Equation for String Vibrations

The equation for vibration of a simple string of variable density is

$$\rho(x)\frac{\partial^2\phi}{\partial t^2} = T\frac{\partial^2\phi}{\partial x^2},$$

where t is the time, x is the spatial variable, T is the constant tension of the string, $\rho(x)$ is the spatial distribution of the density along the string for $0 \le x \le 1$, and it is assumed that the string is clamped at both ends so $\phi(0,t) = \phi(1,t) = 0$ for all t.

Equation for the Modes or Eigenfunctions

We can decompose the string vibration into modes by taking the Fourier transform of the wave equation in t. Taking $\phi(x,t) = \tilde{\phi}(x)\, exp(-i\omega t)$, we find

$$-\omega^2 \rho(x) \tilde{\phi}(x) = T \tilde{\phi}_{xx}(x),$$

where the subscript notation indicates spatial derivatives with respect to x. We will then simplify further by assuming that $T = 1$. Then, the eigenfunctions of the equation satisfy

$$\phi_{n,xx}(x) + \omega_n^2 \rho(x) \phi_n(x) = 0,$$

subject to $\phi_n(0) = \phi_n(1) = 0$, where ω_n is the angular eigenfrequency of the eigenfunction $\phi_n(x)$.

Rayleigh-Ritz Characterization of ω_n

From the preceding equations it is easy to see that the eigenfunctions must satisfy

$$\omega_n^2 = \frac{\int_0^1 \phi_{n,x}^2(x)dx}{\int_0^1 \rho(x)\phi_n^2(x)dx}.$$

Recall that these eigenfrequencies can also be characterized using the variational form of this equation by noting that

$$\omega_n^2 \leq \frac{\int_0^1 v_{n,x}^2(x)dx}{\int_0^1 \rho(x)v_n^2(x)dx},$$

where $v_n(x)$ is a trial eigenfunction satisfying the b.c. $v_n(0) = v_n(1) = 0$ and also subject to $n-1$ constraints.

Rayleigh-Ritz and Feasibility

The first eigenfunction has no constraints. The second eigenfunction is orthogonal to the first. The third is orthogonal to the first two, and so on.

Now the preceding equation can be used more easily for feasibility analysis by inverting it, so that

$$\frac{1}{\omega_n^2} \leq \frac{\int_0^1 \rho(x) v_n^2(x) dx}{\int_0^1 v_{n,x}^2(x) dx}.$$

This version is preferable because now the linearity of the right hand side in the unknown $\rho(x)$ is apparent.

Scale Invariance of the Eigenfunctions

This is a very useful property for the subsequent analysis, for it is now easy to show that the eigenfunctions are independent of density $\rho(x)$ scale, so we can multiply $\rho(x)$ by any positive γ. This will shift the values of the eigenfrequencies, but will not affect the eigenfunctions. This is similar to the statement in traveltime tomography that the ray paths are scale invariant to a shift in the scale of the wave slowness. In optics, the ray paths depend only on the index of refraction, not on c_{vacuum}.