Canonical Functions of Elasticity

For bulk modulus,

$$\Lambda(G) \equiv \left\langle \frac{1}{K(x)+4G/3} \right\rangle^{-1} - \frac{4}{3}G.$$

For shear modulus,

$$\Gamma(F) \equiv \left\langle \frac{1}{G(x)+F} \right\rangle^{-1} - F,$$

where

$$F \equiv G(9K+8G)/6(K+2G).$$

Monotonicity of the Canonical Functions

Both canonical functions are monotonic functions of their arguments.

For example,

$$\frac{d\Lambda(G)}{dG} = \frac{4}{3}\left\langle \frac{1}{K(x)+4G/3} \right\rangle^{-2} \times \left(\left\langle \frac{1}{(K(x)+4G/3)^2} \right\rangle - \left\langle \frac{1}{K(x)+4G/3} \right\rangle^2 \right) \geq 0.$$

Non-negativity follows easily from the Cauchy-Schwartz inequality $\langle a \rangle^2 \leq \langle a^2 \rangle$.

Effective Medium Theory Results

CPA results for bulk modulus are

$$K^{eff} = \Lambda(G^{eff}).$$

CPA results for shear modulus are

$$G^{eff} = \Gamma(F^{eff}).$$

Note that the two equations are coupled, and must be iterated to obtain their solutions.

Hashin-Shtrikman Bounds

Hashin-Shtrikman bulk modulus bounds are

$$K_{HS}^{\pm} = \Lambda(G_{\pm}).$$

Hashin-Shtrikman shear modulus bounds are

$$G_{HS}^{\pm} = \Lambda(F_{\pm}),$$

where $K_+ = max_i\, K_i$, $K_- = min_i\, K_i$, $G_+ = max_i\, G_i$, $G_- = min_i\, G_i$, and $F \equiv G(9K + 8G)/6(K + 2G)$.

Gassmann's Results

Gassmann's results can also be written in terms of the canonical functions

$$K^{sat} = \Lambda_{sat}(G_{dr}),$$
$$G^{sat} = G_{dr},$$

and

$$G_{dr} = \Gamma(K_s, 0; F_{dr}),$$

where F_{dr} is determined by solving the coupled equation for the drained bulk modulus

$$K_{dr} = \Lambda(K_s, 0; G_{dr}).$$