Backus-Gilbert inverse

Backus and Gilbert [1968; 1970] formulated an approach to the inverse problem that is especially well-suited to problems in geophysics and imaging of the Earth. They stress the finiteness of the number of data points available for any inversion process, the imprecision of those data points, and the (virtually) infinite dimensionality of the continuum quantities to be estimated from the data. Backus and Gilbert make the further assumption in their approach that a "good" Earth model is available so the inverse problem may be legitimately linearized with respect to that model.

The mathematical formulation of their approach assumes that the discrete data γ_i are related to an Earth model $E(r)$ and some set of functions $G_i(r)$ that characterize the interaction between the Earth model and the data collected according to an integral relation of the form

$$\int_0^1 G_i(r)E(r)dr = \gamma_i, \quad \text{for} \quad i = 1, \dots, m. \tag{1}$$

The functions $G_i(r)$ are assumed to be known and result from a foward computation using the "good" Earth model. An example of such data might be the frequencies of vibration of either the Earth or the Sun, where i is then an index over various modes of vibration.

To characterize the accuracy and resolution of the final result, Backus and Gilbert introduce a function $A(r; r_0)$ with the desired properties that

$$\int_0^1 A(r; r_0)dr = 1 \quad \text{and} \quad \int_0^1 A(r; r_0)E(r)dr \simeq E(r_0). \tag{2}$$

So A is intended to act much like a Dirac delta function $\delta(r - r_0)$. The point of the method is to try to construct such an A from the known functions G_i in the form of an expansion

$$A(r; r_0) = \sum_{i=1}^{m} a_i(r_0) G_i(r). \tag{3}$$

Substituting (3) into (2) shows that, in terms of the measured data, we must have one constraint on the a_i's

$$\sum_{i=1}^{m} a_i(r_0) \int_0^1 G_i(r) dr = 1 \tag{4}$$

and an approximate formula for the Earth model in terms of the γ_i's

$$\sum_{i=1}^{m} a_i(r_0) \gamma_i \simeq E(r_0). \tag{5}$$

In the Backus-Gilbert theory, the set of equations (1) plays the same role as $\mathbf{Ms} = \mathbf{t}$ in our previous discussions, while equation (5) plays the same role as $\mathbf{XMs} = \mathbf{Xt}$. Thus, the $a_i(r_0)$'s play the role of a row of an approximate inverse operator $\mathbf{X}$, such that $\sum X_{ji}\gamma_i = E_j \equiv E(r_0)$. We will therefore define the vector $\mathbf{a}^T = (\, a_1, a_2, \ldots, a_m\,)$ as a row vector of the Backus-Gilbert approximate inverse $\mathbf{X}$.

So far we do not have enough constraints to estimate the $a_i(r_0)$'s. Making use of the idea that A should resemble a delta function, Backus and Gilbert introduce a spread function

$$S(A; r_0) = 12 \int_0^1 (r - r_0)^2 A^2(r; r_0) dr. \tag{6}$$

If A is a delta function $\delta(r-r_0)$, this integral vanishes. So S is a measure of the deviation of A from a delta function. Substituting the expansion (3) of A in terms of the $G_i(r)$'s into (6), we see that S is given by

$$S(A;r_0) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(r_0)a_j(r_0)N_{ij}(r_0), \tag{7}$$

where the matrix $N_{ij}(r_0)$ is given by

$$N_{ij}(r_0) = 12\int_0^1 (r-r_0)^2 G_i(r)G_j(r)dr. \tag{8}$$

One row of the Backus and Gilbert inverse $\mathbf{X}$ is then found by minimizing (7) with respect to the a_i's, subject to the constraint (4). If we define

$$c_i = \int_0^1 G_i(r)dr, \tag{9}$$

then it is not difficult to show that

$$\mathbf{a}(r_0) = \left(\mathbf{c}^T\mathbf{N}^{-1}(r_0)\mathbf{c}\right)^{-1}\mathbf{N}^{-1}(r_0)\mathbf{c}, \tag{10}$$

where $\mathbf{N}$ is the matrix whose elements are the N_{ij}'s, $\mathbf{a}$ is the vector whose elements are the a_i's, $\mathbf{c}$ is the vector whose elements are the c_i's. We have assumed that $\mathbf{N}$ is invertible. When it is invertible, satisfaction of the constraint (4) by the result (10) is then straightforward to check. Otherwise, we must again invoke some approximation method to solve the Backus-Gilbert inversion problem.