Regularization

The time has come to deal with the realities of our problem. The normal matrix in the least squares problem is probably singular in geophysical problems even for the overdetermined case, where it has a chance of being nonsingular. How then do we deal with these difficulties, short of computing the full SVD of M? Of course, we could do that in some cases, but often the values of either dimension m or n or sometimes both is so large as to preclude this as a rational possiblity. In these cases, the concept of "regularization" is useful.

Damped Least Squares

The idea of damped least squares can be thought of in terms of smoothing the model, or (what is equivalent) of shifting the eigenvalue spectrum of the matrix M so that the shifted matrix can be inverted. Probably the most common way to introduce the concept is to suppose that we have found a model that we like but we want to use our tomographic data to make some improvements without deviating too much from this model.

Damped Least Squares Functional & Solution

We then formulate a least squares problem with a constraint such that

$$\mathcal{F}_\mu(s) = (t - Ms)^2 + \mu(s - s_0)^2,$$

where s_0 is the model we do not want to deviate from very much.

The solution of this problem is

$$(M^T M + \mu I)(s - s_0) = M^T(t - Ms_0)$$

or

$$s = s_0 + (M^T M + \mu I)^{-1} M^T \Delta t,$$

where Δt is $t - Ms_0$.

SVD and Damped Least Squares

It is an elementary exercise now to show what the damping term has done to the spectrum of the singular values. I will leave that as as exercise to the reader.

What values of the damping parameter should be used? The answer to that question is very problem dependent. One approach to answering it is to try a large value, then try smaller values until the result is too noisy then go back to one of the larger values that gave you a result you still liked. More sophisticated and methodical approaches to choosing μ are possible.