Lecture Notes on

Nonlinear Inversion and Tomography:

I. Borehole Seismic Tomography

Developed from a Series of Lectures by

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Contents

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Preface

> In general we look for a new law by the following process: First we guess it. Then we compute the consequences of the guess to see what would be implied if this law that we guessed is right. Then we compare the result of the computation to nature, with experiment or experience, compare it directly with observation, to see if it works. If it disagrees with experiment it is wrong. In that simple statement is the key to science.
>
> — Richard P. Feynman, *The Character of Physical Law*

Nonlinear inverse problems are common in science and engineering. In fact the quotation from Feynman shows clearly that the process of discovering physical laws is itself an inverse problem.

What is an inverse problem? Subtraction is the inverse of addition. Division is the inverse of multiplication. Root is the inverse of power. Given the answer (say, the number 4) find the question ($2+2=?$ or $8/2=?$ or $\sqrt{16}=?$). This last example (commonly seen in the game of Jeopardy) is most important since it is clear that the same answer (*i.e.*, the data) could come from many questions (*e.g.*, models and methods of analysis) — and therefore it is not surprising that a degree of ambiguity (sometimes a very high degree of ambiguity) is an inherent part of most realistic inverse problems. Physical scientists are used to thinking about situations that lead to equations with unique solutions. Because of the traditional training they receive, most scientists are very uncomfortable with mathematical problems that do not have unique solutions. Yet both quantum mechanics and chaos theory provide numerous examples of real experiments (*e.g.*, the double slit experiment and weather) where ambiguities are typically encountered. The subject of inverse problems is another realm where lack of uniqueness commonly occurs. Methods of dealing with the ambiguities therefore play a vital — if not quite central — role in our analysis.

How do we solve an inverse problem? In general, we use the prescription described by Feynman: guess, compute, compare. But one more element is added in the inverse problems we discuss: feedback. When searching for physical laws, we make a guess, compute the consequences, and compare with experiment. If the comparisons are unfavorable, then we have learned that our first guess is bad, but we may not have any constructive procedure for generating a better guess. In contrast, when trying to solve problems in nonlinear inversion and tomography, we often think we know the physical laws (*i.e.*, the general equations governing the processes), but we may not know the precise values of the parameters in the equations. In such circumstances, it may be possible to make use of the observed discrepancies between the measured and computed results to adjust the parameters and, thereby, obtain improved guesses in a systematic way. This process of feeding the errors back to help produce a better set of parameters for the starting equation then becomes the paradigm for nonlinear inversion algorithms. In some cases it may happen that one feedback step is sufficient; in others many iteration steps may be needed to achieve the desired agreement between model and data.

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Chapter 1

Introduction to the Traveltime Inversion Problem

Our main topic is seismic traveltime inversion in 2- and 3-dimensional heterogeneous media. A typical problem is to infer the (isotropic) compressional-wave slowness (reciprocal of velocity) distribution of a medium, given a set of observed first-arrival traveltimes between sources and receivers of known location within the medium. This problem is common for crosswell seismic transmission tomography imaging a 2-D region between vertical boreholes in oil field applications. We also consider the problem of inverting for wave slowness when the absolute traveltimes are not known, as is normally the case in earthquake seismology.

In this Introduction, we define most of the general terminology we will use throughout our analysis.

1.1 Wave Slowness Models

When a sound wave or seismic wave is launched into a medium, it takes time for the influence of the wave to progress from a point close to the source to a more distant point. The time taken by the wave to travel from one point of interest to the next is called the *traveltime*. For a medium that is not undergoing physical or chemical changes during the passage of the sound, the wave has a definite speed with which it always travels between any two points in the medium. We call this speed the *average wave speed* or *wave velocity*. We can also define a *local wave speed* associated with each point in the medium by considering the average wave speed for two points that are very closely spaced. The *local slowness* is the inverse of the local wave speed. It is most convenient to develop inversion and tomography formulas in terms of wave slowness models, because the pertinent equations are linear in slowness.

We consider three kinds of slowness models. Sometimes we allow the slowness to be a general function $s(\mathbf{x})$ of the position $\mathbf{x}$. However, we often make one of two more restrictive assumptions that (i) the model comprises homogeneous cells (in 2-D), or blocks (in 3-D), with s_j then denoting the slowness value of the jth cell, or blocks. Or (ii) the model is composed of a grid with values of slowness assigned at the grid points together with some interpolation scheme (bilinear, trilinear, spline, etc.) to specify the values between grid points. Of course, as cells/blocks become smaller and smaller (down to infinitesimal), we can think of cells/blocks of constant slowness as a special case of continuous models, or of continuous models as a limiting case of cells/blocks.

When it is not important which type of slowness model is involved, we refer to the model abstractly as a vector $\mathbf{s}$ in a vector space $\mathcal{S}$. For a block model with n blocks we have $\mathcal{S} = \mathbf{R}^n$, the n-dimensional Euclidean vector space. ($\mathbf{R}$ denotes the set of real numbers.) A continuous slowness model, on the other hand, is an element of a function space, e.g., $\mathcal{S} = C(\mathbf{R}^3)$, the set of continuous functions of three real variables. No matter how we parameterize the model, we should always keep in mind that, for real materials, our models necessarily have far fewer parameters than the actual medium they are intended to represent. Thus, our models are analogous to cartoon drawings of public figures, trying to capture the main features with the minimum of detail.

1.2 Fermat's Principle and Traveltime Functionals

The traveltime of a seismic wave is the integral of slowness along a ray path connecting the source and receiver. To make this more precise, we will define two functionals[1] for traveltime.

Let P denote an arbitrary path connecting a given source and receiver in a slowness model $\mathbf{s}$. We will refer to P as a *trial ray path*. We define a functional τ^P which yields the traveltime along path P. Letting $\mathbf{s}$ be the continuous slowness distribution $s(\mathbf{x})$, we have

$$\tau^P(\mathbf{s}) = \int_P s(\mathbf{x})\, dl^P, \tag{1.1}$$

where dl^P denotes the infinitesimal distance along the path P.

Fermat's principle [Fermat, 1891; Goldstein, 1950; Born and Wolf, 1980] states that the correct ray path between two points is the one of least overall traveltime, *i.e.*, it minimizes[2] $\tau^P(\mathbf{s})$ with respect to path P.

Let us define τ^* to be the functional that yields the traveltime along the Fermat (least-time) ray path. Fermat's principle then states

$$\tau^*(\mathbf{s}) = \min_{P \in Paths} \tau^P(\mathbf{s}), \tag{1.2}$$

where *Paths* denotes the set of all continuous paths connecting the given source and receiver.[3] The particular path that produces the minimum in (1.2) is denoted P^*. If more than one path produces the same minimum traveltime value, then P^* denotes any particular member in this set of minimizing paths.

Substituting (1.1) into (1.2), we have *Fermat's principle of least time*:

$$\boxed{\tau^*(\mathbf{s}) \quad = \quad \textstyle\int_{P^*} s(\mathbf{x})\, dl^{P^*} \quad = \quad \min_P \int_P s(\mathbf{x})\, dl.^P} \tag{1.3}$$

---

[1] A *functional* is a function which maps a function space or a vector space to the set of real numbers.

[2] Fermat's principle is actually the weaker condition that the traveltime integral is *stationary* with respect to variations in the ray path, but for traveltime inversion using measured first arrivals it follows that the traveltimes must be *minima*.

[3] The notation $P \in Paths$ means that P is a *member* of the set *Paths*.

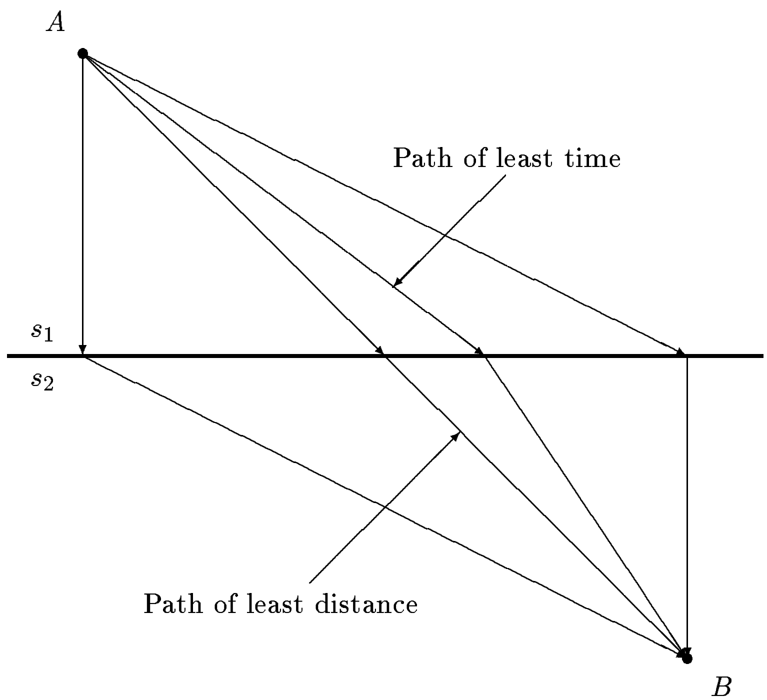

Figure 1.1: Snell's law gives the path of least traveltime from point A to point B. Other paths shown are: least distance through medium 1, least distance through medium 2, and least total distance.

The traveltime functional $\tau^*(\mathbf{s})$ is stationary with respect to small variations in the path $P^*(s)$.

1.3 Snell's Law

Snell's law is a consequence of Fermat's principle [Born and Wolf, 1980]. This result can be derived using a simple geometric argument based on stationarity of the traveltime functional, illustrated in Figures 5.1 and 5.2. The well-known result is

$$\boxed{s_1 \sin\theta_1 \quad = \quad s_2 \sin\theta_2, \qquad \text{(Snell's law)}} \tag{1.4}$$

where θ_1 and θ_2 denote the angles of the ray path from the normal to the boundary that separates the two regions.

A thorough discussion of the physical significance of Fermat's principle and its relation to Snell's law may be found in *The Feynman Lectures* [Feynman, Leighton, and Sands, 1963]. Relations to the *principle of least action* and *Hamilton-Jacobi theory* are discussed by Goldstein [1950], Boorse and Motz [1966], and Born and Wolf [1980]. An interesting and less technical account is given by Gleick [1993].

The main point to be made here is that Snell's law is special. There are various assumptions that go into the derivation such as: the points A and B are far from the boundary, the two media on either side of the boundary are homogeneous with constant isotropic slowness, etc. For general imaging problems, the underlying media may be very complex and it may not be convenient to apply Snell's law. A standard ray tracing method may fail in some circumstances, so it is preferable to consider more robust methods of determining approximate ray paths and traveltimes. Such methods will be discussed in some detail in Chapter 5.

1.4 Seismic Inversion and Tomography

Suppose we have a set of observed traveltimes, $t_1, \ldots, t_m$, from m source-receiver pairs in a medium of slowness $s(\mathbf{x})$. Let P_i be the Fermat ray path connecting the ith source-receiver pair. Neglecting observational errors, we can write

$$\int_{P_i} s(\mathbf{x})\, dl^{P_i} = t_i, \quad i = 1, \ldots, m. \tag{1.5}$$

Given a block model of slowness, let l_{ij} be the length of the ith ray path through the jth cell:

$$l_{ij} = \int_{P_i \cap \text{cell}_j} dl^{P_i}. \tag{1.6}$$

Given a model with n cells, Eq. (1.5) can then be written

$$\sum_{j=1}^{n} l_{ij} s_j = t_i, \quad i = 1, \ldots, m. \tag{1.7}$$

Note that for any given i, the ray-path lengths l_{ij} are zero for most cells j, as a given ray path will in general intersect only a few of the cells in the model. Figure 1.2 illustrates ray path segmentation for a 2-D cell model.

We can rewrite (1.7) in matrix notation by defining the column vectors $\mathbf{s}$ and $\mathbf{t}$ and the matrix $\mathbf{M}$ as follows:

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}, \qquad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mn} \end{pmatrix}. \tag{1.8}$$

Equation (1.7) then becomes the basic equation of forward modeling for ray equation analysis:

$$\boxed{\mathbf{M}\mathbf{s} \;=\; \mathbf{t}.} \tag{1.9}$$

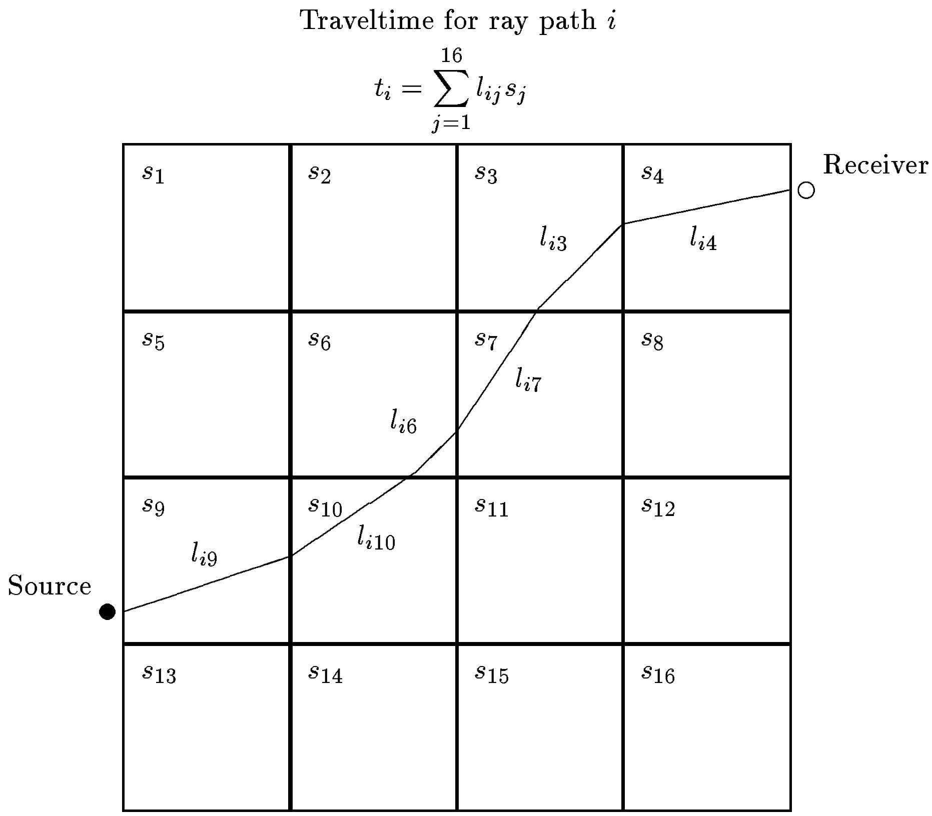

Figure 1.2: Schematic illustration of ray paths through a cell slowness model.

Note that equation (1.9) may be viewed as a numerical approximation to equation (1.3), *i.e.*, it is just a discretized form of the equation. We will study equation (1.9) at great length. Equation (1.9) may be used for any set of ray paths, whether those ray paths minimize (1.3) or not. If the ray paths used to form the matrix $\mathbf{M}$ actually are minimizing ray paths, then we should keep in mind that $\mathbf{M}$ is then implicitly a function of $\mathbf{s}$.

The methods developed apply to both two-dimensional and three-dimensional imaging applications. We use the term *inversion* for either 2-D or 3-D applications. When discussing only 2-D applications, we will use the term *tomography*. The prefix *tomo* is Greek for *slice* and therefore implies a 2-D reconstruction. Similarly, the cells in 2-D are sometimes called *pixels* since they are 2-D picture elements, while the cells or blocks in 3-D are sometimes called *voxels* since they are 3-D volume elements. Thus, traveltime tomography reconstructs the slowness values in model pixels (or cells), while traveltime inversion reconstructs the values in model voxels (or blocks or cells).

1.5 Backprojection for Bent Rays

The term *backprojection* will be used to mean a one-step approximate inversion scheme to solve (1.9) for the slowness vector $\mathbf{s}$.

The physical idea behind backprojection is this: If we have measured a traveltime t_i along the ith ray-path and we know the total path length along that path is $L_i = \sum_{j=1}^{n} l_{ij}$, then the path-average slowness along the ray path is

$$\langle s \rangle_i = \frac{t_i}{L_i} = \frac{\int_{P_i} s \, dl^{P_i}}{\int_{P_i} dl^{P_i}}. \tag{1.10}$$

The ith ray path passes through the jth cell if $l_{ij} > 0$ and misses the cell if $l_{ij} = 0$. An estimate of the slowness in cell j can be obtained by finding the mean of the path-average slownesses $\langle s \rangle_i$ for all the rays that do traverse the jth cell. This averaging process is *backprojection*: accumulating (summing) all the path-averages and then dividing by the total number of contributions.

We can formalize this procedure by introducing the sign function such that $\text{sgn}(l_{ij}) = 1$ if $l_{ij} > 0$ and $\text{sgn}(l_{ij}) = 0$ if $l_{ij} = 0$. Then, the total number of ray paths passing through the jth cell is $N_j = \sum_{i=1}^{m} \text{sgn}(l_{ij})$, and the mean slowness is

$$s_j \simeq N_j^{-1} \sum_{i=1}^{m} \text{sgn}(l_{ij}) \frac{t_i}{L_i}. \tag{1.11}$$

We call (1.11) the formula for *elementary backprojection*. Variations on this formula are explored in the PROBLEMS.

The formula (1.11) provides a fast but inaccurate estimate of the cell slowness, based on available data. The formula is so simple that it can easily be evaluated by hand or using a pocket calculator, whereas the more accurate methods of inverting for slowness (see Chapter 4) are not really practical unless modern computational facilities are available.

There are many possible modifications of the physical arguments for backprojection formulas. Each new choice seemingly leads to a new estimate, showing that the interpretation of these estimates is ambiguous and these methods should not be used for work requiring high accuracy reconstruction. For example, suppose that the slowness in cell j is determined by a weighted sum of the products $l_{ij}t_i$. This approach seems to be an improvement over the preceding one, since it still accounts for our expectation that the ith ray path should not contribute to the estimate if $l_{ij} = 0$ but in addition weights a ray path more heavily when it samples more of the cell. The formula then becomes

$$s_j \simeq \sum_{i=1}^{m} w_i l_{ij} t_i, \tag{1.12}$$

where some choice of the weights w_i must be made. Substituting (1.12) into (1.9), we find that

$$\sum_{j=1}^{n} l_{ij} s_j = \sum_{k=1}^{m} w_k \left(\sum_{j=1}^{n} l_{ij} l_{kj} \right) t_k \simeq t_i. \tag{1.13}$$

Our initial choice of the form of the weights was too simple to allow a rigorous solution to be developed this way, but an approximate solution is obtained by choosing

$$w_i = \left(\sum_{j=1}^{n} l_{ij}^2 \right)^{-1} . \tag{1.14}$$

Defining a diagonal matrix $\overline{\mathbf{D}}$ whose diagonal elements are given by

$$\overline{\mathbf{D}}_{ii} = (\mathbf{M}\mathbf{M}^T)_{ii} = \sum_{j=1}^{n} l_{ij}^2, \tag{1.15}$$

we see that (1.12) and (1.14) lead to the estimate

$$\mathbf{s} \simeq \overline{\mathbf{D}}^{-1}\mathbf{M}^T\mathbf{t}. \tag{1.16}$$

This result is not so simple as the formula (1.11) for elementary backprojection, but the implied computations are still manageable without using very sophisticated computers.

Formulas (1.11), (1.16), and numerous variations are all *backprojection* formulas. As we attempt to compute accurate inverses, these backprojection formulas will frequently reappear as the starting point of rigorous iteration schemes.

Problems

PROBLEM 1.5.1 *Define the hit matrix* $\mathbf{H}$ *such that* $H_{ij} = sgn(l_{ij})$ *and the diagonal matrices* $\mathbf{N}$ *such that* $N_{jj} = \sum_{i=1}^{m} sgn(l_{ij})$*, and* $\mathbf{L}$*, such that* $L_{ii} = \sum_{j=1}^{n} l_{ij}$*. Show that (1.11) is equivalent to*

$$\mathbf{s} \simeq \mathbf{N}^{-1}\mathbf{H}^T\mathbf{L}^{-1}\mathbf{t}.$$

PROBLEM 1.5.2 *Elementary backprojection can be applied either to wave slowness as in (1.11) or to wave velocity. Then,*

$$v_j = \frac{1}{s_j} \simeq N_j^{-1} \sum_{i=1}^{m} sgn(l_{ij}) \frac{L_i}{t_i}. \tag{1.17}$$

It is a general result that the harmonic mean of a set of numbers $\{x_i\}$ *(given by* $x_{harm}^{-1} = N^{-1}\sum_{i=1}^{N} x_i^{-1}$*) is always less than or equal to the mean (*$x_{mean} = N^{-1}\sum_{i=1}^{N} x_i$*), i.e.,*

$$x_{harm} \leq x_{mean}. \tag{1.18}$$

Use this result to determine a general relation between the backprojection formulas (1.11) and (1.17).

PROBLEM 1.5.3 *Consider an elementary backprojection formula based on weighting the average ray slowness* $\langle s \rangle_i = t_i/L_i$ *with respect to the path length* l_{ij} *– instead of with the cell hit factor* $sgn(l_{ij})$. *Show that an alternative to (1.11) is then given by*

$$\mathbf{s} \simeq \mathbf{C}^{-1}\mathbf{M}^T\mathbf{L}^{-1}\mathbf{t}, \tag{1.19}$$

where $\mathbf{C}$ *is the diagonal matrix whose diagonal elements are given by* $C_{jj} = \sum_{i=1}^{m} l_{ij}$. *Define a corresponding estimate for the velocity, then use (1.18) to obtain a relation between these two estimates. Using (1.19), show that*

$$\sum_{i=1}^{m} (\mathbf{Ms})_i = \sum_{i=1}^{m} t_i,$$

*demonstrating that this backprojection formula is an unbiased estimator (*i.e., *average predicted traveltime agrees with average measured traveltime). What can be said about the bias of the backprojection formula for velocity?*

PROBLEM 1.5.4 *Construct a backprojection formula by supposing that the slowness may be determined in the form*

$$s_j \simeq \sum_{i=1}^{m} w_i l_{ij} \frac{t_i}{L_i}$$

and by finding a useful set of weights w_i. *Compare the resulting formula to (1.16).*

1.6 Diffraction Tomography and Full Waveform Inversion

Geophysical diffraction tomography [Devaney, 1983; Harris, 1987; Wu and Toksöz, 1987; Lo, Duckworth, and Toksöz, 1990] consists of a collection of methods including Born [Born, 1926; Newton, 1966] and Rytov [Rytov, 1937; 1938; Keller, 1969; Born and Wolf, 1980] inversion that make use of full waveform information in seismic data. An example of real crosswell transmission data is shown in Figure 1.3. Successful inversion of real data has also been performed using both microwave and ultrasonic diffraction tomography [Tabbara, Duchêne, Pichot, Lesslier, Chommeloux, and Joachimowicz, 1988]. Instead of using only the first arrival traveltimes as the data in the inversion, amplitude and phase in the waveform following the first arrival are used. It is necessary to use full waveform information whenever the wavelengths of the probing waves are comparable in size to the anomalies present in the region to be imaged. The ray approximation is strictly valid only for very high frequencies or equivalently for wavelengths "small" compared with the size of the anomalies (there will be a discussion of the eikonal equation in a Chapter 5). The term "small" is subject to interpretation, but extensive experience with the asymptotic analysis of wave propagation problems [Bleistein, 1984] has shown that, if the largest wavelength found in bandlimited data is $\lambda_{\max}$, then the ray approximation is valid when the anomalies are of size $\simeq 3\lambda_{\max}$ or larger. If this relationship is violated by the tomographic experiment, then diffraction tomography should play an important role in the reconstruction.

Figure 1.3: Example of real data for crosswell seismic tomography showing the result of a single fan beam with a source at 700 feet in one borehole and receivers spaced 10 feet apart in the other hole. (Courtesy of CONOCO Inc.)

Diffraction tomography is both more and less ambitious than traveltime tomography. As it exists today, diffraction tomography is a strictly linear tomography method. A starting model is required. The usual starting model is a constant, because this method requires a comparison between predicted wave fields (planewaves for a constant background) and the measured wave fields. If a nonconstant starting model is used, then "distorted wave" diffraction tomography may be applied to the differences between the computed complex wave field and the measured wave field. In either case, it is possible to prove convergence of diffraction tomography to a solution of the inversion problem if the comparison wave field differs by a small enough amount from the measured wave field. Thus, diffraction tomography is one type of *linear tomography* — although in this case the "rays" may not be straight, it is still linear in the mathematical sense that the perturbations from the starting model must be very small in some sense. So diffraction tomography is less ambitious than traveltime tomography in the sense that it is inherently limited to be *linear tomography*.[4]

On the other hand, diffraction tomography is more ambitious than traveltime tomography, because it tries to make use of more of the information contained in the measured seismic waveforms. There are serious problems involved with this process, because amplitude information can be ambiguous. It is well known that wave attenuation, scattering, three-dimensional geometrical spreading, mode conversion, and reflection/transmission effects can all mimic each other — producing similar effects in the waveform. Thus, to be successful, diffraction tomography must achieve the ambitious goal of solving all of these problems simultaneously for real data. To date, most of the work in diffraction tomography has been limited to two-dimensional inversions and the most successful applications have used ultrasound for medical imaging or microwaves for imaging metallic reinforcements in concrete.

I view diffraction tomography and full waveform inversion as challenging long-term goals. The wave slowness results obtained from our traveltime tomography analysis may be used as the required starting model for "distorted wave" diffraction tomography. So the potential benefits of diffraction tomography provide an additional motivation for improving traveltime inversion and tomography.

1.7 Linear vs Nonlinear Inversion and Tomography

We now define three problems in the context of Eq. (1.9). Each of these problems will be studied at some length in this book.

In the *forward* problem, we are given $\mathbf{s}$; the goal is to determine $\mathbf{M}$ and $\mathbf{t}$. This entails computing the ray path between each source and receiver (*e.g.*, using a ray tracing algorithm) and then computing the traveltime integral along each path.

In *linear tomography or inversion* problems, we are given $\mathbf{M}$ and $\mathbf{t}$; the objective is to determine $\mathbf{s}$. The assumption here is that the ray paths are known *a priori*, which is justified under a linear approximation that ignores the dependence of the ray paths on the slowness distribution. Typically, the ray paths are assumed to be straight lines connecting

[4] An iterative method for diffraction tomography has been proposed recently by Ladas and Devaney [1991]; a nonlinear least-squares approach to full waveform inversion has been proposed by Tarantola and Valette [1982] and Tarantola [1984]. Such methods are "nonlinear" in the sense used here.

sources and receivers, adding a second connotation to the term *linear*. Linear tomography is commonly practiced in medical imaging and in many geophysical situations as well.

In *nonlinear tomography or inversion* problems, we are given only $\mathbf{t}$ (along with the source and receiver locations); the goal is to infer $\mathbf{s}$, and (for most of the methods considered) incidently $\mathbf{M}$. In this problem, the dependence of ray paths on the slowness distribution strongly influences the design of the inversion algorithm. Nonlinear inversion is required for problems with significant slowness variations across the region of interest, including many seismic inversion problems. The ray paths in such media will show large curvature (*i.e.*, be nonlinear) which cannot be known before the inversion process begins.

Linear tomography and inversion problems can be solved approximately using *backprojection* techniques (see Section 1.5). Linear inversion problems can also be solved more accurately using a variety of optimization techniques. In the standard least-squares method (see Section 3), for example, the normal solution for $\mathbf{s}$ is expressed analytically as

$$\hat{\mathbf{s}} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{t}, \tag{1.20}$$

assuming the matrix inverse exists. If the inverse does not exist, then (1.20) must be *regularized*. Typically, regularization is accomplished by adding a positive matrix to $\mathbf{M}^T\mathbf{M}$ and replacing the singular inverse in (1.20) by the inverse of the modified matrix.

For nonlinear inversion, an iterative algorithm is generally needed to find an approximate solution $\hat{\mathbf{s}}_b$. The basic structure of such an algorithm (see Figure 1.4) is as follows:

1. Set $\hat{\mathbf{s}}_b$ to a given initial model (a constant or the previously best-known geological model).

2. Compute the ray-path matrix $\mathbf{M}$ and traveltimes $\hat{\mathbf{t}}_b$ for $\hat{\mathbf{s}}_b$ and set $\Delta\mathbf{t} = \mathbf{t} - \hat{\mathbf{t}}_b$.

3. If $\Delta\mathbf{t}$ is sufficiently small, output $\hat{\mathbf{s}}_b$ and stop.

4. Find a model correction $\widehat{\Delta\mathbf{s}}$ as the solution to the linear inversion problem: $\mathbf{M}\widehat{\Delta\mathbf{s}} = \Delta\mathbf{t}$.

5. Update $\hat{\mathbf{s}}_b$ to the new model obtained by adding the model correction $\widehat{\Delta\mathbf{s}}$ to the previous model $\hat{\mathbf{s}_b}$.

6. Return to Step 2.

This algorithm looks very reasonable and in fact sometimes it actually works! But not always. For models with low slowness contrasts, the algorithm will converge to a sensible result. When the method fails, the failure mode is usually a divergence to a highly oscillatory model. *Ad hoc* procedures to reduce the possible range of slowness values and to guarantee a high degree of smoothness in the reconstructed model have commonly been introduced to deal with this instability. Such smoothness constraints come from external considerations (like the class of models in which we want the solution to lie), not from the data. But a really satisfactory method of stabilizing the iteration scheme based on information in the data itself has been lacking.

Analyzing the algorithm, we see that there are really only two significant calculations contained in it. Step 2 is just the solution of the *forward problem* for $\hat{\mathbf{s}}_b$. This step should not introduce any instability, since it can be performed essentially as accurately as desired (if the computing budget is large enough and the computers fast enough). Step 4, on the other hand, is a *linear inversion* step imbedded in a nonlinear algorithm. We should be skeptical of this step. Linear inversion implicitly assumes that the updated model (after adding the model correction) is not so different from the previous model that the ray-path matrix $\mathbf{M}$ should change significantly from one iteration to the next. If this implicit assumption is violated, then this step is not justified, and steps 4 and/or 5 in the algorithm must be modified.

Feasibility analysis supplies a set of rigorous physical constraints on the reconstruction process. Experience has shown that constraints on smoothness or limits on the maximum and minimum values of the model are generally not needed if feasibility constraints are applied.

In the Chapters that follow, these problems will be analyzed in some detail, and several methods of stabilizing the nonlinear inversion problem will be developed.

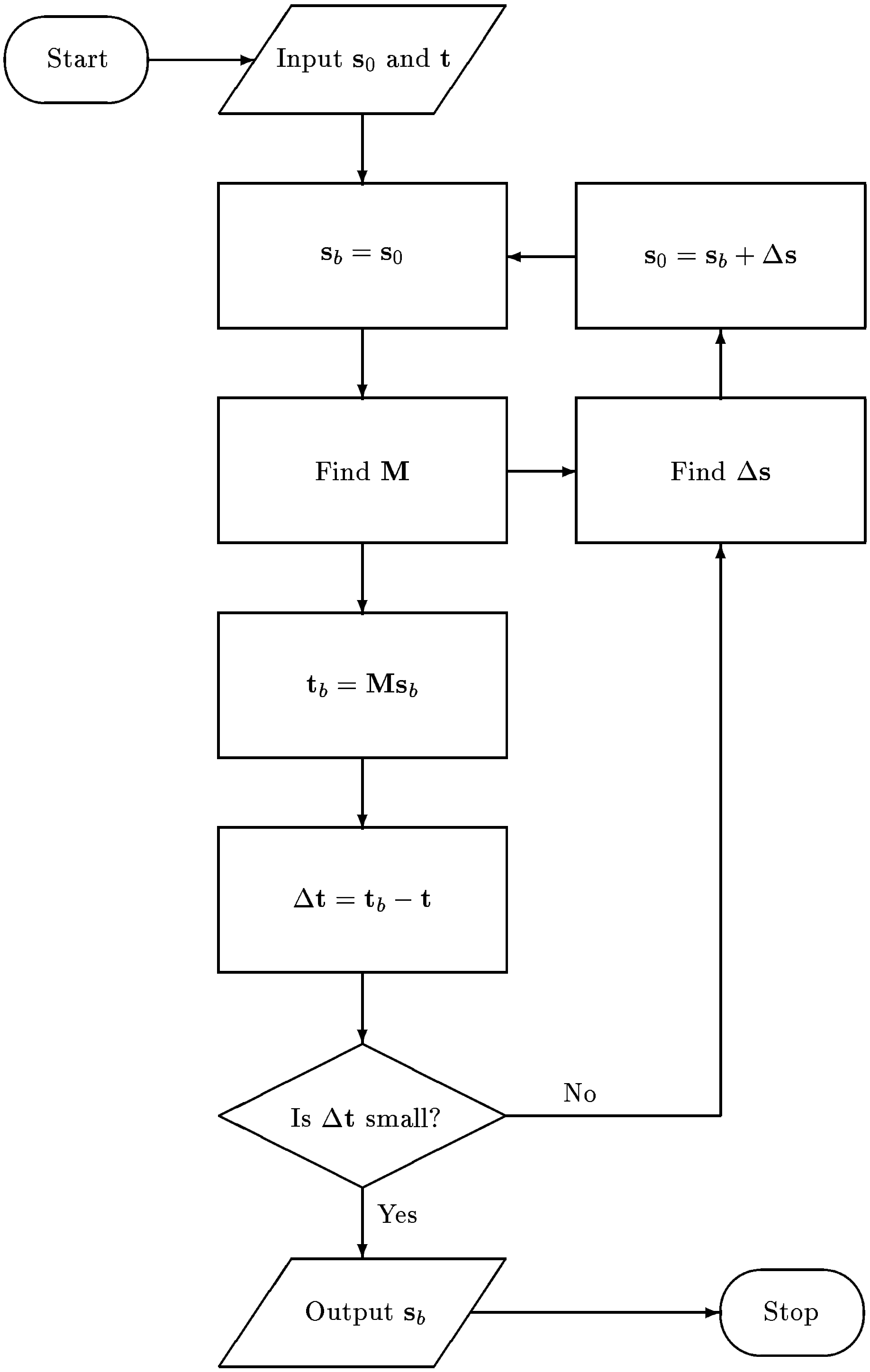

Figure 1.4: Iterative algorithm for traveltime inversion.