Lecture Notes on

Nonlinear Inversion and Tomography:

I. Borehole Seismic Tomography

From a Series of Lectures by

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Chapter 6

Ghosts in Traveltime Inversion and Tomography

A ghost in seismic traveltime inversion is a model perturbation that does not affect the agreement between the predicted and measured first arrival traveltimes. For example, if

$$\mathbf{M}\mathbf{s} = \mathbf{t}, \quad \mathbf{M}(\mathbf{s} + \mathbf{g}) = \mathbf{t}, \tag{6.1}$$

then subtracting shows that

$$\boxed{\mathbf{M}\mathbf{g} \quad = \quad 0,} \tag{6.2}$$

so $\mathbf{g}$ lies in the null space of $\mathbf{M}$, *i.e.*, in the null space of the traveltime functional. A careful analysis of the ghosts shows that, while some are unavoidable due to the limited view angles used when the data were collected, others are caused by unfortunate choices made when discretizing the model. Thus, some ghosts may be eliminated by making unusual choices for the model parametrization.

It is important to realize from the outset that it may not be either possible or even desirable to eliminate all the ghosts. In fact, the normal solution to the least-squares problem cannot be found if $\mathbf{M}^T\mathbf{M}$ is not invertible. Lack of invertibility is caused by the presence of a right null space for $\mathbf{M}$ and the members of that null space we call *ghosts*. In some cases, simple tricks can be developed to eliminate the ghosts, but not always.

One reference on this topic is Ivansson [1986].

6.1 Feasibility Constraints and Ghosts

Because feasibility constraints depend on the traveltime data while ghosts are independent of the traveltime data (depending instead only on the ray path matrix), feasibility constraints are always orthogonal to all ghost vectors. This fact is illustrated in Figure 6.1. Actually, this statement is somewhat oversimplified in the context of nonlinear inversion algorithms, where we may want to consider many ray-path matrices simultaneously, but

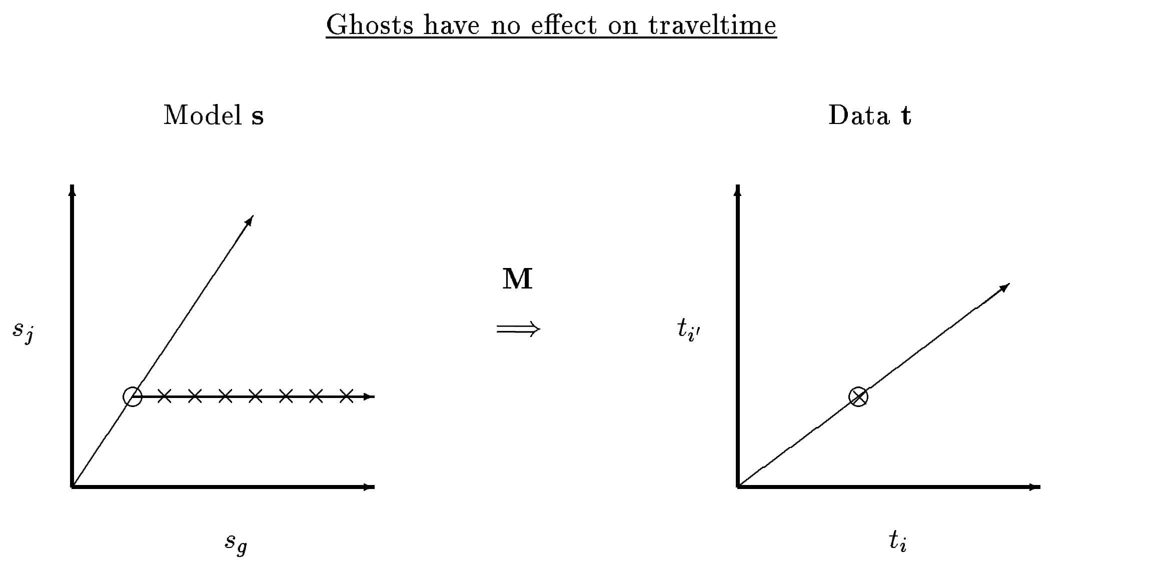

Figure 6.1: Feasibility constraints are orthogonal to all ghost vectors.

we will ignore this complication here. Thus, the remainder of this Chapter will describe various kinds of ghost and other means of dealing with them.

6.2 Types of Ghosts

We now carry out an in-depth analysis of a few common ghosts occurring in seismic inverse problems.

6.2.1 Single cell ghost

A single cell ghost occurs when no ray passes through a certain cell. That cell is uncovered, not illuminated, not hit by any of the rays in the data set (at least for the current choice of ray paths). Thus, the slowness of that cell is arbitrary, as it has no effect on determining any of the traveltimes in the data set.

The proper way to deal with such a cell is to assign it some arbitrary value, like the average slowness of all cells or the average of all contiguous cells.

6.2.2 Two cells with only one ray

When any two cells are covered by one and only one ray, a ghost arises because the increment of traveltime δt_i through these cells is invariant to a perturbation of the form

$$\mathbf{g}^T = (0, \ldots, 0, l_{ik}, 0, \ldots, 0, -l_{ij}, 0, \ldots, 0), \tag{6.3}$$

since

$$\delta t_i = l_{ij}s_j + l_{ik}s_k = l_{ij}(s_j + \alpha l_{ik}) + l_{ik}(s_k - \alpha l_{ij}), \tag{6.4}$$

where α is an almost arbitrary scalar. The one constraint on α is that the perturbed slowness vector

$$\mathbf{s}' = \mathbf{s} + \alpha\mathbf{g} \tag{6.5}$$

must be positive. Note that there is no ghost associated with a single cell having only one covering ray.

The proper way to deal with such pairs of cells (especially if they are contiguous) is to treat them as if they were combined into one larger cell, *i.e.*, assign the same value of slowness to both cells. This approach has the effect of eliminating the ghost while simultaneously reducing the size of the model space by one dimension.

If more than two cells are covered by one and only one ray, then there will be multiple ghosts (for p cells there will be $p-1$ ghosts). Again, one way to eliminate this problem is to treat all such cells as a single cell. This approach may not be the best one if the cells are not contiguous. Other approaches will be discussed in the section on eliminating ghosts.

6.2.3 Underdetermined cells in an overdetermined problem

The preceding discussion is a special case of a more general problem: underdetermined cells imbedded in an overdetermined inversion problem. Underdetermination means having fewer equations than unknowns. The example of two cells with only one ray is a common example of this effect. Others would be three cells with two rays, 20 cells with 15 rays, etc. The existence of underdetermined cells may be the result of poor experimental design, of physical limitations at the experimental site that reduce the possible range of view angles significantly (as in crosshole geometry), or they may be caused by severe ray bending effects when high contrasts in the slowness values are present. In the latter situation, we expect that rays will tend to avoid very slow regions (Fermat's principle says to take the fastest path, which may mean to go around the slow region). Since experiments will normally be planned to achieve the desired resolution assuming straight-ray coverage, the actual coverage in slow regions is smaller than planned and may be so reduced by these ray bending effects to the extent of causing underdetermination.

This problem with ghosts can now be reduced to

$$\mathbf{M}'\mathbf{s}' = \delta\mathbf{t}', \tag{6.6}$$

where $\mathbf{M}'$ is an $m' \times n'$ matrix with $m' < n'$, $\mathbf{s}'$ is the subvector of the slowness model of length n', and $\delta\mathbf{t}'$ is the subvector of the traveltime increment of length m'. A particular solution of (6.6) is given by

$$\mathbf{s}' = \mathbf{M}'^T(\mathbf{M}'\mathbf{M}'^T)^{-1}\delta\mathbf{t}', \tag{6.7}$$

if the matrix $\mathbf{M}'\mathbf{M}'^T$ is invertible. But the general solution of (6.6) is a vector of the form

$$\mathbf{s}' = \mathbf{M}'^T(\mathbf{M}'\mathbf{M}'^T)^{-1}\delta\mathbf{t}' + \mathbf{g}', \tag{6.8}$$

where $\mathbf{g}'$ is any vector from the right null space of $\mathbf{M}'$. This null space must have dimension at least $n' - m'$.

The preferred solution to this problem is again to combine contiguous cells until the number of equations is at least equal to the number of unknowns. Then $n' - m' = 0$, and the null space is eliminated. Another method of dealing with the problem if the cells are not contiguous is to assign a slowness value to $n' - m'$ of those cells that have the least coverage, thus removing them from the inversion problem.

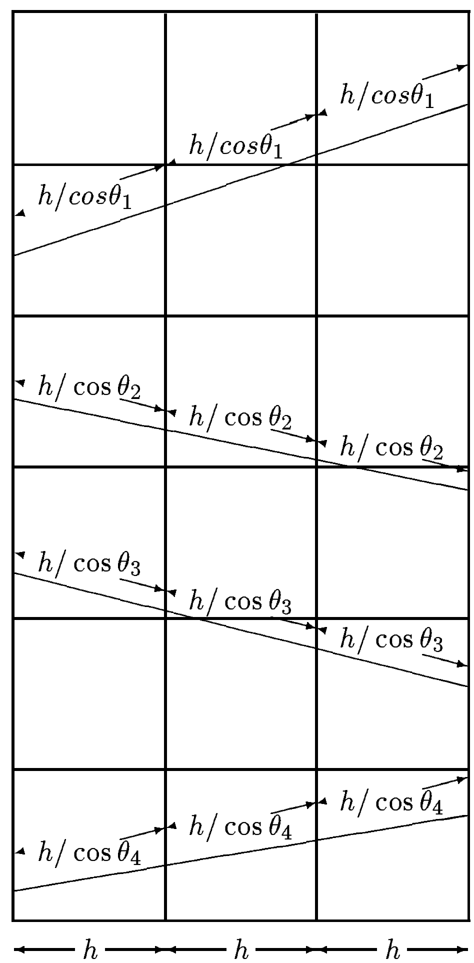

Figure 6.2: Stripes are caused by straight rays used in crosshole geometry.

6.2.4 Stripes

One of the most common types of ghosts in borehole-to-borehole tomography is the vertical stripe. Stripes are ghosts caused by an unfortunate resonance of the model parametrization, the limited set of view angles possible in the crosshole geometry, and the use of straight rays in the reconstruction (see Figure 6.2).

To see how the problem arises, consider the geometry of two vertical boreholes with square cells of dimension h in the image plane. For purposes of illustration, suppose that the borehole separation is just three cell widths and the borehole depth is two cell heights. To get from one borehole to the other, the rays must cross the lines forming the vertical boundaries between cells. Assuming straight rays, each ray is characterized by a total ray-path length $L_i = 3h/\cos\theta_i$, where θ_i is the angle the ray makes with the horizontal. So the total path length in each vertical column is just $h/\cos\theta_i$. This part of the path length is shared among the cells in a column differently for each ray path, but that is not so important. What is important is that the sum is constant in each column for every ray.

The ray-path matrix takes the form

$$\mathbf{M} = \begin{pmatrix} \frac{d_{11}h}{\cos\theta_1} & \frac{d_{12}h}{\cos\theta_1} & \frac{e_{13}h}{\cos\theta_1} & \frac{e_{14}h}{\cos\theta_1} & \frac{f_{15}h}{\cos\theta_1} & \frac{f_{16}h}{\cos\theta_1} \\ \frac{d_{21}h}{\cos\theta_2} & \frac{d_{22}h}{\cos\theta_2} & \frac{e_{23}h}{\cos\theta_2} & \frac{e_{24}h}{\cos\theta_2} & \frac{f_{25}h}{\cos\theta_2} & \frac{f_{26}h}{\cos\theta_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{d_{m1}h}{\cos\theta_m} & \frac{d_{m2}h}{\cos\theta_m} & \frac{e_{m3}h}{\cos\theta_m} & \frac{e_{m4}h}{\cos\theta_m} & \frac{f_{m5}h}{\cos\theta_m} & \frac{f_{m6}h}{\cos\theta_m} \end{pmatrix}, \tag{6.9}$$

where the ds, es, and fs are nonnegative fractions satisfying

$$\sum_{j=1}^{2} d_{ij} = \sum_{j=3}^{4} e_{ij} = \sum_{j=5}^{6} f_{ij} = 1, \tag{6.10}$$

for every ray path $1 \le i \le m$. The ds are associated with the cells in the first column; the es with the second column; and the fs with the third column. Then it is clear that these three vectors

$$\mathbf{g}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{g}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{g}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \tag{6.11}$$

are ghosts for this problem, since in each case we find that

$$\mathbf{Mg} = \begin{pmatrix} \frac{h}{\cos\theta_1} - \frac{h}{\cos\theta_1} \\ \frac{h}{\cos\theta_2} - \frac{h}{\cos\theta_2} \\ \vdots \\ \frac{h}{\cos\theta_m} - \frac{h}{\cos\theta_m} \end{pmatrix} = 0 \tag{6.12}$$

follows from (6.10).

These ghosts show up in the reconstructed slowness as vertical stripes — a constant slowness perturbation is subtracted from one column and added to any other column.

To eliminate these ghosts, we need to break the unfortunate symmetry that has caused this artifact to arise. These ghosts would not exist if the cells were not lined up perfectly with both of the vertical boreholes. So one solution is to use cells that are not square or rectangular, *i.e.*, odd shapes like hexagons, triangles, etc. Using a rectangular but staggered grid would also remove the degeneracy. Or, combining a few of the poorly covered cells near the top and bottom would also break the symmetry. A still simpler method of eliminating the problem (at least conceptually) is to use bent rays, rather than the artificially straight rays that are often incorrectly assumed to be adequate.

PROBLEMS

FORMAT FOR PROBLEMS 6.2.1–6.2.4: *The next four problems consider a* 3×3 *model with the layout*

s_1	s_2	s_3
s_4	s_5	s_6
s_7	s_8	s_9

PROBLEM 6.2.1 *A ray-path matrix for a* 3×3 *slowness model is*

$$\mathbf{M} = \begin{pmatrix} 1.414 & 0 & 0 & 0 & 1.414 & 0 & 0 & 0 & 1.414 \\ 0 & 0.9 & 1.118 & 1.118 & 0.218 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.054 & 1.054 & 1.054 \end{pmatrix}.$$

Find the ghosts for this set of ray paths. If the measured traveltime vector is

$$\mathbf{t} = \begin{pmatrix} 4.666 \\ 3.488 \\ 3.794 \end{pmatrix},$$

which of the following statements is true of the slowness model?

1. *The model is not constant.*

2. *The average slowness of cells 7, 8, & 9 is 1.2.*

3. *The model is*

0.9	*1.2*	*0.9*
1.0	*1.3*	*1.2*
1.1	*1.4*	*1.1*

PROBLEM 6.2.2 *A ray-path matrix for a* 3×3 *slowness model is*

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 1.045 & 1.045 & 1.045 & 0 & 0 & 0 \\ 0 & 0 & 0.403 & 1.052 & 1.052 & 0.649 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.021 & 1.057 & 1.057 & 1.036 & 0 & 0 \\ 1.011 & 0 & 0 & 0.101 & 1.112 & 0.101 & 0 & 0 & 1.011 \\ 0 & 0 & 0 & 0 & 0.502 & 1.004 & 1.004 & 0.502 & 0 \end{pmatrix}.$$

Find the ghosts for this set of ray paths.

PROBLEM 6.2.3 *A ray-path matrix for a* 3×3 *slowness model is*

$$\mathbf{M} = \begin{pmatrix} 1.012 & 0.502 & 0 & 0 & 0.510 & 1.012 & 0 & 0 & 0 \\ 0 & 0 & 0.444 & 1.012 & 1.012 & 0.568 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.005 & 1.005 & 1.005 \\ 0.503 & 1.103 & 1.103 & 0.600 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.001 & 0 & 0 & 0.009 & 1.010 & 1.010 \end{pmatrix}.$$

Find the ghosts for this set of ray paths.

PROBLEM 6.2.4 *A ray-path matrix for a* 3×3 *slowness model is*

$$\mathbf{M} = \begin{pmatrix} 1.033 & 1.033 & 1.033 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.049 & 1.049 & 1.049 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.101 & 1.101 & 1.101 \\ 0 & 0 & 1.414 & 0 & 1.414 & 0 & 1.414 & 0 & 0 \\ 1.118 & 0 & 0 & 0 & 1.118 & 1.118 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.052 & 0 & 0 & 0.025 & 1.077 & 1.077 \\ 0 & 0 & 0 & 0 & 1.044 & 1.055 & 1.055 & 0.011 & 0 \\ 0 & 0 & 0.352 & 1.068 & 1.068 & 0.716 & 0 & 0 & 0 \\ 0.022 & 0 & 0 & 1.053 & 1.075 & 1.075 & 0 & 0 & 0 \end{pmatrix}.$$

Find the ghosts for this set of ray paths, if any.

PROBLEM 6.2.5 *Consider an* $m \times n$ *ray-path matrix* $\mathbf{M}$ *for straight rays through a rectangular model with* q *columns and* r *rows, so* $n = q \times r$*. Suppose that all the rays considered are crosshole. Show that the diagonal elements of the model resolution matrix* $\mathcal{R} = \mathbf{M}^\dagger\mathbf{M}$ *satisfy*

$$\mathcal{R}_{jj} \leq \frac{n+1-q}{n}. \tag{6.13}$$

Make a table of these resolution bounds for different choices of the number of columns and rows. What strategy for model design leads to the best resolution when the rays are straight and crosshole?

PROBLEM 6.2.6 *Consider a slowness model composed of vertical stripes (see Figure 6.3). Suppose that all slowness values and thicknesses of the stripes are known, but the spatial order is not known. Use Snell's law to show that the spatial order of the stripes cannot be determined from vertical crosshole transmission traveltime data. [Hint: Consider the ray parameter (5.17).]*

6.2.5 Linear dependence

Ghosts arise from the linear dependence of $\mathbf{M}$ or of submatrices such as $\mathbf{M}'$. The example of stripes arises from a gross linear dependence of all the rows of the full matrix $\mathbf{M}$. The examples of underdetermined group of cells arise because of poor coupling (or coverage)

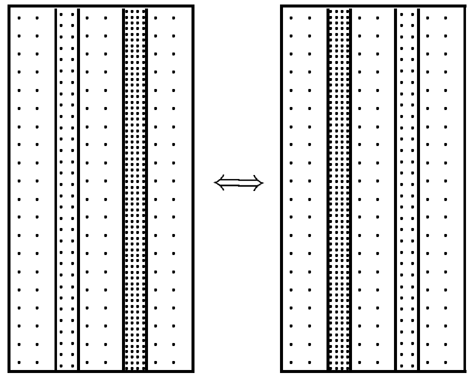

Figure 6.3: Two indistinguishable models: the spatial ordering of the stripes cannot be determined from crosshole transmission data using straight rays.

between the rays and the cells. Other, more subtle and complex, linear dependencies may also occur.

Indeed the defining equation for a ghost

$$\mathbf{Mg} = 0 \tag{6.14}$$

is a statement of linear dependence of the rows of $\mathbf{M}$. Equation (6.14) contains m equations for the n unknown components of $\mathbf{g}$, with $m > n$. Any n of these m equations are sufficient to determine $\mathbf{g}$ and the remaining $m-n$ equations can be determined from these n equations. An exception to this statement occurs when the ghost is caused by complete decoupling as in the case of the single cell ghost.

6.3 Eliminating Ghosts (Ghostbusting)

Ghosts may be removed by using a variety of techniques, some of which have already been described. Although ray bending can introduce ghosts in situations where very slow regions are avoided by most rays, it can also provide a simple solution to some of the problems created by the limited view angles available in crosshole measurements. Methods of improving the coupling between rays and cells include implementations using fat rays [Kak, 1984; Michelena and Harris, 1991].

6.3.1 Fat rays

The fact that ray paths are stationary (*i.e.*, that small variations in the ray path have no effect on the traveltime to first order) means that each ray actually has a bundle of rays close to it, all with virtually the same traveltime. We can improve the coupling between the rays and cells in the model by taking advantage of this fact. One possible approach is to use more than one ray between each source and receiver pair: for example, during the computation of the ray paths, we could save not only the ray we found with the least traveltime, but also save several other trial rays that were close to the best one. Then, in place of the single row of the ray-path matrix for the ith ray, we now insert multiple rows

$$\mathbf{M}_i = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{\mu 1} & l_{\mu 2} & \cdots & l_{\mu n} \end{pmatrix}, \tag{6.15}$$

where μ is the multiplicity of the the ith ray path. This approach is relatively easy to implement either for bending methods or for shooting methods of ray tracing, and has the effect of multiplying the size of the data set by the number of rays (μ) saved per source/receiver pair. The disadvantage of using multiple ray paths per source/receiver pair is that the data storage problem also gets multiplied by the number of rays saved per pair.

The method of fat rays is an alternative using the same underlying physics without increasing the size of our matrices. In this approach, we treat each ray as if it has a finite thickness. Then, instead of measuring the linear increment of the ray that has passed through a cell, in 2-D the ray now has an area associated with it and we measure the overlap of the ray area with the cell area. In three dimensions, these areas all become volumes. If the ray width in 2-D is Δw and the ray cross section in 3-D has area Δa, then the ray-path matrix becomes

$$\mathbf{M} = \frac{1}{\Delta w} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \tag{6.16}$$

with the a_{ij}s being overlap areas in two dimensions and

$$\mathbf{M} = \frac{1}{\Delta a} \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{pmatrix}, \tag{6.17}$$

with the v_{ij}s being overlap volumes in three dimensions. For traveltime tomography, it is still important that the sums of the rows of $\mathbf{M}$ result in sensible ray-path lengths, otherwise the reconstructed slowness values will not be meaningful. The disadvantage of the method just outlined is that the overlap areas and volumes are often tedious to compute.

Another method that has the advantages of both of the previous methods is first to obtain the set of near-ray-path lengths shown in (6.15) and then average them according to

$$\bar{l}_{ij} = \frac{1}{\mu} \sum_{i'=1}^{\mu} l_{i'j}. \tag{6.18}$$

With this approach, we end up with a single effective ray path and so do not increase the ray storage problem, but we have the advantage that the individual contributions $l_{i'j}$ leading to $\bar{l}_{ij}$ through (6.18) are comparatively easy to compute. The resulting ray-path matrix is

$$\overline{\mathbf{M}} = \begin{pmatrix} \bar{l}_{11} & \bar{l}_{12} & \cdots & \bar{l}_{1n} \\ \bar{l}_{21} & \bar{l}_{22} & \cdots & \bar{l}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{l}_{m1} & \bar{l}_{m2} & \cdots & \bar{l}_{mn} \end{pmatrix}. \tag{6.19}$$

It is clear that fat rays will accomplish the goal of improving the coupling between the rays and cells. The matrices in (6.16) and (6.17) will certainly be significantly less sparse than the the usual $\mathbf{M}$ based on skinny rays. Whether this change will be sufficient to make a significant improvement in the reconstructions will, of course, depend on the particular application. In general, fat rays should be used in addition to (not instead of) the other methods of ghostbusting described in this section.

6.3.2 Damping

The general damped least-squares solution $\mathbf{s}$ to the inversion problem, including general weight matrices, was shown in (4.100) to satisfy

$$\left(\mathbf{M}^T\mathbf{F}^{-1}\mathbf{M} + \mu\mathbf{G}\right)(\mathbf{s} - \mathbf{s}_0) = \mathbf{M}^T\mathbf{F}^{-1}(\mathbf{t} - \mathbf{M}\mathbf{s}_0). \tag{6.20}$$

Since damping is designed to reduce the effects of contributions from eigenvectors with low eigenvalues, we should check to see if this approach eliminates ghosts. Multiplying (6.20) on the right by $\mathbf{g}^T$ and using the definition $\mathbf{M}\mathbf{g} = 0$, we find that

$$\mathbf{g}^T\mathbf{G}(\mathbf{s} - \mathbf{s}_0) = 0. \tag{6.21}$$

When the weight matrix $\mathbf{G} = \mathbf{I}$, (6.21) shows that no ghosts can contribute to $\mathbf{s}$ unless $\mathbf{s}_0$ already contains such terms. Thus, norm damping successfully eliminates ghosts in this one case. However, if $\mathbf{G} \neq \mathbf{I}$, (6.21) is a conjugacy condition, showing that $\mathbf{g}$ is orthogonal to the correction vector $\mathbf{s} - \mathbf{s}_0$ relative to the weight matrix $\mathbf{G}$. Although (6.21) is consistent with the absence of ghosts in the correction vector, it does not guarantee they are eliminated in all cases. When $\mathbf{G} = \mathbf{C}$ (the coverage matrix), the interpretation of (6.21) is almost as simple as that for the identity matrix; when $\mathbf{G} = \mathbf{K}^T\mathbf{K}$ where $\mathbf{K}$ is based on gradients or Laplacians of the the model vector, its interpretation becomes more complex.

6.3.3 Summary

Methods of eliminating ghosts can be divided into two main categories: (*i*) experimental design and (*ii*) model design together with analytical tricks.

No amount of analysis can salvage a badly designed experiment. When designing a tomographic experiment, it is important to gather data from as many view angles as possible. It is also important to gather enough data so that the cells we can resolve from our data analysis are about the same size as the anomalies we want to detect. A rule of thumb is that the number of source/receiver pairs should be about *twice* the number of cells we want to resolve in our experiment. Another useful rule of thumb is to choose the average cell size to be about $3\lambda_{\max}$, where $\lambda_{\max} = 1/f_{\min}s_{\min}$ is the maximum expected wavelength associated with the minimum frequency $f_{\min}$ in the pulse propagation data and the minimum slowness $s_{\min}$ expected in the region to be imaged. This rule arises from extensive experience with the asymptotic analysis of wave propagation which we will not present here.

The analyst must design the model to take optimum advantage of the data gathered, while accounting for any prior knowledge of the medium to be imaged. The shapes and sizes of the model cells are ours to choose, and should be used to advantage to solve any problems that cannot be eliminated through good experimental design. We are always free to choose cells larger than the expected resolution of the traveltime data. We may delete some cells if they have poor ray coverage, or some contiguous cells with poor coverage may be combined into a single cell for purposes of reconstruction. Cells can be of any shape we choose; the choice of square or rectangular cells is often made for ease of display and for ease of computation of ray paths, but other considerations may drive us to use odd shapes for cells in some applications. Analytical tricks can be applied during the reconstruction process once we have the data at home. Smoothing and clipping the slowness model values can be done to force the reconstructed values to lie within reasonable limits. Fat rays are a last resort if the other methods are not sufficient to eliminate the ghosts.

6.4 Significance of Ghosts

It is important to recognize that elimination of all ghosts may be neither possible nor desirable. In our efforts to solve the inverse problem

$$\mathbf{Ms} = \mathbf{t} \tag{6.22}$$

for the slowness model $\mathbf{s}$, we should keep in mind that there are really three stages in the inversion process. The first stage is to find, if possible, a particular model $\mathbf{s}$ that satisfies the data. The second stage is to analyze the null space of the operator $\mathbf{M}$. We may use standard numerical techniques at this point in the analysis to perform a singular value decomposition of $\mathbf{M}$ and obtain a full characterization of the null space. Having finished both of these steps, we can finally provide the complete solution to the inversion problem. In fact, it may be that we need perturbations from the null space to satisfy various physical or geological boundary conditions present at the site where the tomographic data were gathered. This process is entirely analogous to the process of solving an ordinary differential equation by finding a particular solution, computing a set of homogeneous solutions, and finally producing a linear combination that satisfies the initial or boundary conditions.

Problems

PROBLEM 6.4.1 *An inhomogeneous linear differential equation of first order is*

$$\frac{dx}{dt} + \lambda x = f(t),$$

with the initial condition $x(t) = x(0)$ for $t = 0$. The solution of this equation is well known to be

$$x(t) = x(0)\exp(-\lambda t) + \exp(-\lambda t)\int_0^t f(t')\exp(\lambda t')dt'.$$

Which term of this solution is analogous to $\mathbf{s}_{\mathrm{LS}} = \mathbf{M}^{\dagger}\mathbf{t}$*? Which term is analogous to a ghost satisfying* $\mathbf{Mg} = 0$*? Which term is analogous to the scalar* γ *in the full solution* $\mathbf{s} = \mathbf{s}_{\mathrm{LS}} + \gamma\mathbf{g}$*?*

PROBLEM 6.4.2 *Solve* $\mathbf{Ms} = \mathbf{t}$ *for* $\mathbf{s}$ *when*

$$\mathbf{M} = \begin{pmatrix} 13/12 & 13/12 & 5/6 \\ 13/12 & 13/12 & 5/6 \\ 5/6 & 5/6 & 4/3 \end{pmatrix}$$

and $\mathbf{t}^T = (49/16, 49/16, 23/8)$*, subject to the constraint* $s_2 = s_1 + 1/4$*. If the constraint is changed to* $s_2 = s_1 + 3$*, does the problem still have a physical solution (*$\mathbf{s} > 0$*)? What is the range of permissible values of* σ *in the constraint* $s_2 = s_1 + \sigma$*?*

PROBLEM 6.4.3 *Reconsider* PROBLEM 6.2.1 *in light of the following additional information:*

1. *The total variation in the model from top to bottom is less than 20%.*
2. *The model is probably horizontally stratified, or nearly so.*
3. *The given ray-path matrix is only a straight ray approximation to the true ray path.*

First, solve the problem by considering only the first two of the new constraints while using the straight ray approximation. Then, try to solve the full problem using one-step backprojection based on bent rays.

PROBLEM 6.4.4 *Formulate a definition of the "best approximate solution" of the matrix equation* $\mathbf{MZ} = \mathbf{Y}$ *when constraints on* $\mathbf{Z}$ *are given. Compare and contrast this definition with the one given in* PROBLEM 4.1.25*. Is there a unique best approximate solution of* $\mathbf{MZ} = \mathbf{Y}$ *that satisfies the new definition?*