Derivation of Gassmann's Equations

The presence of a saturating pore fluid in porous media introduces an additional control field p_f and an additional type of strain variable ζ. p_f is the fluid pressure, and ζ is the change in amount of fluid mass contained in the pores.

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ -\zeta \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{12} & -\beta \\ S_{12} & S_{11} & S_{12} & -\beta \\ S_{12} & S_{12} & S_{11} & -\beta \\ -\beta & -\beta & -\beta & \gamma \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -p_f \end{pmatrix}$$

Gassmann Derivation (2)

The meanings of the compliances S_{11} and S_{12} do not change in any way from their meanings in elasticity:

$$S_{11} = \frac{1}{E_{dr}} = \frac{1}{9K_{dr}} + \frac{1}{3G_{dr}}$$

$$S_{12} = -\frac{\nu_{dr}}{E_{dr}} = \frac{1}{9K_{dr}} - \frac{1}{6G_{dr}}$$

K_{dr} and G_{dr} are the bulk and shear moduli of the drained (almost dry) porous medium. E_{dr} and ν_{dr} are the drained Young's modulus and Poissons's ratio.

Gassmann Derivation (3)

β and γ are poroelastic constants connecting p_f to the elastic strains e_{ii}, and connecting the elastic stresses σ_{ii} and p_f to the fluid increment ζ.

Now consider the saturated (and undrained, meaning that the liquid is trapped and cannot escape from the volume) case so that

$$\zeta \equiv 0.$$

Then, the previous equations show that

$$p_f = -\frac{\beta}{\gamma}\,(\sigma_{11} + \sigma_{22} + \sigma_{33}).$$

Gassmann Derivation (4)

The preceding equation is known as the pore pressure build-up equation, for the obvious reason that it shows how the pore pressure increases when an external compressive load is applied to a closed system.

We can now use this equation to eliminate both ζ and p_f from the poroelastic equations and we find

$$\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \end{pmatrix} = \left[\begin{pmatrix} S_{11} & S_{12} & S_{12} \\ S_{12} & S_{11} & S_{12} \\ S_{12} & S_{12} & S_{11} \end{pmatrix} - \frac{\beta^2}{\gamma} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right] \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix}$$

Gassmann Derivation (5)

The resulting set of equations shows that, for isotropic poroelastic media, the compliances for the saturated system are given by

$$S_{11}^{sat} = S_{11} - \frac{\beta^2}{\gamma}$$

and

$$S_{12}^{sat} = S_{12} - \frac{\beta^2}{\gamma}.$$

Using our previous relations, we find

$$\frac{1}{9K^{sat}} + \frac{1}{3G^{sat}} = \frac{1}{9K_{dr}} + \frac{1}{3G_{dr}} - \frac{\beta^2}{\gamma}$$

and

$$\frac{1}{9K^{sat}} - \frac{1}{6G^{sat}} = \frac{1}{9K_{dr}} - \frac{1}{6G_{dr}} - \frac{\beta^2}{\gamma}.$$

Gassmann Derivation (6)

Taking the difference, we find $1/2G^{sat} = 1/2G_{dr}$, or

$$G^{sat} = G_{dr}.$$

This shows that the saturated shear modulus is not influenced at all mechanically by the liquid.

Then, substituting the shear modulus result back into either of the previous equations gives

$$\frac{1}{K^{sat}} = \frac{1}{K_{dr}} - \frac{9\beta^2}{\gamma},$$

which is one form of the result known as Gassmann's formula for the bulk modulus of a poroelastic system.