Effective Stress for Transport Properties of Inhomogeneous Porous Rock

James G. Berryman
Lawrence Livermore National Laboratory
P. O. Box 808 L-202
Livermore, CA 94551-9900

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Abstract

General effective-stress rules are derived for various physical properties of inhomogeneous porous rocks. Some rigorous relations arising in the analysis show that the fluid (pore) pressure p_f is least effective at counteracting the changes caused by confining pressure for the solid (grain) volume; p_f is more effective for the total (solid plus pore) volume; p_f is still more effective for the pore volume; and p_f is most effective at maintaining the fluid content of the pores. Although these results are expected intuitively, this analysis provides the first rigorous demonstration. During analysis of coefficients, care is taken to distinguish between rigorous inequalities (following from thermodynamics) and empirical inequalities (commonly observed, but not required by thermodynamics). For microscopically homogeneous rocks (the Gassmann limit), it is shown that the confining pressure is *always* at least as effective as the fluid pressure at changing the fluid permeability; therefore, it is impossible to use any "equivalent homogeneous rock" to explain experimental results of Zoback and Byerlee [1975] and others (wherein it has been shown experimentally that the permeability sometimes is more strongly influenced by fluid pressure than confining pressure). We show that the "equivalent homogeneous rock" paradigm may be successfully replaced by the "two-constituent porous medium" paradigm. In principle, the new paradigm can explain the data, but new measurements of pore compressibilities are required before quantitative comparisons can be made.

1 Introduction

When two or more strain producing fields may be applied independently to the same material, an important qualitative question often arises while analyzing experimental data: Which of the fields has greatest effect on a given physical property? If the material property of interest is found to be a linear function of each applied field, then another way of asking the same question is this: What linear combination of the fields (if any) will produce no measurable change in a physical property even though the strength of the fields themselves is changing? These questions lead naturally to the concept of effective stress [Terzaghi, 1936; Skempton, 1960; Terzaghi and Peck, 1967; Robin, 1973; Carroll, 1980].

Effective-stress measurements have been made on various physical properties of porous rocks. The most commonly measured feature is the compressional wave velocity [Brandt, 1955; Van der Knapp, 1959; King, 1966; Todd and Simmons, 1972; Nur, Walls, Winkler, and DeVilbiss, 1980; Whiting, 1982; Coyner, 1984; Christensen and Wang, 1985; Coyner and Cheng, 1985; Han, Nur, and Morgan, 1986]. Some data on shear wave velocities is also available [Gardner, Wyllie, and Droschak, 1965; Banthia, King, and Fatt, 1965; King, 1966; Nur, Walls, Winkler, and DeVIlbiss, 1980; Christensen and Wang, 1985; Coyner and Cheng, 1985]. Effective-stress coefficients for bulk and pore compressibilities were measured by Fatt [1959], Van der Knapp [1959], and Zimmerman, Somerton, and King [1986]. Experimental studies of electrical conductivity g or formation factor F have been performed as a function of confining pressure at fixed fluid pressure [Wyble, 1958; Dobrynin, 1962; Brace, Orange, and Madden, 1965; Brace and Orange, 1968a,b; Brace, 1972; Trimmer, Bonner, Heard, and Duba, 1980; Daily and Lin, 1984; Walsh and Brace, 1984; Longeron, Argaud, and Feraud, 1986], but little work seems to have been performed to determine the pertinent effective stress for this property [Dey, 1986]. A few studies of the effective stress for fluid permeability k are available [Brace, Walsh, and Frangos, 1968; Zoback, 1975; Zoback and Byerlee, 1975; Nur, Walls, Winkler, and DeVilbiss, 1980; Coyner, 1984; Bernabé, 1986; Dey, 1986; Bernabé, 1987], while many others are available for the behavior of k as a function of the confining pressure for fixed fluid pressure [Fatt and Davis, 1952; Fatt, 1953; McLatchie, Hemstock, and Young, 1958; Wyble, 1958; Ferrell, Felsenthal, and Wolfe, 1962; Knutson and Bohor, 1963; Vairogs, Hearn, Dareing, and Rhoades, 1971; Vairogs and Rhoades, 1973; Walsh and Brace, 1984]. Brace and Martin [1968] analyzed effective stress for fracture strength of brittle rocks. Related work on thermoelastic response of porous materials has been performed by Palciauskas and Domenico [1982], McTigue [1986], and Palciauskas and Domenico [1989].

Virtually all previous theoretical analyses of effective-stress relations for rocks [Nur and Byerlee, 1971; Carroll, 1980; Walsh, 1981; Zimmerman, Somerton, and King, 1986] have used the same restrictive assumption used by Gassmann [1951], postulating a microscopically homogeneous solid frame. Since natural rocks are often quite heterogeneous and therefore obviously do not satisfy the homogeneity condition, the validity of such analyses is founded on an implicit assumption that an "equivalent homogeneous rock" can be constructed and that the analysis of this fictitious homogeneous rock will satisfactorily explain all available data. However, we give a rigorous demonstration that effective-stress data on fluid transport through porous rocks cannot be explained in terms of any "equivalent homogeneous rock." This counterexample to common wisdom of composite science shows clearly that more sophisticated methods are required to explain the behavior of porous media.

We base our analysis not on specific models of porous media, but rather on very general scaling rules that such media must obey. For example, an insulating porous rock saturated with a conducting brine solution is known to have the conductivity $g = g_f/F$, where g_f is the conductivity of the brine and F is called the formation factor. Neglecting some small internal surface conduction effects, the formation factor is a bulk property depending only on the twisted shape of the internal pore space of the rock. Furthermore, F is a scale invariant property of the rock; if the rock and its pore space could be uniformly expanded or contracted everywhere, then neither the porosity ϕ nor the formation factor F would change. Although, in general, a change of confining pressure and fluid pressure in a rock does not produce uniform swelling or shrinking, there is one set of circumstances where this happens: Consider a microhomogeneous rock (one containing a single type of solid grain) with no change in differential (confining minus fluid) pressure. Then, it is well-known [Carroll, 1980] that this ideal rock does undergo a uniform expansion or contraction (implying constant ϕ and F), and therefore the formation factor of such a rock can only be a function of the change in differential pressure. Corresponding arguments for the bulk and shear moduli (both of which are also scale invariant properties) show that they must just be functions of the differential pressure, assuming only that the material bulk and shear constants K_m and μ_m for the grains do not change significantly as a function of the ambient pressure. We consider some elementary examples of the special results available in the Gassmann limit as prototypes, and then proceed to construct some more sophisticated examples for fluid permeability and porous mixtures using the same type of general arguments.

To present the simplest version of the analysis, we limit the scope of the paper mostly to a discussion of clean and clay-rich sandstones. This limitation is imposed by making a strong assumption of linearity in the differentials. By this, we mean that the coefficients of all differentials are assumed to change very slowly compared to the differentials themselves. Thus, highly fractured rocks – wherein a small change in applied pressure may produce a large change in the frame bulk modulus K – are excluded from consideration. For such materials, the linear regime is reached only at very high confining pressures, where all the fractures are essentially closed. Although the fundamental ideas presented here still apply to highly fractured materials, the mathematical description is sufficiently different (made more difficult by the nonlinearities) that we postpone their treatment to a later paper. A brief outline of the changes required in the analysis is given in the discussion section.

We begin by presenting the general stress-strain relations that must hold, as shown by Brown and Korringa [1975]. Then, we use arguments similar to those of Carroll [1980] to find the most general forms of the effective stress principles for porous materials. Next, we use a rigorous inequality derived by Berryman and Milton [1991] and Berryman [1992] to show that the more common effective-stress coefficients for volume properties satisfy a general set of inequalities among themselves. Then, we reconsider the homogeneous frame limit and determine the effective-stress coefficient for fluid permeability. This coefficient is proven always to be less than unity in the Gassmann limit, showing that experimental data such as that of Zoback and Byerlee [1975] with effective-stress coefficient greater than unity cannot be explained using any "equivalent homogeneous rock." Finally, using some exact results from Berryman and Milton [1991], we generalize the present analysis to materials composed of two types of porous components (for example, a clay and sand mixture) and derive effective-stress coefficients for both the formation factor and the fluid permeability. We show that Coyner's [1984] data on the jacketed and unjacketed frame moduli as a function of pressure can be understood in terms of an equivalent two-component porous rock. The paper concludes with a section discussing the results.

2 Stress-Strain Relations

Three bulk moduli characteristic of the porous frame are defined by Brown and Korringa [1975] using the expressions:

$$\frac{1}{K} = -\frac{1}{V}\left(\frac{\partial V}{\partial p_d}\right)_{p_f}, \tag{1}$$

$$\frac{1}{K_s} = -\frac{1}{V}\left(\frac{\partial V}{\partial p_f}\right)_{p_d}, \tag{2}$$

and

$$\frac{1}{K_\phi} = -\frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial p_f}\right)_{p_d}, \tag{3}$$

where V is the total sample volume, $V_\phi = \phi V$ is the pore volume, $p_c = -\frac{1}{3}\mathrm{Tr}(\tau) = -\frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz})$ is the confining (external) pressure, p_f is the fluid (pore) pressure, and $p_d = p_c - p_f$ is the differential pressure.

The first constant K is just the bulk modulus of the drained porous frame, a commonly measured quantity [Fatt, 1959; Van der Knapp, 1959; Brace, 1965; Nur and Byerlee, 1971; Coyner, 1984] sometimes called the "jacketed modulus." The second constant K_s (sometimes called the "unjacketed modulus") is also relatively easily measured [Fatt, 1959; Van der Knapp, 1959; Nur and Byerlee, 1971; Coyner, 1984], since it requires an observation of the change in total volume while the confining pressure and fluid pressure are incremented equally. If the frame is homogeneous (*i.e.*, composed of only one solid material), then this constant is equal to the bulk modulus K_m of its single constituent, so

$$K_s = K_m. \tag{4}$$

If the frame is inhomogeneous (*i.e.*, composed of two or more solids), then the constant K_s is certainly some average of the bulk moduli of the constituents. What that average should be is generally not known, but the Voigt-Reuss-Hill average [Hill, 1952] has often been employed in this context to provide estimates [Brace, 1965]. For a porous medium with just two distinct porous constituents having material moduli $K_m^{(1)}, K_m^{(2)}$ and frame moduli $K^{(1)}, K^{(2)}$, Berryman and Milton [1991] have shown that the correct average is

$$\frac{1}{K_s} = \frac{x^{(1)}}{K_m^{(1)}} + \frac{x^{(2)}}{K_m^{(2)}}, \tag{5}$$

where the weights satisfy

$$x^{(1)} = 1 - x^{(2)} = \frac{1/K^{(2)} - 1/K}{1/K^{(2)} - 1/K^{(1)}}. \tag{6}$$

Since elementary bounds on the bulk modulus K show it must lie between the moduli $K^{(1)}$ and $K^{(2)}$, it follows that the weights $x^{(1)}, x^{(2)}$ are both nonnegative and lie in the range $[0, 1]$. Thus, K_s is truly a weighted average of the material moduli, albeit an unusual one. The third constant K_ϕ is more difficult to measure than the other two, since it involves determining the change in the pore volume while the confining pressure and fluid pressure are incremented equally. A few attempts at measuring K_ϕ have been made by Hall [1953], Greenwald [1980], and Green and Wang [1986]; however, all of these reported results are subject to criticism. (See Zimmerman [1984] and Zimmerman, Somerton, and King [1986] for a discussion and for an example of experimental apparatus that might be used for this measurement. Also, see the review by Knutson and Bohor [1963].) If the frame is homogeneous, then the constant $K_\phi = K_m$ due to the fact that porosity remains constant – in this very special case – if $p_d = constant$. However, if the frame is inhomogeneous, then K_ϕ has a complicated dependence on the material properties. Berryman and Milton [1991] again find an exact expression for K_ϕ when only two porous constituents are present. We will discuss this poorly understood constant in greater detail later in the paper.

A fourth constant may be defined by

$$\frac{1}{K_p} = -\frac{1}{V_\phi}\left(\frac{\partial V_\phi}{\partial p_d}\right)_{p_f}, \tag{7}$$

but reciprocity [Brown and Korringa, 1975; Berryman and Milton, 1991] shows that K_p is not independent of the other three constants. It is known in general that

$$\frac{1}{K_p} = \frac{1}{\phi}\left(\frac{1}{K} - \frac{1}{K_s}\right). \tag{8}$$

Defining a new constant

$$\alpha \equiv 1 - K/K_s, \tag{9}$$

we can rewrite (8) as

$$K_p = \phi K/\alpha. \tag{10}$$

Since K, ϕ, and α are all nonnegative quantities, K_p is also a nonnegative quantity. Measurements of K_p may be found in Geertsma [1957], Fatt [1958], Van der Knapp [1959], Knutson and Bohor [1963], and Zimmerman, Somerton, and King [1986]. However, it is important to note that, although a measurement of K_p may be used to find K_s if K and ϕ are known, knowledge of this modulus does not help us to find the other pore modulus K_ϕ.

Using these definitions, the isotropic-stress/volume-strain relations become

$$-\frac{\delta V}{V} = \frac{\delta p_d}{K} + \frac{\delta p_f}{K_s} \tag{11}$$

for the total volume strain and

$$-\frac{\delta V_\phi}{V_\phi} = \frac{\delta p_d}{K_p} + \frac{\delta p_f}{K_\phi} \tag{12}$$

for the pore volume strain.

3 Effective Stress Principles

In this section, the analysis being presented is very similar to an analysis presented by Carroll [1980]. The main difference is that we treat the general problem for inhomogeneous porous materials. Carroll's results may be recovered by replacing K_s and K_ϕ everywhere by the material bulk modulus K_m when only one solid constituent is present. We derive the exact effective stress that follows from the general stress-strain relations (11) and (12).

In the next section, we analyze the resulting coefficients and establish relationships among them using elementary bounding arguments. It is important to keep in mind that all the results of the next two sections depend on only the porosity ϕ, the three frame moduli K, K_s, K_ϕ, and the fluid modulus K_f.

3.1 Total volume

The effective-stress principle for total volume follows immediately from the general stress-strain relation (11), giving

$$-\frac{\delta V}{V} = \frac{\delta p_d}{K} + \frac{\delta p_f}{K_s} = \frac{1}{K}\left(\delta p_c - \alpha \delta p_f\right), \tag{13}$$

where the coefficient $\alpha = 1 - K/K_s$ was defined previously in (9). This coefficient is often measured [Van der Knapp, 1959; Nur and Byerlee, 1971; Coyner, 1984]. For example, using the measured on data on the jacketed and unjacketed bulk moduli as shown in Figures 1–4, we can compute α as shown in Table 1. The usual range of values for α is $\phi \le \alpha \le 1$, which is completely consistent with the data in Figures 1–4 and Table 1.

3.2 Pore volume

The effective-stress principle for pore volume follows immediately from the general stress-strain relation (12), giving

$$-\frac{\delta V_\phi}{V_\phi} = \frac{\delta p_d}{K_p} + \frac{\delta p_f}{K_\phi} = \frac{1}{K_p}\left(\delta p_c - \beta \delta p_f\right), \tag{14}$$

where the coefficient is

$$\beta \equiv 1 - K_p/K_\phi. \tag{15}$$

This coefficient has seldom been measured directly (but see Fatt [1958]). Note that, if $K_\phi = K_s$ (which is true in the Gassmann limit), then $\beta = 1 - \phi(1/\alpha - 1)$. Although K_p is a nonnegative quantity, the theory shows that in some exceptional circumstances K_ϕ may be negative; thus, while the usual range of values for β is $\phi \le \beta \le 1$, it is possible that $\beta > 1$.

3.3 Fluid content

Let N be the number of fluid molecules in a volume V. The number density within the fluid volume is given by $\bar{\rho}_f = N/V_f$, while the number density within the total volume is $\rho_f = N/V = \phi\bar{\rho}_f$. If the initial number of fluid molecules in the volume is N_0 when $V_\phi^{(0)} = V_f^{(0)}$, then the number after an applied stress is $N = N_0 V_\phi/V_f$. The increment in fluid content $\delta\zeta$ in the volume V is defined by Biot [1962; 1973] as the fluid mass injected into a unit element of unit initial volume divided by the initial fluid density, which is equivalent to the definition

$$\delta\zeta \equiv \frac{\delta N/V}{N_0/V_f} = \frac{\delta V_\phi - \delta V_f}{V}. \tag{16}$$

Then, if the fluid bulk modulus is K_f, the relative change in the number of fluid molecules in the volume is given by

$$-\delta\zeta/\phi_0 = -\frac{\delta N/V}{\phi\bar{\rho}_f} = -\frac{\delta N}{N_0} = \frac{1}{K_p}\left(\delta p_c - \gamma \delta p_f\right), \tag{17}$$

where the effective-stress coefficient is

$$\gamma \equiv \beta + K_p/K_f. \tag{18}$$

The constant γ is just the inverse of Skempton's constant B [Skempton, 1954], which has also often been measured [Lambe and Whitman, 1969; Palciauskas and Domenico, 1982; Green and Wang, 1986].

Clearly, $\beta \leq \gamma$ in all cases, since $0 \leq K_f, K_p$. For well-consolidated porous materials, it will also normally be true that $1 < \gamma$, since the fluid bulk modulus (say for water or air) is generally much smaller than the frame bulk modulus so $1 \leq K_p/K_f$. However, if the frame is exceptionally weak so that $K \to 0$, then $\gamma \to \beta \to \alpha \to 1$; thus, in this limit (common in soils [Lambe and Whitman, 1969]), all three of these effective stresses reduce to p_d.

Equation (18) shows that the pore volume effective-stress coefficient β may be computed if measurements have been made of $\alpha, \gamma, \phi, K_f$, and K.

3.4 Porosity

Next we consider variations in porosity $\phi \equiv V_\phi/V$. Since

$$\frac{\delta V_\phi}{V_\phi} = \frac{\delta\phi}{\phi} + \frac{\delta V}{V}, \tag{19}$$

we have

$$-\frac{\delta\phi}{\phi} = \left(\frac{\alpha-\phi}{\phi K}\right)\delta p_d + \left(\frac{1}{K_\phi} - \frac{1}{K_s}\right)\delta p_f = \left(\frac{\alpha-\phi}{\phi K}\right)\left(\delta p_c - \chi\delta p_f\right), \tag{20}$$

where the coefficient is given by

$$\chi \equiv 1 - \frac{1/K_\phi - 1/K_s}{1/K_p - 1/K} = \beta + \left(\frac{\beta-\alpha}{\alpha-\phi}\right)\phi = \left(\frac{\beta-\phi}{\alpha-\phi}\right)\alpha. \tag{21}$$

Some measurements of the coefficient $(\alpha-\phi)/\phi K$ may be inferred from data of Fatt [1953] and Brandt [1955]. The experiments consistently show that an increase in confining pressure results in a decrease in porosity, so this coefficient is positive. It follows that $\phi \leq \alpha$ is an empirical result.

Note that, when only one solid constituent is present, $K_s = K_\phi = K_m$, so $\chi \equiv 1$; thus, in the Gassmann limit, the effective pressure for porosity is just the differential pressure p_d, as expected. Furthermore, the coefficient χ equals unity only when $K_s = K_\phi$ or when $K = 0$. If the moduli satisfy $K_s > K_\phi > 0$, then $\chi < 1$; however, if $K_\phi > K_s$ or if $K_\phi < 0$, then $\chi > 1$. If $\beta > 1$, then we must have $K_\phi < 0$ and therefore $\chi > 1$. Finally, since the last identity in (21) may be solved for β and gives

$$\beta = \chi - \phi\left(\chi/\alpha - 1\right), \tag{22}$$

measurements of χ may be used in conjunction with measurements of ϕ and α to find β.

3.5 Solid volume

The solid volume is related to the total volume and the porosity by $V_s = (1-\phi)V$. Since

$$-\frac{\delta V_s}{V_s} = \frac{\delta V}{V} - \frac{\delta\phi}{(1-\phi)}, \tag{23}$$

we have

$$-\frac{\delta V_s}{V_s} = \frac{1}{(1-\phi)K_s}\delta p_d + \frac{1}{(1-\phi)}\left(\frac{1}{K_s} - \frac{\phi}{K_\phi}\right)\delta p_f = \frac{1}{(1-\phi)K_s}\left(\delta p_c - \sigma\delta p_f\right). \tag{24}$$

The coefficient σ is given by

$$\sigma \equiv \frac{\phi K_s}{K_\phi} = \left(\frac{1-\beta}{1-\alpha}\right)\alpha = \alpha - \left(\frac{\chi-\alpha}{1-\alpha}\right)(\alpha-\phi), \tag{25}$$

where we used (22) in the last step to eliminate β. This coefficient is no easier to measure than β. However, in the Gassmann limit, (25) simplifies since $K_\phi \to K_s$ and $\chi \to 1$ so $\sigma \to \phi$ [Carroll, 1980]. (Note also that, if $\beta > 1$, then $\sigma < 0$.) A physical argument for the deviation of σ from ϕ follows from the observation that, during the deformation, the porosity generally does not remain constant unless $K_\phi = K_s$. The effective-stress coefficient σ accounts correctly for the changing value of porosity during a typical deformation process.

3.6 Undrained response

The undrained response of the saturated porous medium to a change in the confining pressure δp_c is found by setting

$$-\frac{\delta V_\phi}{V_\phi} = -\frac{\delta V_f}{V_f} = \frac{\delta p_f}{K_f}, \tag{26}$$

where K_f is the fluid bulk modulus and the undrained bulk modulus K_u is defined by

$$-\frac{\delta V}{V} \equiv \frac{\delta p_c}{K_u}. \tag{27}$$

Substituting (12) into (26) yields

$$\frac{\delta p_c}{\delta p_f} = \beta + K_p/K_f = \gamma, \tag{28}$$

showing that the change in fluid pressure during a confining measurement is $\delta p_f = \delta p_c/\gamma$. The pore pressure buildup coefficient for the undrained response is therefore $B = 1/\gamma$ [Skempton, 1954; Palciauskas and Domenico, 1982; Green and Wang, 1986]. Substituting (11) into (27) yields

$$K_u = \frac{K}{(1 - \alpha\delta p_f/\delta p_c)} = \frac{K}{(1 - \alpha/\gamma)}, \tag{29}$$

showing that the undrained bulk modulus is simply related to the drained modulus K and the effective-stress coefficients α and γ, for total volume and fluid content respectively.

The formula (29) is equivalent to the generalized Gassmann's equation derived by Brown and Korringa [1975]

$$K_u = K + \alpha^2 \left[\alpha/K_s + \phi(1/K_f - 1/K_\phi)\right]^{-1}. \tag{30}$$

If there is only one solid constituent so $K_s = K_\phi = K_m$, it is not hard to show that (29) and (30) reduce correctly to Gassmann's equation [Gassmann, 1951]. Gassmann's result has also been rederived by various other authors, including Biot and Willis [1957], Geertsma [1957], Nagumo [1965a], and Nur and Byerlee [1971]. Rice and Cleary [1976] obtained general results essentially equivalent to those of Brown and Korringa [1975].

4 General Relations Among Effective Stress Coefficients

Berryman and Milton [1991] and Berryman [1992] give a thermodynamic stability argument to show, in general (not just for one or two constituent porous media), it must be true that

$$\frac{\alpha}{K_s} - \frac{\phi}{K_\phi} \geq 0. \tag{31}$$

The derivation of (31) requires the assumption that the moduli K_s and K_ϕ are independent of the pore fluid modulus K_f. Multiplying (31) by K/α, we find

$$(1-\alpha) = K/K_s \geq K_p/K_\phi = (1-\beta), \tag{32}$$

so in general we have

$$\sigma \leq \alpha \leq \beta \leq \gamma. \tag{33}$$

The first inequality follows from the definition (25) of σ without restriction when $\alpha \leq \beta$, which we have just shown follows from (31). The final inequality follows from nonnegativity of the ratio K_p/K_f. Since

$$\alpha \leq \beta = \chi - \phi(\chi/\alpha - 1) \tag{34}$$

follows from (22) and (33), we can multiply (34) by α and rearrange terms to obtain the rigorous result

$$0 \leq (\alpha - \phi)(\chi - \alpha), \tag{35}$$

which also follows from (25) and $\sigma \leq \alpha$. If in addition we know (as is observed) that $\phi \leq \alpha$, then (35) implies $\alpha \leq \chi$ and therefore that $\phi \leq \alpha \leq \beta = \chi - \phi(\chi/\alpha - 1) \leq \chi$. However, the empirical inequality $\phi \leq \alpha$ is not known to be rigorous in all circumstances at present.

The relations in (33) are some of the main results of this paper. To summarize the significance of the general inequalities contained in (33), we see that the fluid (pore) pressure p_f is least effective at counteracting the changes caused by the confining pressure for the solid volume; p_f is more effective for the total volume; p_f is still more effective for the pore volume; and p_f is most effective at maintaining the fluid content of the pores.

To make further progress on understanding effective stress for transport properties, we will need to be more specific about the nature of the porous materials considered. The two examples we study are (a) the homogeneous frame or Gassmann limit and (b) the inhomgeneous frame composed of two porous constituents.

5 Gassmann Limit: Homogeneous Solid Frame

It is instructive to consider the relations among the various effective-stress coefficients in the Gassmann limit, when there is only one solid constituent so $K_s = K_\phi = K_m$. Then, it is known [Voigt, 1928; Hill, 1952; Hill, 1963; Watt, Davies, and O'Connell, 1976] that the Voigt average $(1-\phi)K_m \geq K$, providing a rigorous upper bound on the frame modulus K. Rearranging the formula for the Voigt bound shows that

$$\phi \leq 1 - K/K_m = \alpha \leq 1, \tag{36}$$

where the final inequality follows from the nonnegativity of both K and K_m (as required for thermodynamic stability of the frame and grain material respectively). Using (8) and (36), we find that

$$\phi K \leq K_p \leq K, \tag{37}$$

so that

$$0 \leq \phi \leq \alpha \leq \beta \leq 1 - \phi + \phi\alpha \leq 1. \tag{38}$$

Then, it follows from the positivity of K_p and K_f that

$$\phi \leq \alpha \leq \beta \leq \gamma, \tag{39}$$

so (29) shows the undrained modulus K_u is always greater than or equal to K. It also follows easily from (21), (25), and (38) that

$$0 \leq \sigma = \phi \leq \alpha \leq \beta = 1 - \phi(1/\alpha - 1) \leq \chi = 1. \tag{40}$$

Most of the results derived so far in this section have been obtained previously by Zimmerman [1984] and Zimmerman, Somerton, and King [1986]. In some cases, they were also reported by Berryman and Thigpen [1985] and Berryman [1986].

We emphasize that the results in this section are special to the Gassmann limit. Some remarkably different results may apply when the solid frame is inhomogeneous.

Fatt [1958] used results of Brandt [1955] to show that $\chi \simeq 0.85$ in some sandstone reservoir rocks (also see Ferrell, Felsenthal, and Wolfe [1962], Knutson and Bohor [1963], Nagumo [1965a,b], Wagner and Voigt [1971], and Schopper [1982]). Since $\chi \equiv 1$ is required for all homogeneous solid frames, this result is our first indication that no equivalent homogeneous rock or set of rocks can be used to explain available effective-stress data. This particular result is not definitive, however, since the error bars on the measured values of χ are quite large (Fatt says this coefficient probably varies in the range $0.75 \leq \chi \leq 1$ for various rocks and pressures). Another set of measurements to be discussed shortly does not suffer from this possible criticism.

5.1 Electrical conductivity

Electrical conductivity for inhomogeneous media is a scale invariant material property: If the medium undergoes an expansion or contraction without change of shape, then the electrical conductivity remains constant – assuming only that the properties of the constituents are also independent of strain. So, if we take the grains in a porous rock as insulators and inject a conducting fluid into the pores, we expect that the overall conductivity of the saturated medium will be of the form

$$g = g_f G, \tag{41}$$

where g_f is the conductivity of the pore fluid (g_f may be a function of the fluid pressure [Daily and Lin, 1985]), and – for the present application – $G \leq 1$ is a real function depending only on the relative geometry of the pore space – not on the absolute scale. In general, G may have a complicated dependence on the confining and fluid pressures, and there may not be any combination of p_c and p_f that leaves G invariant. However, considering the Gassmann limit, we see that relative positioning is dependent only on the differential pressure so G is a function only of $p_d = p_c - p_f$. The function G is therefore rigorously scale invariant (whereas the pressure dependence of g_f must be factored out of g to make it scale invariant for realistic experiments). The porosity ϕ is another rigorously scale invariant property of the porous material, so in general we might suppose that G could be expressed as a function of the porosity.

Archie's law [Archie, 1942; Sen, Scala, and Cohen, 1981] for the electrical conductivity g of a brine saturated porous medium is

$$g = \frac{g_f}{F} = g_f \phi^m, \tag{42}$$

where g_f is the electrical conductivity of the saturating fluid, $F \equiv g_f/g$ is the formation factor, ϕ is the porosity of the porous formation, and m is Archie's cementation exponent (generally in the range $1 < m \leq 2$, but occasionally m as high as 2.3 has been observed). An additional constant factor is sometimes included in (42), but its presence would make no difference to the arguments that follow. We see that in the Gassmann limit

$$G \equiv \frac{1}{F} \simeq \phi^m, \tag{43}$$

and therefore

$$\frac{\delta G}{G} \simeq m\frac{\delta\phi}{\phi} = -m\left(\frac{\alpha-\phi}{\phi K}\right)\delta p_d. \tag{44}$$

These results are special to the Gassmann limit, because *only* for homogeneous frames is it true that the pore space swells or shrinks at the same rate as the bulk volume.

Neglecting the pressure dependence of g_f (see Daily and Lin [1985]), we see that the effective pressure for g (or equivalently for the formation factor) is just the differential pressure. So the effective stress for the electrical conductivity is $\delta p_c - \epsilon\delta p_f = \delta p_d$, where the value of the effective-stress coefficient ϵ is therefore always $\epsilon = \chi = 1$ for homogeneous frames.

5.2 Fluid permeability

Fluid permeability for porous media is not a scale invariant material property: Darcy's constant k has the dimensions of length squared, so a uniform swelling or shrinking of the isotropic porous medium changes the value of the permeability proportional to $V^{2/3}$ (since V has dimensions of length cubed). The dependence of the permeability on geometry may therefore be expressed in general as

$$k = const \times H \times V^{2/3}, \tag{45}$$

where H depends only on the relative positioning of the grains and is therefore rigorously scale invariant. Like G, the factor H will generally be a complicated function of the confining and fluid pressures with no combination leaving it invariant. However, also like G, the pore space swells or shrinks at the same rate as the grains in the Gassmann limit so H is rigorously seen to be a function only of the differential pressure. In analogy with the arguments leading to (44), we suppose that

$$\frac{\delta H}{H} \simeq n\frac{\delta\phi}{\phi} = -n\left(\frac{\alpha-\phi}{\phi K}\right)\delta p_d. \tag{46}$$

The constant n may be related approximately to Archie's cementation exponent m through the Kozeny-Carman relation [Paterson, 1983; Walsh and Brace, 1984]

$$k \simeq \frac{\phi^2}{2s^2F}, \tag{47}$$

where s is a measure of the specific surface area (for an equivalent smooth-walled pore) and therefore $s^{-2} = const \times V^{2/3}$. (Avellaneda and Torquato [1991] provide a different insight into the role of formation factor in estimates of permeability.) Thus, $H \simeq \phi^2/F = \phi^{2+m}$ and $n \simeq 2+m$. Recalling that $m \simeq 2$, we find this estimate is in good agreement with empirical results of Adler, Jacquin, and Quiblier [1990] who found the permeability correlated well with the relation $k \propto \phi^n$ where $n \simeq 4.15$. Bourbié, Coussy and Zinszner [1987] find $n \simeq 7$ for $\phi \le 0.05$, $4 \le n \le 5$ for $0.10 \le \phi \le 0.25$, and $n \simeq 3$ for sintered glass and some sandstones with porosities in the range $0.15 \le \phi \le 0.30$. A nominal value of $n \simeq 4$ is therefore reasonable, based on experimental evidence.

To find the effective stress for the permeability, we combine (13), (45), and (46). We then find

$$\frac{\delta k}{k} = \frac{\delta H}{H} + \tfrac{2}{3}\frac{\delta V}{V} = -\left[n\left(\frac{\alpha-\phi}{\phi K}\right) + \frac{2}{3K}\right](\delta p_c - \kappa\delta p_f), \tag{48}$$

where the effective-stress coefficient for permeability is

$$\kappa = 1 - \frac{2\phi(1-\alpha)}{3n(\alpha-\phi)+2\phi} \le 1. \tag{49}$$

The inequality follows from the facts that $\phi \le \alpha \le 1$ for the homogeneous frame and that the denominator is always positive as long as $\phi > 0$.

The bound (49) has great practical and conceptual significance for analysis of rocks. If we suppose that any porous rock can be well approximated by an "equivalent homogeneous rock," then (49) makes a definite prediction that the effective-stress coefficient κ must be less than unity for microhomogeneous porous materials. Considering Table 2, we see that this prediction is verified by the data on two Al_2O_3 samples with no clay content. However, (49) is in direct conflict with all the other experimental results in Table 2, showing that the effective-stress coefficient κ for fluid permeability can be significantly greater than unity for a variety of rocks containing multiple constituents [Zoback, 1975; Zoback and Byerlee, 1975; Nur, Walls, Winkler, and DeVilbiss, 1980; Coyner, 1984]. Thus, we have found that it is actually impossible to explain this aspect of the behavior of these clay-bearing porous rocks under stress in terms of an equivalent homogeneous frame. This result does not imply that it is never appropriate to use "an equivalent homogeneous frame" postulate to analyze such rock data, but it does show that circumstances can arise in very inhomogeneous rocks that invalidate such a postulate.

This negative result provides a strong motivation (based on existing experimental evidence) to attempt a more rigorous analysis of porous media containing at least two constituents.

6 Two Porous Constituents

Suppose the solid frame is composed of two distinct porous constituents (say, type-1 and type-2), each of which obeys a volume stress-strain relation analogous to (11) so that, microscopically, we have

$$-\frac{\delta V^{(1)}}{V^{(1)}} = \frac{\delta p_d^{(1)}}{K^{(1)}} + \frac{\delta p_f^{(1)}}{K_m^{(1)}}, \tag{50}$$

and

$$-\frac{\delta V^{(2)}}{V^{(2)}} = \frac{\delta p_d^{(2)}}{K^{(2)}} + \frac{\delta p_f^{(2)}}{K_m^{(2)}}. \tag{51}$$

For two constituents, Berryman and Milton [1991] have shown that there exists a ratio of the macroscopic pressure increments $\delta p_c/\delta p_f \equiv \theta$ such that the relative change in the volumes of each constituent (and therefore of the composite) is the same. Thus, the composite porous medium undergoes a uniform swelling or shrinking so the shapes and relative positions of all the porous constituents remain fixed while the overall size increases or decreases. Furthermore, the microscopic pressure changes equal the macroscopic ones, so $\delta p_d = \delta p_d^{(1)} = \delta p_d^{(2)}$ and $\delta p_f = \delta p_f^{(1)} = \delta p_f^{(2)}$. In terms of the total volume effective stresses, we find

$$\begin{aligned}&\frac{1}{K}(\delta p_c - \alpha\delta p_f) \\ &\quad = \frac{1}{K^{(1)}}(\delta p_c - \alpha^{(1)}\delta p_f) \\ &\quad = \frac{1}{K^{(2)}}(\delta p_c - \alpha^{(2)}\delta p_f),\end{aligned} \tag{52}$$

showing that

$$\begin{aligned}\frac{\delta p_c}{\delta p_f} = \theta &= \frac{\alpha^{(1)}/K^{(1)} - \alpha^{(2)}/K^{(2)}}{1/K^{(1)} - 1/K^{(2)}} \\ &= \frac{\alpha^{(1)}/K^{(1)} - \alpha/K}{1/K^{(1)} - 1/K} \\ &= \frac{\alpha/K - \alpha^{(2)}/K^{(2)}}{1/K - 1/K^{(2)}}.\end{aligned} \tag{53}$$

The result (53) shows how the volume effective-stress coefficient α for the composite depends on the various constants of its constituents. It is not difficult to rearrange (53) to recover the version (5) quoted earlier.

An elementary result for the frame bulk modulus

$$\min(K^{(1)}, K^{(2)}) \leq K \leq \max(K^{(1)}, K^{(2)}), \tag{54}$$

when combined with (53) and the results of the preceding section, shows that

$$\min(\phi^{(1)}, \phi^{(2)}) \leq \min(\alpha^{(1)}, \alpha^{(2)}) \leq \alpha \leq \max(\alpha^{(1)}, \alpha^{(2)}) \leq 1. \tag{55}$$

These estimates are all elementary but rigorous.

Berryman and Milton [1991] have also shown that, for two component composite porous media, the remaining independent constant K_ϕ is given by

$$\frac{\phi}{K_\phi} = \frac{\alpha}{K_s} - \left\langle \frac{\alpha(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle - \langle \alpha(\mathbf{x}) - \alpha \rangle \left(\frac{\alpha^{(1)} - \alpha^{(2)}}{K^{(1)} - K^{(2)}} \right), \tag{56}$$

where $\langle \cdot \rangle$ is the volume average of the quantity in brackets. Multiplying (56) by K/α, we find that

$$\beta = \alpha + \left(\frac{K}{\alpha} \right) \left[\left\langle \frac{\alpha(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle + \langle K(\mathbf{x}) - K \rangle \left(\frac{\theta - \alpha}{K} \right)^2 \right], \tag{57}$$

where we used (53) [or equivalently see (59) in the next subsection] to simplify the expression. Equation (57) may be substituted into the general expression (21) to show that

$$\chi = \alpha + \left(\frac{K}{\alpha - \phi} \right) \left[\left\langle \frac{\alpha(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})} \right\rangle + \langle K(\mathbf{x}) - K \rangle \left(\frac{\theta - \alpha}{K} \right)^2 \right]. \tag{58}$$

6.1 Effective stress for uniform contraction or expansion

One important point to emphasize here is that the analysis just presented clearly demonstrates the existence of yet another effective-stress principle. For the two component medium, relative positions of the porous constituents remain unchanged if the quantity $\delta p_c - \theta \delta p_f = const$, so changes in geometry depend on this new effective stress. Then, either (52) or (53) may be rearranged to show that the effective-stress coefficient θ satisfies

$$\frac{\alpha - \theta}{K} = \frac{\alpha^{(1)} - \theta}{K^{(1)}} = \frac{\alpha^{(2)} - \theta}{K^{(2)}} = \frac{\alpha^{(1)} - \alpha^{(2)}}{K^{(1)} - K^{(2)}}, \tag{59}$$

where the final equality in (59) follows easily from the others. This result shows that either $\theta \geq \max(\alpha^{(1)}, \alpha^{(2)})$ or $\theta \leq \min(\alpha^{(1)}, \alpha^{(2)})$. Thus, using (55), θ is generally bounded away from α. In the Gassmann limit, it is straightforward to show that $\theta = \chi = 1$, so $\theta \geq \alpha$ in this special case. Conversely, if $\theta = 1$, then $K_m^{(1)} = K_m^{(2)} = K_s$ follows directly from (59), showing in general that $\theta \neq 1$ if $K_m^{(1)} \neq K_m^{(2)}$. If one of the constituents (say the second one) is nonporous and essentially incompressible, then $K^{(2)} \to \infty$ and $\theta = \alpha = \alpha^{(1)}$ follows directly from (59).

Now, if we suppose that the drained bulk modulus K could be varied without changing the properties of the two constituents (for example, by changing the volume fractions), then it is easy to see that (59) implies

$$\frac{\delta \alpha}{\delta K} = \frac{\alpha^{(1)} - \alpha^{(2)}}{K^{(1)} - K^{(2)}}. \tag{60}$$

Substituting (60) back into (59) shows that $K\delta\alpha/\delta K = \alpha - \theta$ and therefore that

$$\theta = \alpha - K\frac{\delta\alpha}{\delta K}. \tag{61}$$

Similarly, using the definition of α, we find

$$\frac{\delta\alpha}{\delta K} = -\frac{1}{K_s} + \frac{1}{K}\frac{\delta(1/K_s)}{\delta(1/K)}. \tag{62}$$

Substituting into (61) gives

$$\theta = 1 - \frac{\delta(1/K_s)}{\delta(1/K)}. \tag{63}$$

These two rules can be used to compute θ from experimental data on K and K_s as a function of confining pressure [Coyner, 1984] to the extent that both properties are in fact changing due to variations in the volume fractions of the constituents. We have found that (63) is more robust than (61) for estimating θ from real data on K and K_s. For this approach to be valid, it is necessary (but not sufficient) to find that the value of θ computed this way remains constant over some finite range of variation δK, or equivalently that $1/K_s$ is a linear function of $1/K$. (Experimental constancy of θ could be accidental and, therefore, is not sufficient to establish validity of all these assumptions.) It is normally observed that $\delta K_s/\delta K \geq 0$ and, therefore, that $\delta(1/K_s)/\delta(1/K) \geq 0$. By substituting these rules into (61) and (63), we obtain the empirical result $\alpha \leq \theta \leq 1$.

To clarify these issues further, a more general and more rigorous derivation of the relation determining θ is obtained by considering the integrability conditions required to guarantee reversibility, *i.e.*, so that the total volume is a function $V = V(p_c, p_f)$ independent of the particular stress-strain path used to achieve the final pressures p_c and p_f. The present argument is similar to one given by Zimmerman [1984] for the equations in the Gassmann limit. The Euler conditions for integrability following from (11) are

$$-\frac{\partial^2 \ln V}{\partial p_c \partial p_f} = \frac{\partial}{\partial p_f}\left(\frac{1}{K}\right) = \frac{\partial}{\partial p_c}\left(\frac{1}{K_s} - \frac{1}{K}\right). \tag{64}$$

Assuming that the bulk modulus of the drained porous frame has the functional dependence $K = K(p_c - \theta p_f)$ with θ constant, then

$$\frac{\partial}{\partial p_f}\left(\frac{1}{K}\right) = -\theta\frac{\partial}{\partial p_c}\left(\frac{1}{K}\right). \tag{65}$$

Substituting (65) into (64) and solving for θ, we find

$$\theta = 1 - \frac{\partial}{\partial p_c}\left(1/K_s\right)/\frac{\partial}{\partial p_c}\left(1/K\right). \tag{66}$$

This rigorous result (with assumed constancy of θ) should be compared with the result obtained in (63), based on the two-component model. The most important feature of this second derivation is that it requires none of the assumptions contained in the two-component model. It also shows clearly that stress-strain data such as Coyner's data on K and K_s as a function of p_c may be used to compute the effective-stress coefficient θ for real materials.

To check whether these ideas agree with experiment, we have replotted some of Coyner's [1984] data on K_s and K (see Figure 5) for various rocks to illustrate the linear dependence of $1/K_s$ on $1/K$ for confining pressures less than 30 MPa. We have purposely excluded data on all the materials for higher pressures since these rocks may not be expected to satisfy the simple linear model presented here at the higher pressures. To validate the "homogeneous equivalent rock" paradigm, these curves should all be constant. To validate the "two-constituent porous rock" paradigm, the curves only need to be linear over some small range of pressures. Although some of the curves are apparently constant (Berea sandstone, Bedford limestone), all the curves are observed to be nearly linear over this range of pressures. Table 1 summarizes the results for θ.

6.2 Volume fraction changes

For later developments, it will be important to know the behavior of the regional volume fractions

$$v_A \equiv \frac{V^{(1)}}{V} \quad \text{and} \quad v_B \equiv \frac{V^{(2)}}{V}, \tag{67}$$

as functions of applied pressure. (See Figure 6.) Note that, by assumption, $v_A + v_B = 1$ initially, but in general $v_A + v_B \leq 1$, since cracks and other voids may open up between the constituents when an effective stress $\delta p_c - \theta\delta p_f \neq 0$ is applied to the composite. The overall porosity is determined by

$$1 - \phi = v_A(1-\phi_1) + v_B(1-\phi_2). \tag{68}$$

Because of the new effective-stress principle for uniform swelling and shrinking, we know that both volume fractions can only be functions of the effective pressure $p_c - \theta p_f$ in the linear regime. Using (13) and (50), we find crudely that

$$\delta v_A \simeq v_A \left[\frac{1}{K}(\delta p_c - \alpha\delta p_f) - \frac{1}{K^{(1)}}\left(\delta p_c^{(1)} - \alpha^{(1)}\delta p_f\right)\right], \tag{69}$$

where we have assumed that the microscopic and macroscopic fluid pressures have equilibrated. Except when $\delta p_c = \theta\delta p_f$, the microscopic confining pressure $\delta p_c^{(1)}$ is unknown and need not equal δp_c (in fact, the local confining stress fluctuates and will generally not even be scalar). We will ignore these difficulties here and suppose that $\delta p_c^{(1)} = \delta p_c$ or equivalently that, in compression,

$$\delta v_A \equiv \frac{1}{K_A}(\delta p_c - \theta\delta p_f) \quad \text{and} \quad \delta v_B \equiv \frac{1}{K_B}(\delta p_c - \theta\delta p_f), \tag{70}$$

where

$$\frac{1}{K_A} \simeq v_A\left(\frac{1}{K} - \frac{1}{K^{(1)}}\right) \quad \text{and} \quad \frac{1}{K_B} \simeq v_B\left(\frac{1}{K} - \frac{1}{K^{(2)}}\right), \tag{71}$$

to a reasonable approximation.

Equations (70) and (71) are somewhat oversimplified. In fact, we know that v_A and v_B are functions of the pressures only through the combination $\delta p_c - \theta\delta p_f$; however, the actual functions could also be dependent on the geometrical arrangement of the components, as well as the constituents' moduli and the bulk modulus. The dependence of these functions on the effective stress could also be at least weakly nonlinear. To see this, note that

$$\frac{1}{K_A} + \frac{1}{K_B} \simeq \frac{1}{K} - \frac{v_A}{K^{(1)}} - \frac{v_B}{K^{(2)}} \leq 0, \tag{72}$$

where the inequality follows from the fact that the harmonic mean of $K^{(1)}$ and $K^{(2)}$ is a lower bound on K [Hill, 1952; Hill, 1963; Watt, Davies, and O'Connell, 1976]. Inequality (72) shows that in compression $\delta v_A + \delta v_B \leq 0$, so that new cracks or voids must open when $\delta p_c - \theta\delta p_f > 0$. However, in tension or when existing compression is released, the same result shows that $\delta v_A + \delta v_B > 0$, which implies the whole volume ($= 1$) is less than the sum of its parts ($1 + \delta v_A + \delta v_B > 1$). This clearly unphysical result means that our initial assumption of linearity in (70) may need to be reexamined [Garg and Nur, 1973; Zimmerman, Somerton, and King, 1986]. In particular, we might expect to find that K, K_A, and/or K_B take different values in compression and tension as has often been observed in rocks [Jaeger and Cook, 1976]. For example, if the materials in the sample are poorly cemented (as in a soil), the modulus in tension may be vitually zero, while the modulus in compression may be indistinguishable from that of a well-cemented sample. We wish to emphasize that, while (70) holds rigorously in compression, (71) is merely an approximation we need when making comparisons with experiment. Ideally, K_A and K_B can be measured directly, but little information about these coefficients is available at present.

The unphysical results noted here may also be eliminated by considering a two-solid-component model that includes cracks or voids in the unstressed state. Then, $v_A + v_B < 1$ initially and either sign of the change $\delta v_A + \delta v_B$ is allowed. Berryman and Milton [1992] have recently shown how to obtain exact results for such models.

6.3 Clayey sandstone

Now we consider a special case that we will call the "clayey-sandstone model." (See Figure 7.) One of the constituents of this model has no porosity, so $K^{(2)} = K_m^{(2)}$ and $\alpha^{(2)} = \phi^{(2)} = 0$. The other constituent has a very soft frame modulus, so $K^{(1)} \to 0$ and $\alpha^{(1)} \to 1$. Substituting these limits into (59), we find that

$$\alpha \simeq 1 - \frac{K}{K_m^{(2)}} \tag{73}$$

and

$$\theta \to \alpha^{(1)} \to 1. \tag{74}$$

So we have $K_s \simeq K_m^{(2)}$ and

$$\frac{1}{K_\phi} \simeq \frac{1}{K_m^{(1)}} + \frac{1}{\phi^{(1)}} \left(\frac{1}{K_m^{(2)}} - \frac{1}{K_m^{(1)}} \right). \tag{75}$$

Note that the magnitude of $K_m^{(2)}$ compared to that of $K_m^{(1)}$ has not yet been specified in the clayey-sandstone model and may be considered arbitrary. Thus, K_ϕ can take a wide range of values. Then, we find that

$$\chi \simeq 1 + \left(\frac{v_A - \phi}{\alpha - \phi} \right) \left(\frac{1}{K_m^{(1)}} - \frac{1}{K_m^{(2)}} \right) K \tag{76}$$

and that

$$\frac{1}{v_A K_A} \simeq -\frac{1}{K^{(1)}}, \tag{77}$$

while

$$\frac{1}{v_B K_B} \simeq \frac{\alpha}{K}. \tag{78}$$

The porosity is given by $\phi = v_A \phi^{(1)}$ initially.

We will use these results in the next section when we need to evaluate formulas for the effective-stress coefficients of electrical conductivity and fluid permeability.

7 Transport Properties

In this section, we use the results of the last section to analyze electrical conductivity and fluid permeability in a two-constituent porous medium.

7.1 Electrical conductivity

The Bergman-Milton [Bergman, 1980; Milton, 1980; Bergman, 1981; Milton, 1981; Bergman, 1982; Korringa and LaTorraca, 1986; Milton, 1986; Stroud, Milton, and De, 1986] analytical approach to understanding the effective electrical conductivity g of two component inhomogeneous media shows that

$$g = G(g_1, g_2) = g_1 G(1,0) + g_2 G(0,1) + \int_0^\infty \frac{dx \mathcal{G}(x)}{\frac{1}{g_1} + \frac{x}{g_2}}, \tag{79}$$

where $G(1,0)$ and $G(0,1)$ are constants depending only on the geometry and $\mathcal{G}(x) \geq 0$ is a resonance density also depending only on the geometry. The integral in (79) is known as a Stieltjes integral [Baker, 1975]. Although the representation (79) has usually been employed to study the behavior of g in the complex plane when g_1 and g_2 are themselves complex (corresponding to mixtures of conductors and dielectrics), we will restrict consideration here – as Bergman did in his earlier work [Bergman, 1978] – to pure conductors so that g_1, g_2, and g are all real and nonnegative.

If one of the components is an insulator (say $g_2 = 0$), then (79) reduces to

$$g = \frac{g_1}{F_A}, \tag{80}$$

where F_A is the formation factor found by taking all of region A to be pore space and all of region B to be insulator. A similar result follows by taking g_1 to be the insulator, so (79) may be rewritten as

$$g = \frac{g_1}{F_A} + \frac{g_2}{F_B} + \int_0^\infty \frac{dx \mathcal{G}(x)}{\frac{1}{g_1} + \frac{x}{g_2}}, \tag{81}$$

and the fact that $G(1,1) = 1$ leads to the sumrule

$$\frac{1}{F_A} + \frac{1}{F_B} + \int_0^\infty \frac{dx \mathcal{G}(x)}{1+x} = 1. \tag{82}$$

Because the electrical conductivity is a scale invariant quantity, the geometry dependent terms F_A, F_B, and $\mathcal{G}(x)$ can depend only on the relative geometry. Since the relative geometry remains fixed when $\delta p_c - \theta \delta p_f = const$, we see that all these terms must be functions only of the effective stress $p_c - \theta p_f$. As the terms F_A and F_B are formation factors, we may use Archie's law to show that

$$F_A \simeq v_A^{-m_A} \quad \text{and} \quad F_B \simeq v_B^{-m_B}, \tag{83}$$

where v_A and v_B are precisely the volume fractions defined in (70) and the exponents approximately satisfy $1 < m_A, m_B \leq 2$. Little is known at present about the resonance density function $\mathcal{G}(x)$, so we can only say for certain that it is some (probably very weak) function of the effective stress $\delta p_c - \theta \delta p_f$.

To make further progress, we must specialize. Consider a clayey sandstone, so one component is impermeable sand and the other component is a permeable clay that essentially fills the void space among the sand grains. Let $g_2 = 0$ represent the conductivity of the insulating sand grains. Then, we suppose the clay is composed of insulating particles of a single material so the porous clay by itself satisfies Gassmann's equation. If g_f is the conductivity of the conducting fluid in the pores, the corresponding effective conductivity of the saturated clay is

$$g_1 = \frac{g_f}{F_1}, \tag{84}$$

where F_1 is the formation factor of the porous clay and satisfies $F_1 = \phi_1^{-m_1}$ with ϕ_1 being the porosity of the clay and m_1 is an appropriate Archie cementation exponent. Finally, we find that the effective conductivity of the clayey sandstone should be given by

$$g = \frac{g_f}{F_1 F_A} = g_f \phi_1^{m_1} v_A^{m_A} = g_f \phi^{m_1} v_A^{m_A - m_1}, \tag{85}$$

recalling that the total porosity for this model is $\phi = \phi_1 v_A$. We see that (85) will show some deviations from Archie's law for the composite only if the exponents m_1 and m_A differ significantly. Otherwise, $m_1 \simeq m_A \simeq m$ will give

$$g \simeq g_f \phi^m. \tag{86}$$

Combining these results with (20), (69), and (70), we find from (85) that the stress rule for effective conductivity is

$$\frac{\delta g}{g} = -\left[m_1\left(\frac{\alpha-\phi}{\phi K}\right) + \frac{m_1 - m_A}{v_A K_A}\right](\delta p_c - \epsilon\delta p_f), \tag{87}$$

where

$$\begin{aligned}\epsilon &= 1 - \frac{m_1(\alpha-\phi)(1-\chi) + (m_1-m_A)\phi K(1-\theta)/v_A K_A}{m_1(\alpha-\phi) + (m_1-m_A)\phi K/v_A K_A} \\ &= \alpha + \frac{m_1(\alpha-\phi)(\chi-\alpha) + (m_1-m_A)\phi K(\theta-\alpha)/v_A K_A}{m_1(\alpha-\phi)+(m_1-m_A)\phi K/v_A K_A}.\end{aligned} \tag{88}$$

Recall that (35) shows the product $(\alpha-\phi)(\chi-\alpha)$ is always nonnegative.

In the Gassmann limit, $\theta = \chi = 1$ so $\epsilon = 1$ as expected. If $m_1 = m_A$, then $\epsilon = \chi$ and the value of θ does not affect the conductivity; the effective stress for conductivity is then the same as that for porosity. If $m_1 \neq m_A$, then the values of both θ and K_A are important in determining ϵ. At present, little experimental evidence for the deviation of ϵ from unity is available (see Dey [1986]), since most effective-stress experiments on electrical conductivity known to the author have been performed on fairly clean sandstones [Longeron, Argaud, and Feraud, 1986]. Thus, (88) is a definite new prediction of this analysis and suggests that some new experiments should be performed to check its accuracy.

For the clayey-sandstone model, we find that if $m_1 \simeq m_A$ then $\epsilon \simeq \chi$, but if $m_1 \neq m_A$ then $\epsilon \simeq \theta \simeq 1$. Since we expect $\theta \leq \chi$ for this model, the general conclusion is that

$$\theta \leq \epsilon \leq \chi, \tag{89}$$

but these bounds are only approximate. Evaluating the formulas for the clayey sandstone model, suppose that $m_1 = 2$, $m_A = 2.15$, $\alpha = 0.85$, $\phi^{(2)} = 0.8$, $v_A = 0.25$, $\phi = 0.2$, and $K/K^{(1)} = 10$. Then, $\chi > 0.99$ (assuming only that $K_m^{(1)} < \infty$) and $\epsilon \simeq 1.3\chi - 0.3$, showing that the effective-stress coefficient for electrical conductivity can be greater than unity if $\chi > 1$.

7.2 Fluid permeability

Dagan [1979] shows that the effective permeability of an inhomogeneous porous medium is determined by the same equations as those for the effective conductivity of a similar inhomogeneous medium. Dagan's argument is correct for two component porous media as long as both constituents have finite and comparable permeabilities k_1 and k_2. If one of the regions is impermeable (as it would be if composed of solid grains), then the no-slip condition at the boundaries of these solid grains introduces a new physical effect not found in the electrical conduction problem. Assuming that both k_1 and k_2 are bounded away from both zero and infinity, the effective permeability of a two component composite porous medium is therefore given quite accurately by

$$k \simeq G(k_1, k_2) = \frac{k_1}{F_A} + \frac{k_2}{F_B} + \int_0^\infty \frac{dx\mathcal{G}(x)}{\frac{1}{k_1} + \frac{x}{k_2}}, \tag{90}$$

where G and $\mathcal{G}$ are – not just analogous functions but – actually the same functions as in the electrical conduction problem. When $k_1 = k_2$, (82) shows that $k = k_1$ as expected; thus, in the uniform frame limit, (90) reduces correctly to (45). The most important insight we gain from introducing (90) is the observation that the terms F_A, F_B, and $\mathcal{G}(x)$ are dependent *only* on the effective stress $p_c - \theta p_f$.

To make further progress we must specialize again. Considering the clayey-sandstone model once more, we want to take $k_2 = 0$, indicating the sand grains are impermeable. However, this limit is precisely the one for which the formula (90) is not strictly valid. Physically, we know that the introduction of the no-slip boundary condition must reduce the rate of fluid transport and thus decrease the permeability. Therefore, when we take the $k_2 \to 0$ limit of (90), we have found at best an upper bound on the permeability rather than a direct estimate. So we have

$$k \le k^{(+)} = \frac{k_1}{F_A}. \tag{91}$$

This bound is expected to be quite close to the value of the permeability k as long as the intrinsic permeability of the clay k_1 is so low that the presence of the sand-grain boundaries has little influence on the overall fluid transport.

Now we suppose that, as in (45) and (47),

$$k_1 \simeq \frac{\phi_1^2}{2s_1^2 F_1} = const \times \phi_1^{n_1} \times V_1^{2/3}, \tag{92}$$

where $n_1 \simeq 2 + m_1$. Then, since $V_1 = v_A V$, the effective-stress formula for the permeability bound is given by

$$\begin{aligned}\frac{\delta k^{(+)}}{k^{(+)}} &= n_1 \frac{\delta\phi_1}{\phi_1} + \tfrac{2}{3}\left(\frac{\delta v_A}{v_A} + \frac{\delta V}{V}\right) + m_A \frac{\delta v_A}{v_A} = n_1 \frac{\delta\phi}{\phi} - q\frac{\delta v_A}{v_A} + \tfrac{2}{3}\frac{\delta V}{V} \\ &= -\left[n_1\left(\frac{\alpha - \phi}{\phi K}\right) + \frac{2}{3K} + \frac{q}{v_A K_A}\right](\delta p_c - \kappa \delta p_f),\end{aligned} \tag{93}$$

where $\phi = \phi_1 v_A$, $q = n_1 - m_A - \frac{2}{3}$, and

$$\begin{aligned}\kappa &= 1 - \frac{3n_1(\alpha-\phi)(1-\chi) + 2\phi(1-\alpha) + 3q\phi K(1-\theta)/v_A K_A}{3n_1(\alpha - \phi) + 2\phi + 3q\phi K/v_A K_A} \\ &= \alpha + \frac{3n_1(\alpha-\phi)(\chi-\alpha) + 3q\phi K(\theta-\alpha)/v_A K_A}{3n_1(\alpha-\phi) + 2\phi + 3q\phi K/v_A K_A}.\end{aligned} \tag{94}$$

Again recall that (35) shows $(\alpha - \phi)(\chi - \alpha)$ is always nonnegative, while the empirical result for θ is $\alpha \le \theta$. Rigorous bounds on κ are made difficult to obtain by the fact that the terms in the denominator of (94) do not always have the same sign. (For clayey sandstone, the last term is usually negative.) When all terms are positive, we can show $\alpha \le \kappa$, but the value of an upper bound on κ depends on whether $\chi < 1$ or $\chi > 1$. In the homogeneous frame limit, since $\theta = \chi = 1$, $v_A \equiv 1$, and $K_A \to \infty$ (since $K \to K^{(1)}$), we obtain

$$\kappa = 1 - \frac{2\phi(1-\alpha)}{3n_1(\alpha-\phi) + 2\phi} \le 1, \tag{95}$$

in agreement with (49).

In general, the total volume effective-stress coefficient satisfies $\alpha < 1$, but the porosity coefficient χ can have values either less than or greater than unity. For the general two component problem, K_A will normally be negative, while θ is restricted by the empirical inequalities $\alpha \le \theta \le 1$. Thus, we find that the expression (94) can take a wide variety of values because of the variability of χ and K_A. For the clayey-sandstone model, considering the poorly consolidated limit where $K^{(1)} \to 0$ we find that $\kappa \to \theta \to 1$, since then $v_A K_A \simeq -K^{(1)} \to 0$. However, if $K^{(1)}/K << 1$ but remains finite, then we can get a magnification (or resonance) effect due to some cancellation in the denominator of (94). In the case of most interest, the result for κ is given approximately by

$$\kappa \simeq \alpha + M(\chi - \theta), \tag{96}$$

where the magnification factor $M \simeq 40$ if $n_1 = 4$, $\alpha = 0.85$, $v_A = 0.25$, $\phi = 0.2$, $q = 4/3$, and $K/K^{(1)} = 10$. Evaluating (94) using $\chi \simeq 1.1$, the result for the effective-stress coefficient is $\kappa \simeq 5$. This estimate agrees reasonably well with the experimental result for the Berea sandstone considered by Zoback and Byerlee [1975] and Coyner [1984]. Thus, if the effective grain modulus of the pore-filling material $K_m^{(1)}$ is sufficiently smaller than that of the sand grains $K_m^{(2)}$, we can easily find that both $\chi > 1$ and $\kappa > 1$.

The theory shows that it is possible for the effective-stress coefficient κ to be greater than unity as observed by Zoback and Byerlee [1975], Nur, Walls, Winkler, and DeVilbiss [1980], and Coyner [1984]. To obtain better quantitative agreement between theory and experiment, we need to know values of constants usually not measured, such as K_ϕ, β, or χ.

8 Discussion

It has sometimes been speculated [Walsh, 1981] that the effective-stress coefficient for the variation of permeability κ should be the same as that for the pore volume β. For homogeneous frames, we can make a direct comparison of these two coefficients, since

$$1 - \beta = \frac{\phi(1-\alpha)}{\alpha} \quad \text{and} \quad 1 - \kappa = \frac{2\phi(1-\alpha)}{3n(\alpha - \phi) + 2\phi}. \tag{97}$$

Then, it is straightforward to show that, in this limit,

$$\beta \leq \kappa \leq \chi = 1, \tag{98}$$

where the first inequality is true as long as $\frac{2}{3} \leq n$. Since the expected range of n is $3 < n \leq 4$, $\beta < \kappa$ will generally be valid for homogeneous frames. Thus, although these coefficients should behave similarly, we expect the fluid pressure to have a somewhat stronger effect on the permeability than on the pore volume for the same value of confining pressure.

In both (43) and (46), we implicitly assume that Archie's law (or its equivalent for H) is a good approximation. To estimate how accurate this approximation is, we may consider the formula

$$G \equiv \phi^{m(\phi)} \tag{99}$$

to be a defining equation for the exponent m as a function of ϕ. Then, the variation of (99) shows that

$$\frac{\delta G}{G} = \left[m(\phi) + \phi \ln \phi \frac{\delta m}{\delta \phi} \right] \frac{\delta \phi}{\phi}. \tag{100}$$

Thus, by supposing Archie's law holds, we are implicitly assuming that

$$\left| \phi \ln \phi \frac{\delta m}{\delta \phi} \right| << m(\phi), \tag{101}$$

or that the variation of $m(\phi)$ with the respect to ϕ is negligible. A similar argument holds for H with n replacing m in (101).

Another issue arises when the porosity is low and approaches the percolation threshold ϕ_c. Then, (99) should be modified to $G \equiv (\phi - \phi_c)^{m(\phi)}$ for $\phi \geq \phi_c$. This change results in only a minor modification of the preceding analysis, but may nevertheless be required for some applications.

The analysis presented here applies directly to porous materials such as clean sandstones and clay-rich sandstones. Our explicit assumption of the existence of a linear regime precludes the direct application of these results to highly fractured rocks, wherein a small change in confining pressure brings about a large change in the bulk modulus K [Fatt and Davis, 1952; Walsh, 1965; Morlier, 1971; Snow, 1968; Gangi, 1978; Tsang and Witherspoon, 1981; Walsh, 1981]. For such media, the approach introduced here may be generalized by treating differential ratios like $\delta\phi/\delta p_d$ as defining equations for curve tangents and subsequently integrating all the resulting equations simultaneously. In such applications, at least one additional equation is needed for the behavior of the bulk modulus itself. One reasonable choice for this equation in the Gassmann limit is $K = (1-\phi)^b K_m$, where $b \geq 1$. This choice is consistent with the Voigt bound, and may also be consistent with the Hashin-Shtrikman bounds [Hashin and Shtrikman, 1961] if b is large enough ($b \geq 1 + 3K_m/4\mu$). From the present point of view, the most important characteristic of this choice is its consistency with the scale invariance property of K. Similarly, the analysis of elastic moduli for two-component composite porous media can be pursued using the approach of Kantor and Bergman [1984], just as we used the Bergman-Milton approach to analyze the electrical conductivity. Combining the analysis of the elastic properties with that for the transport properties should yield a rigorous treatment valid for fractured rock. We will leave this line of enquiry to be pursued at a later time.

9 Conclusions

In this paper, we have established bounds on and general relations among effective-stress coefficients for various physical properties. For example, the inequalities summarized in (33) show that the fluid pressure p_f is least effective at counteracting the changes induced by confining pressure for the solid volume V_s; p_f is more effective for the total volume V; p_f is still more effective for the pore volume V_ϕ; and p_f is most effective at maintaining the fluid content ζ of the pores. Although these results seem reasonable based on physical intuition, our analysis provides the first rigorous demonstration. During this analysis, care was taken to distinguish between rigorous inequalities (following from thermodynamics) and empirical inequalities (commonly observed, but not known to be required by thermodynamics).

For microscopically homogeneous rocks (the Gassmann limit), it was shown that the permeability effective-stress coefficient $\kappa \leq 1$, so the confining pressure is *always* at least as effective as the fluid pressure at changing the fluid permeability. We concluded from this result that it is impossible to use any "equivalent homogeneous rock" to explain experimental results of Zoback and Byerlee [1975] and Nur, Walls, Winkler, and DeVilbiss [1980] showing that $\kappa > 1$ for some clay-rich sandstones. The "equivalent homogeneous rock" paradigm was then replaced by the "two-constituent porous medium" paradigm. We have shown the new paradigm predicts, in some circumstances, that $\kappa > 1$ will occur for clay-rich sandstones, but these results at best establish plausibility of this explanation. New measurements of pertinent pore compressibilities are required before definitive quantitative comparisons can be made.

One of the main conclusions to be drawn from this work is the need for measurements of the pore bulk modulus K_ϕ and the effective-stress coefficients β and χ depending on this constant. Similarly, measurements of the effective-stress coefficient ϵ for electrical conductivity are presently lacking. Such measurements are needed in order to turn some of the plausibility arguments presented here into definitive predictions.

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TABLE 1. Values of ϕ, K_s, K, and α for two different confining pressures applied to various rocks measured by Coyner [1984]. The value of the effective-stress coefficient θ for expansion and contraction is computed from the given values of K_s and K.

	$p_c = 0$	$p_c = 10$ MPa			$p_c = 25$ MPa			
Rock Sample	$\phi(\%)$	K_s(GPa)	K(GPa)	α	K_s(GPa)	K(GPa)	α	θ
Weber Sandstone	9.5	37.0	4.0	0.89	38.0	10.0	0.74	0.995
Navajo Sandstone	11.8	34.0	13.0	0.62	34.5	16.5	0.52	0.974
Berea Sandstone	17.8	39.0	6.0	0.85	39.0	10.0	0.74	1.000
Bedford Limestone	11.9	66.0	23.0	0.65	66.0	27.0	0.59	1.000
Barre Granite	0.7	54.5	13.5	0.75	55.5	21.5	0.61	0.988
Westerly Granite (red)	0.8	53.0	24.0	0.55	54.0	34.0	0.37	0.971
Chelmsford Granite	1.1	54.5	8.0	0.85	55.5	17.0	0.69	0.995

TABLE 2. Sample properties from Zoback and Byerlee [1975] and Nur *et al* [1980].

Porous Sample	$\phi(\%)$	*Clay Content*(%)	k_0(md)	κ
Al_2O_3(A)	26	0	817	0.43
Al_2O_3(B)	29	0	50	0.86
St. Peter	20	0.5	944	1.2
Berea 350	22	6.0	382	2.2
Brownstone	21	4.5	624	2.4
Berea 350	22	6.0	261	2.9
Berea 500	21	4.3	470	3.2
Berea 200	22	6.3	363	3.2
Massilon	23	5.0	737	3.2
Berea 100	20	8.6	52	3.3
Massilon	24	6	995	3.5
Berea	19	8	42	4.0
Berea 500	21	4.3	617	4.6
Bandera	16	20.0	0.4	7.1

Figure Captions

Figure 1: Jacketed bulk modulus K, unjacketed bulk modulus K_s, and porosity ϕ as a function of hydrostatic confining pressure for Navajo sandstone. Data from Coyner [1984] with the author's permission.

Figure 2: Same as Figure 1 for Weber sandstone. Bars represent probable error. Data from Coyner [1984] with the author's permission.

Figure 3: Same as Figure 1 for Berea sandstone. Data from Coyner [1984] with the author's permission.

Figure 4: Same as Figure 1 for Westerly granite. Bars represent probable error. Data from Coyner [1984] with the author's permission.

Figure 5: The unjacketed compressibility $1/K_s$ as a function of the jacketed compressibility $1/K$ for six of Coyner's [1984] suite of seven rocks. (Barre granite is not shown since its curve is close to that for Chelmsford granite.) Pressure variation is illustrated on the curve for Navajo sandstone, showing that the high end corresponds to lower confining pressure (10 MPa) and the low end to the higher pressures (25 MPa). In fact, Coyner's measurements continue to 100 MPa and almost all the curves begin to deviate from linearity for the higher pressures, but this behavior is beyond the scope of the present study and therefore is not shown.

Figure 6: In a two-constituent mixture, regions A and B could be filled with solids, fluids, fluid-saturated or unsaturated porous solids, or any pair of such materials. For example, in the Gassmann limit, region B might be filled with a pure-grain solid, while region A is filled with a fluid.

Figure 7: In the clayey-sandstone model, region A from Figure 6 is filled with fluid-saturated porous clay-like solid of type-1, while region B is filled with solid sand grains of type-2.

List of Symbols		
g	=	overall electrical conductivity
g_f	=	pore-fluid electrical conductivity
k	=	fluid permeability
m	=	Archie's cementation exponent for conductivity
n	=	exponent for fluid permeability
p_c	=	confining pressure
p_d	=	$p_c - p_f$, the differential pressure
p_f	=	fluid pressure
s	=	a measure of specific surface area
v_A, v_B	=	regional volume fractions
F	=	formation factor
G	=	analytic conductivity function
H	=	scale invariant part of the fluid permeability
K	=	bulk modulus of drained porous frame (jacketed)
K_f	=	fluid bulk modulus
K_m	=	material (or grain) bulk modulus
K_p	=	$\phi K/\alpha$, an effective pore bulk modulus
K_s	=	an effective solid bulk modulus (unjacketed)
K_u	=	bulk modulus of undrained porous frame
K_ϕ	=	an effective pore bulk modulus
$K^{(1)}, K^{(2)}$	=	drained frame moduli of porous constituents
$K_m^{(1)}, K_m^{(2)}$	=	material moduli of solid constituents
V	=	total volume
V_s	=	$(1-\phi)V$, the solid volume
V_ϕ	=	ϕV, the pore volume
α	=	total volume effective-stress coefficient
β	=	pore volume effective-stress coefficient
γ	=	inverse of Skempton's constant B
ϵ	=	electrical conductivity effective-stress coefficient
ζ	=	increment of fluid content
θ	=	relative change effective-stress coefficient for two-component porous medium
κ	=	fluid permeability effective-stress coefficient
μ	=	shear modulus of drained porous frame
μ_m	=	material (or grain) shear modulus
σ	=	solid volume effective-stress coefficient
ϕ	=	porosity
χ	=	porosity effective-stress coefficient