mechanical changes (*i.e.*, applied stress or strain increments) in the system happen much more rapidly than the fluid motion can respond, the system is “undrained” — a situation commonly modeled by taking $\zeta = 0$ in the layer, $p_f = B\sigma$, and $e = (1 - \alpha B)\sigma/K_{dr}$. So the undrained bulk modulus of the layer is

$$K_u \equiv \frac{K_{dr}}{(1 - \alpha B)}. \quad (57)$$

Then, it is not hard to show that

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u \\ \lambda_u & \lambda_u + 2\mu & \lambda_u \\ \lambda_u & \lambda_u & \lambda_u + 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \end{pmatrix}, \quad (58)$$

where the undrained Lamé constant is given by

$$\lambda_u = K_u - \frac{2\mu}{3}. \quad (59)$$

For our purposes, the language used here to describe drained/undrained layers is intended to convey the same meaning as opened/closed pores at the boundaries separating the layers. If pores are open, fluid can move between layers, so pressure can equilibrate over long periods of time. If the pores are closed at these interfaces, pressure equilibration can occur within each layer, but not between layers — no matter how long the observation time.

With (58) and (59), we can now repeat the Backus analysis for a layered system in which each layer is undrained. The only difference is that everywhere λ appeared explicitly before, now λ_u is substituted. Furthermore, the constants resulting from the Backus lamination analysis can now be distinguished as a_u , b_u , c_u , and f_u for the undrained system. The constants without u subscripts are assumed to be drained, *i.e.*, $a = a_{dr}$, etc. The constants l and m are the same in both drained and undrained systems, since they do not depend on either λ or λ_u .

Carrying through this analysis for our main estimate $G_{eff}^{(2)}$, we find, after some rearrangement of terms, that

$$G_{eff}^{(2)} = m - \frac{4c_u}{3} \left[\left\langle \frac{\mu^2}{\lambda_u + 2\mu} \right\rangle \left\langle \frac{1}{\lambda_u + 2\mu} \right\rangle - \left\langle \frac{\mu}{\lambda_u + 2\mu} \right\rangle^2 \right]. \quad (60)$$

Equation (60) is the main result of this paper. It incorporates all the earlier work by making use of the effective shear modulus $G_{eff}^{(2)}$ selected in the end by using energy estimates. Then it is evaluated explicitly here for the simple layered system. The term in square brackets is in Cauchy-Schwartz form (*i.e.*, $\langle \alpha^2 \rangle \langle \beta^2 \rangle \geq \langle \alpha\beta \rangle^2$), and thus this term is always nonnegative. The contribution of this term is therefore a nonpositive correction to the leading term m . Further, it is clear that the correction is second order in the fluctuations in the shear modulus throughout the layered material. This fact means that fluctuations must be fairly large before any effect can be observed, since there are second order subtractions but no first order corrections at all. Finally, we note that the effect of an increase in the undrained constant λ_u is to reduce the magnitude of these correction terms. Thus, since a reduction in a negative contribution leads

to a positive contribution, it is also clear that an increase in Skempton’s coefficient B always leads to an increase in the effective shear modulus $G_{eff}^{(2)}$, as we expected.

Since each layer in the stack is isotropic, the result (60) may appear to contradict the earlier results of Berryman and Wang (2001), showing that shear modulus dependence on fluid properties does not and cannot occur in microhomogeneous isotropic media. But these layered media are not isotropic everywhere. In particular, if we consider a point right on the boundary between any two layers, the elastic properties look very anisotropic at these points. So it is exactly at these boundary points where deviations from Gassmann’s results can and do arise, leading to the result (60).

We will check these ideas by showing some numerical examples in the next section.

7 Numerical Examples

As we learned in the previous section, large fluctuations in the layer shear moduli are required before the predicted deviations from Gassmann’s quasi-static constant shear modulus result will become noticeable. To generate a model that demonstrates these results, I made use of a code of V. Grechka [used previously in a joint publication (Berryman *et al.*, 1999)] and then I arbitrarily picked one of the models that seemed to be most interesting for the present purposes. The parameters of this model are displayed in TABLE 1. The results of the calculations are shown in Figures 1 and 2.

The model calculations were simplified in one way, which is that the value of the Biot-Willis parameter was chosen to be a uniform value of $\alpha = 0.8$ in all layers. We could have actually computed a value of α from the other layer parameters, but to do so would require another assumption about the porosity values in each layer. Doing this seemed of little use because we are just trying to show in a simple way that the formulas given here really do produce the results predicted. Furthermore, if α is a constant, then it is only the product αB that matters. Since we are using B as the plotting parameter, whatever choice of constant $\alpha \leq 1$ is made, it mainly determines the maximum value of the product αB for B in the range $[0, 1]$. So, for a parameter study, it is only important not to choose too a small value of α , which is why the choice $\alpha = 0.8$ was made. This means that the maximum amplification of the bulk modulus due to fluid effects can be as high as a factor of 5 for the present example.

TABLE 1. Layer parameters for the three materials in the simple layered medium used to produce the examples in Figures 1 and 2.

<i>Constituent</i>	K (GPa)	μ (GPa)	z (m/m)
1	9.4541	0.0965	0.477
2	14.7926	4.0290	0.276
3	43.5854	8.7785	0.247

The results for bulk modulus in Figure 1 show that $K = K_R$ and $\omega_+/3$ are always quite close in value. The Voigt upper bound does bound both K_R and ω_+ from above as expected.

Furthermore, the simple Gassmann estimate based on the value of the drained constant ($B = 0$) gives a remarkably good fit to these estimates with no free fitting parameters. This good agreement with Gassmann may depend in part on our choosing α to be a constant. (Without this choice, a direct comparison in fact could not have been made here.)

The results for the effective shear moduli $G_{eff}^{(2)}$, $G_{eff}^{(1)}$, and the quasi-shear mode eigenvalue contribution $\omega_-/2$, show that $G_{eff}^{(2)} \simeq \omega_-/2$, although it actually does bound this parameter from above. The third shear modulus estimate $G_{eff}^{(1)}$ is far from the other two. We also show a constant value of shear modulus based on the Gassmann-like prediction of constant shear modulus equaling the drained modulus, but it is clear that this labeling is not fair to Gassmann as his result was for microhomogeneous materials, and therefore does not strictly speaking apply to layered materials at all. Nevertheless, we see that Gassmann's result for bulk modulus did a very creditable job of matching the bulk modulus values, even though it also was not a fair comparison for exactly the same reasons. This shows that Gassmann's results are much more sensitive to heterogeneity in shear modulus than they are to heterogeneity in the bulk modulus for these layered materials.

8 Conclusions

Our main conclusion is that the two best constants to study for our present purposes are the ones derived from the Reuss and Voigt bounds, $G_{eff}^{(1)}$ and $G_{eff}^{(2)}$. Furthermore, these estimates are naturally paired with the Voigt and Reuss bounds on bulk modulus through the product formulas (see the Appendix for a derivation)

$$(3K_V)(2G_{eff}^{(1)}) = (3K_R)(2G_{eff}^{(2)}) = \omega_+\omega_- . \quad (61)$$

The product formulas are true for both drained and undrained constants, but of course the numerical values of these constants differ in going from drained to undrained constants. Of these two estimates, $G_{eff}^{(2)}$ has the simplest form and further is paired with the Reuss bound on bulk modulus, which is actually the true bulk modulus of the simple (not polycrystalline) layered system. So we believe, based on theoretical considerations – especially using energy estimates, that this choice is worthy of special attention. But the numerical experiments show that $G_{eff}^{(1)}$ and $\omega_-/2$ have very similar values in the cases studied. So the practical advantages of this choice over the others may not be overwhelming.

Eq. (60) shows explicitly that $G_{eff}^{(2)} = m$ together with a generally negative correction whose magnitude depends strongly on fluctuations in layer shear modulus. The magnitude of these correction terms decreases as Skempton's coefficient B increases, so $G_{eff}^{(2)}$ is a monotonically increasing function of B . This is exactly the behavior we were trying to explicate in the present paper, so (60) is one example of the types of aid-to-intuition that we were seeking. The leading term in $G_{eff}^{(2)}$ is also easy to understand, as $G_{eff}^{(2)}$ was first obtained here using the Voigt average, which is an upper bound on the overall behavior; so it is natural that the leading term is m , which is the Voigt average of the μ 's in the layers.

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Appendix: Product Formulas

This Appendix will clarify the derivation of the product formulas relating the bulk modulus $K = K_R$ and $G_{eff}^{(2)}$. The product formula for K_V and $G_{eff}^{(1)}$ is just the dual, and is obtained in a very similar fashion.

Since the relevant excitation for $G_{eff}^{(2)}$ has been shown to be a shear strain proportional to $(1, 1, -2)^T$, consider

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & C_{13} \\ C_{13} & C_{13} & C_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} C_{11} + C_{12} - 2C_{13} \\ C_{11} + C_{12} - 2C_{13} \\ -2(C_{33} - C_{13}) \end{pmatrix}. \quad (62)$$

Then, by applying the inverse of the matrix in (62) to the left side of the equation, we get the useful formula:

$$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & S_{13} \\ S_{13} & S_{13} & S_{33} \end{pmatrix} \begin{pmatrix} C_{11} + C_{12} - 2C_{13} \\ C_{11} + C_{12} - 2C_{13} \\ -2(C_{33} - C_{13}) \end{pmatrix}, \quad (63)$$

which supplies two independent identities among the elastic coefficients. These are

$$1 = (S_{11} + S_{12})(C_{11} + C_{12} - 2C_{13}) - 2S_{13}(C_{33} - C_{13}) \quad (64)$$

and

$$-1 = S_{13}(C_{11} + C_{12} - 2C_{13}) - S_{33}(C_{33} - C_{13}). \quad (65)$$

Adding these together and switching to the a, b, c notation, we find

$$S_{11} + S_{12} + S_{13} = (S_{33} + 2S_{13}) \frac{c - f}{a + b - 2f}. \quad (66)$$

Recalling that

$$\frac{1}{K} = 2(S_{11} + S_{12} + S_{13}) + (S_{33} + 2S_{13}), \quad (67)$$

and then substituting (66), we find

$$\frac{1}{K} = (S_{33} + 2S_{13}) \frac{(a - m - f) + (c - f)}{a - m - f}. \quad (68)$$

Then, since

$$S_{33} + 2S_{13} = \frac{2(a - m - f)}{\omega_+ \omega_-}, \quad (69)$$

we find immediately that

$$6KG_{eff}^{(2)} = \omega_+\omega_-, \quad (70)$$

because $G_{eff}^{(2)} = [(a - m - f) + (c - f)]/3$.

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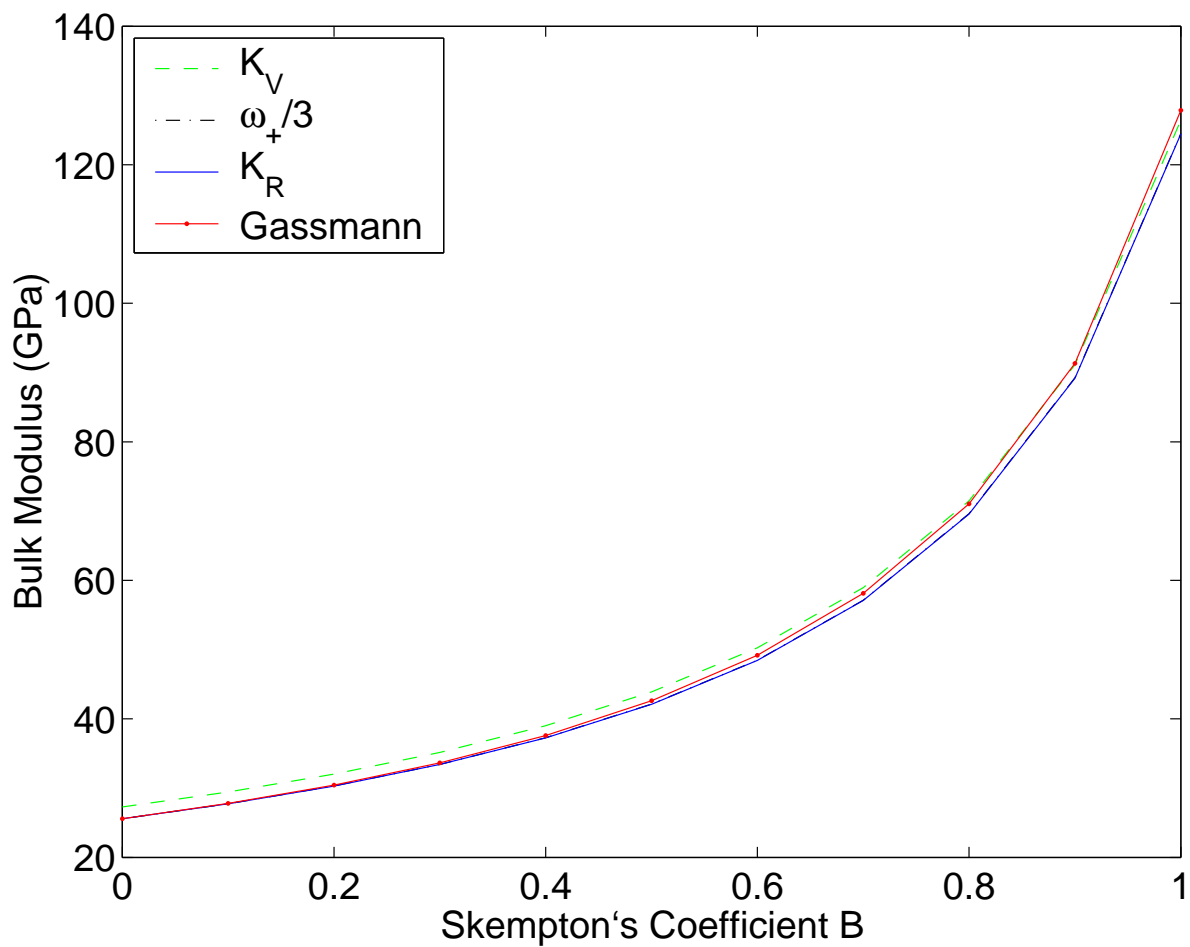


Figure 1: Bulk modulus as a function of Skempton's coefficient B . The Biot-Willis parameter was chosen to be $\alpha = 0.8$, constant in all layers.

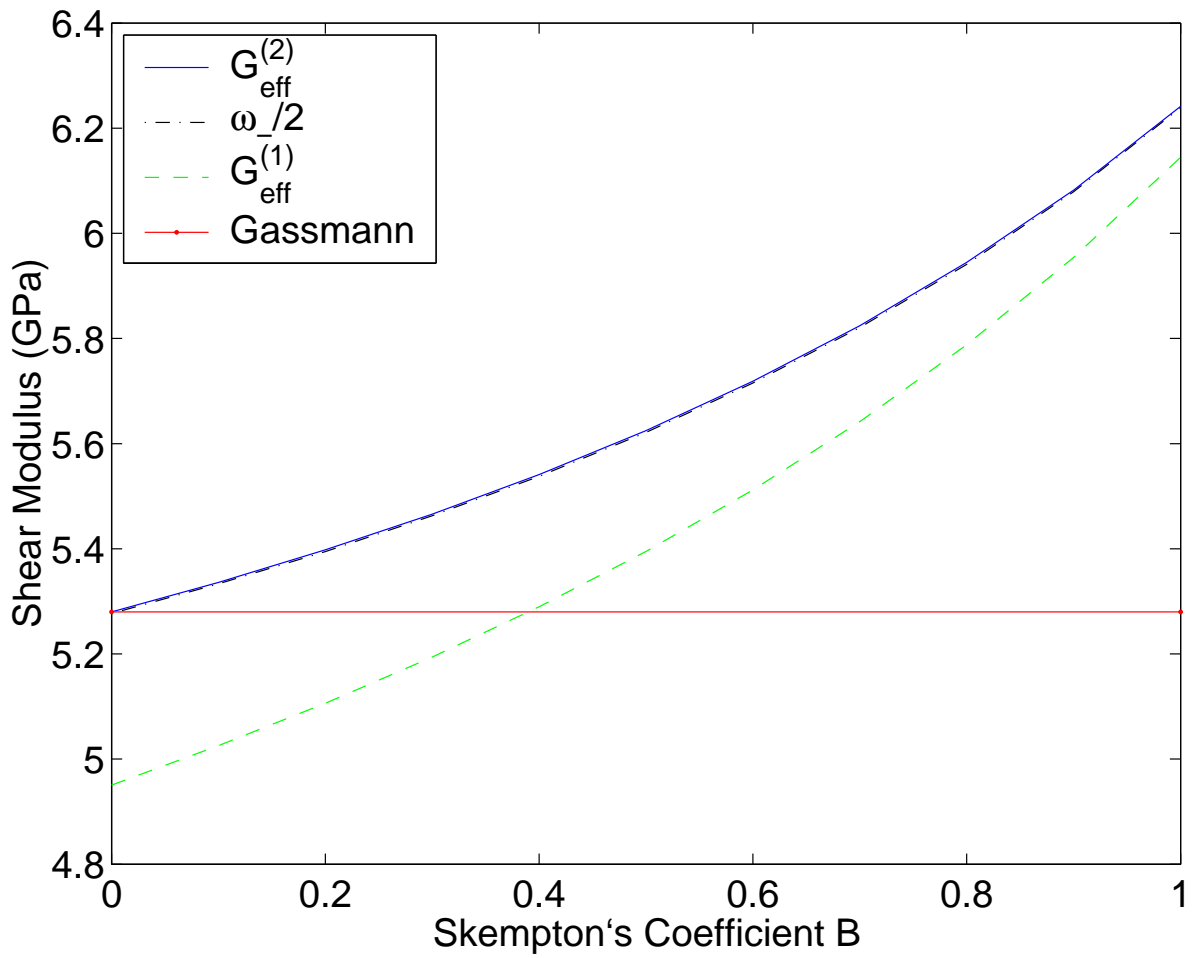


Figure 2: Shear modulus as a function of Skempton's coefficient B . The Biot-Willis parameter was chosen to be $\alpha = 0.8$, constant in all layers.