Geometrical Optics: $n(x) \neq$ constant

When $n(x) \neq$ constant but changes only gradually in space (or is piecewise constant in space), we may seek a solution much like the plane wave:

$$\phi = exp\,[A(x) + ik_0(L(x) - c_0 t)] \qquad .$$

A and L are functions of position to be determined, and are real. The amplitude of the wave is therefore $exp[A(x)]$. If n were constant, L would reduce to nz, where z is the distance in the direction of propagation. L is therefore called the "optical path length" or the "eikonal" (from the Greek for "image").

Some Intermediate Steps

To derive the eikonal equation, we first substitute ϕ into the scalar wave equation. The gradient operator gives

$$\nabla\phi = \phi\nabla(A + ik_0L)$$

$$\nabla^2\phi = \phi[\nabla^2(A + ik_0L) + (\nabla(A + ik_0L))^2]$$

The second equation can be rearranged to give

$$\nabla^2\phi = \phi[\nabla^2A + (\nabla A)^2 - k_0^2(\nabla L)^2] + i\phi k_0[\nabla^2L + 2\nabla A \cdot \nabla L]$$

Almost the Eikonal Equation

The remaining term in the wave equation is

$$-\frac{n^2}{c_0^2}\frac{\partial^2 \phi}{\partial t^2} = k_0^2 n^2 \phi.$$

Substituting all into the wave equation and setting the real and imaginary parts equal to zero separately, we find

$$\nabla^2 A + (\nabla A)^2 + k_0^2[n^2 - (\nabla L)^2] = 0$$
$$\nabla^2 L + 2\nabla A \cdot \nabla L = 0.$$

The Assumption of Geometrical Optics

Both equations are formally exact. No approximations have been made (assuming that the derivatives exist).

Now we want to introduce a standard assumption of geometrical optics: Suppose that the wavelength is small compared to the dimensions of any changes in the medium as reflected in the index of refraction n. The factor $k_0^2 = 4\pi^2/\lambda_0^2$ is therefore large when compared to the spatial gradients of A.

The Eikonal Equation

Neglecting these gradients of A, we finally arrive at

$$(\nabla L)^2 = n^2,$$

which is commonly known as the "eikonal equation of geometrical optics." The surfaces of constant L thus determined are the surfaces of constant optical phase and therefore define the wave fronts. The ray paths are everywhere perpendicular to the wave fronts, and are also determined by the eikonal equation.