A Concept from Hamilton-Jacobi Theory

We will now jump into the middle of Hamilton-Jacobi theory, in order to make connection to the lecture to follow on numerical methods for ray tracing and solution of the eikonal equation. The physical concept we will introduce arises naturally in Hamilton-Jacobi theory but we will not have sufficient time to derive it here. The same concept also plays an important role in path integral analysis used in quantum mechanics and sometimes in wave propagation through random media.

The Action

This concept is that of “the action”

$$S = \int \mathcal{L}\, dt,$$

where S is the action, $\mathcal{L}$ is the Lagrangian of the system, and the integral is an indefinite one over time. Other terms for this quantity are “Hamilton’s (first) principal function” or, for problems with time independent Hamiltonia, “Hamilton’s characteristic function.”

The Action and the Lagrangian

The significance of the action in our present context is that it is stationary along the classical path or ray path. Just as Fermat's "principle of least time" is really a stationary principle, so too the "principle of least action" is a stationary principle.

The Lagrangian is the difference between kinetic and potential energy of the system, so

$$\mathcal{L} = \mathcal{T} - \mathcal{V},$$

where $\mathcal{T}$ is kinetic and $\mathcal{V}$ is potential energy.

Lagrangian for the Scalar Wave Equation

What is the Lagrangian for the scalar wave equation? In fact, there may be more than one choice of Lagrangian. But we will take the kinetic energy to be

$$\mathcal{T} = \frac{1}{2c_0^2}\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t}$$

and the potential energy to be

$$\mathcal{V} = \frac{1}{2n^2}\nabla\phi\cdot\nabla\phi^*,$$

where the complex conjugate wave field ϕ^* is introduced so both energies remain real. The Lagrangian is then

$$\mathcal{L} = \mathcal{T} - \mathcal{V}.$$

Lagrangian and the Eikonal Equation

Substituting for $\partial\phi/\partial t$ and $\nabla\phi$ using our previous expressions, we find that

$$\mathcal{L} = \frac{k_0^2}{2}\phi\phi^*[1 - (\nabla L)^2/n^2] + \frac{\phi\phi^*}{2n^2}(\nabla A)^2.$$

but the last term on the right hand side can be neglected because k_0 is large compared to the gradient of A.

So the final expression for the Lagrangian is

$$\mathcal{L} \simeq \frac{k_0^2}{2}\phi\phi^*[1 - (\nabla L)^2/n^2].$$

Note the factor in $\mathcal{L}$ inside the square brackets.

Eikonal Equation and the Action

It follows that, if the eikonal equation is satisfied

$$(\nabla L)^2 = n^2,$$

then

$$\mathcal{L} \simeq 0$$

and

$$S = \int \mathcal{L} dt \simeq 0 \qquad .$$

So we conclude that if the eikonal equation is satisfied, then the action S vanishes along a ray path.

The Hamiltonian

A quantity related to the Lagrangian that will appear in subsequent lectures is the Hamiltonian $\mathcal{H}$.
In our present notation, this is defined as

$$\begin{aligned}\mathcal{H} &= \mathcal{T} + \mathcal{V} \\ &\simeq \frac{k_0^2 \phi\phi^*}{2}[\tfrac{1}{n^2}(\nabla L)^2 + 1].\end{aligned}$$

This expression should be compared to

$$H = \tfrac{1}{2} v^2 p^2,$$

with $p = \nabla L / c_0$ and $v = c_0 / n$.

H will appear frequently in Lecture 2.