The Generalized Inverse

Now we are in a good position to discuss the concept of a "generalized inverse." We know that for rectangular matrices, or singular square matrices no inverse matrix exists. (This is why we are having this summer school!) Nevertheless, our need to solve problems of the type

$$Ms = t$$

for s is so great that we need to invent new concepts of what a solution is for this problem, and that is why we need the generalized inverse.

Moore-Penrose Inverse

This concept is relatively new, dating its heavy usage from the papers of Penrose and the book by Lanczos (see the references), both being from the mid-1950's and early 1960's. The most commonly used generalized inverse is surely the Moore-Penrose inverse. The reason it is so common is that it is closely related to the least-squares solution as Penrose showed in his second paper. I strongly recommend that everyone read and work through these two very elegant papers. (Homework!)

New Concept of a Solution

So what is a generalized inverse?

If we want to solve approximately for s in the equation:

$$Ms = t,$$

what we need is an operator (matrix) X with the property that

$$XMs = Xt,$$

implies that

$$s \simeq Xt,$$

in some sense to be determined.

The Moore-Penrose Conditions

Penrose proposed that if the following four equations were satisfied by a matrix X that this unique matrix X would be a good choice for the generalized inverse of M.

$$MXM = M$$
$$XMX = X$$
$$(MX)^T = MX$$
$$(XM)^T = XM$$

He then went on to show that X solves the least-squares problem we have been discussing.

Least-Squares and Moore-Penrose

It is not hard to show directly that the two inverses we have already seen for the overdetermined and underdetermined cases:

$$X = [M^T M]^{-1} M^T$$

and

$$X = M^T [M M^T]^{-1},$$

do in fact satisfy the Moore-Penrose conditions. But Penrose's proof is more general than this and applies even when these matrix inverses do not exist.

Resolution and Generalized Inverse

Now we are in position to understand the concept of the resolution operator and how it applies to least-squares, generalized inverse, Moore-Penrose, etc. The generalized inverse was defined in terms of two equations simply related to the original problem:

$$XMs = Xt$$

and

$$MXMs = MXt.$$

Source of the Resolution Operators

If X is a generalized inverse satisying Penrose's four conditions, then the second equation can be simplified to

$$Ms = MXt,$$

since $MXM = M$.
Then, we see these two equations have products that can be given the interpretation of measures of the resolution. One is the model resolution and the other is the data resolution.

Resolution Defined

The two operators in question are:

$$\mathcal{R}_{model} = XM$$

which we will call the model resolution, and

$$\mathcal{R}_{data} = MX$$

which we will call the data resolution.

Note that the Moore-Penrose conditions include two conditions on exactly these operators, requiring them both to be symmetric.

These products have very different behavior for the over- and underdetermined cases.