*Stanford Exploration Project, Report SERGEY, November 9, 2000, pages 617–**??***

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On nonhyperbolic reflection moveout in anisotropic media

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ABSTRACT

The famous hyperbolic approximation of P-wave reflection moveout is strictly accurate only if the reflector is a plane, and the medium is homogeneous and isotropic. Heterogeneity, reflector curvature, and anisotropy are the three possible causes of moveout nonhyperbolicity at large offsets. In this paper, we analyze the situations where anisotropy is coupled with one of the other two effects. Using the weak anisotropy assumption for transversely isotropic media, we perform analytical derivations and comparisons. Both the case of vertical heterogeneity and the case of a curved reflector can be interpreted in terms of an effective anisotropy, though their anisotropic effects are inherently different from the effect of a homogeneous transversely isotropic model.

INTRODUCTION

Hyperbolic approximation of common-midpoint traveltime curves (reflection moveouts) plays an important role in conventional seismic data processing and interpretation. The hyperbolic formula is exact for homogeneous isotropic media with a plane reflector. Deviations from this simple model result in deviations of the true reflection moveout from the hyperbolic approximation. If the nonhyperbolicity is large enough, we may want to take it into account to correct the errors of conventional processing or to obtain additional information about the medium. One of the important causes of nonhyperbolicity is the seismic anisotropy found in a variety of geological environments. The three other important causes are vertical and lateral heterogeneity and the reflector curvature. Even if nonhyperbolic moveout is not caused by anisotropy, we may consider its presence as evidence of an effective anisotropy. However, in order to provide a correct interpretation, it is important to distinguish among the different kinds of effects. In this paper, we analyze the situations when the effect of anisotropy couples with one of the other three effects. We provide a theoretical description of these effects and compare their influence on P-wave reflection moveouts.

A transversely isotropic medium with a vertical symmetry axis (VTI) is the most commonly used anisotropic model. This model is generally attributed to fine layering in sedimentary basins. One of the first nonhyperbolic approximations for P-wave reflection traveltimes

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in VTI was proposed by Muir and Dellinger (1985) and further developed by Dellinger et al. (1993). In a classic paper, Thomsen (1986) developed a weak anisotropy approximation for describing the transversely isotropic model. Tsvankin and Thomsen (1994) used the weak anisotropy assumption to approximate nonhyperbolic reflection moveout in VTI media.

We start this paper with a brief overview of the weak anisotropy approximation and use this approximation in the following sections for analytical derivations. First, we consider the case of a vertically heterogeneous anisotropic layer. For this case, the three-parameter approximation suggested by Tsvankin and Thomsen (1994) is compared with the shifted hyperbola approximation (Malovichko, 1978; Sword, 1987; Castle, 1988; de Bazelaire, 1988). The second case is a homogeneous anisotropic medium with a curved reflector. In this case, we analyze the cumulative effect of anisotropy, reflector dip, and reflector curvature and develop an appropriate three-parameter approximation of the reflection moveout. Third, we consider the case of a weak lateral heterogeneity. We show that with an appropriate choice of the lateral velocity variation, it can can mimic the effect of transverse isotropy on nonhyperbolic moveout. In conclusion, discuss possible practical applications of the theory.

WEAK ANISOTROPY APPROXIMATION

In a transversely isotropic medium, velocities of seismic waves depend on the direction of propagation measured from the symmetry axis. Thomsen (1986) has introduced a convenient parametrization of this dependence, replacing the general notation of elastic anisotropy in terms of stiffness coefficients $C_{\alpha\beta}$ by P- and S-wave velocities along the symmetry axis and three dimensionless parameters. As shown by Tsvankin (1996), the P-wave seismic signatures in vertically transverse isotropic (VTI) media can be conveniently expressed in terms of Thomsen's parameters ϵ, δ, and γ. A deviation of these paramers from zero characterizes the relative strength of anisotropy. Therefore, the *weak anisotropy* approximation (Thomsen, 1986; Tsvankin and Thomsen, 1994) reduces to simple linearization.

In weakly anisotropic VTI media, the squared group velocity V_g^2 of seismic P-waves can be expressed as a function of the group angle ψ as follows:

$$V_g^2(\psi) = V_z^2\left(1 + 2\,\delta\,\sin^2\psi\,\cos^2\psi + 2\,\epsilon\,\sin^4\psi\right)\,, \tag{1}$$

where $V_z = V_g(0)$ is the vertical velocity, and δ and ϵ are Thomsen's dimensionless anisotropic parameters, which are assumed to be small quantities:

$$|\epsilon| \ll 1,\ |\delta| \ll 1. \tag{2}$$

Both parameters are equal to zero in the isotropic case. Their connection with the stiffness coefficients has the following expressions (Thomsen, 1986):

$$\delta = \frac{(C_{13}+C_{44})^2 - (C_{33}-C_{44})^2}{2\,C_{33}\,(C_{33}-C_{44})}\,, \tag{3}$$

$$\epsilon = \frac{C_{11}-C_{33}}{2\,C_{33}}\,. \tag{4}$$

Equation (1) is accurate up to the second-order terms in ϵ and δ. We retain this level of accuracy throughout this paper without additional clarification. As follows from equation (1), the horizontal velocity V_x corresponding to the horizontal ray propagation is

$$V_x^2 = V_g^2(90^\circ) = V_z^2\,(1+2\epsilon)\;. \tag{5}$$

Equation (5) coincides with the exact expression, which is valid for any strength of anisotropy. Another important quantity is the normal moveout (NMO) velocity for homogeneous VTI media with a horizontal reflector. Its exact expression is (Thomsen, 1986)

$$V_n^2 = V_z^2\,(1+2\delta)\;. \tag{6}$$

One example of a physical anisotropic model is ANNIE, proposed by Shoenberg, Muir, and Sayers (1996) to describe anisotropy of shales. According to this model, the elasticity tensor (stiffness matrix) in transversely isotropic shales is represented by the three parameter approximation

$$C = \left[\begin{array}{cccccc} \lambda+2\mu_H & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu_H & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_H \end{array}\right]\;, \tag{7}$$

where λ, μ, and μ_H are density-normalized elastic parameters. Formula (3) shows that Thomsen's parameter δ in this case is equal to zero, which corresponds to the known fact that the normal moveout velocity for shales is approximately equal to the vertical velocity. The parameter ϵ in this case is defined by the equation

$$\epsilon = \frac{\mu_H - \mu}{\lambda+2\mu}\;. \tag{8}$$

It is convenient to rewrite equation (1) in the form

$$V_g^2(\psi) = V_z^2\,\left(1+2\delta\,\sin^2\psi+2\eta\,\sin^4\psi\right)\;, \tag{9}$$

where

$$\eta = \epsilon - \delta\;. \tag{10}$$

The paramter η is equivalent under the weak anisotropy assumption to the *anelliptic* parameter introduced by Alkhalifah and Tsvankin (1995). For the elliptic anisotropy model, ϵ equals δ, and η is equal to zero. To see why the group velocity function becomes elliptic in this case, note that for small δ,

$$\left.V_g^2(\psi)\right|_{\eta=0} = V_z^2\,\left(1+2\delta\,\sin^2\psi\right) \approx \frac{V_z^2}{\cos^2\psi+(1-2\delta)\,\sin^2\psi}\;. \tag{11}$$

In practical cases of VTI media, ϵ is often greater than δ, so the anelliptic parameter η is positive.

Another equivalent form of equation (1) follows from a substitution of the three characteristic velocities V_z, V_x, and V_n, as follows:

$$V_g^2(\psi) = V_z^2 \cos^2\psi + \left(V_n^2 - V_x^2\right) \sin^2\psi \cos^2\psi + V_x^2 \sin^2\psi \; . \tag{12}$$

It is apparent in formula (12) that, in the linear approximation, the anelliptic behavior of anisotropy is controlled by the difference between the normal moveout and horizontal velocities.

We illustrate different types of anisotropy in Figure 1, which shows the wavefronts for different values of the anisotropic parameters. The wavefront, circular in the isotropic case ($\epsilon = \delta = 0$), appears elliptic if $\epsilon = \delta \neq 0$. If ϵ is greater than zero, and δ is smaller than zero, the three characteristic velocities satisfy the inequality $V_x > V_z > V_n$.

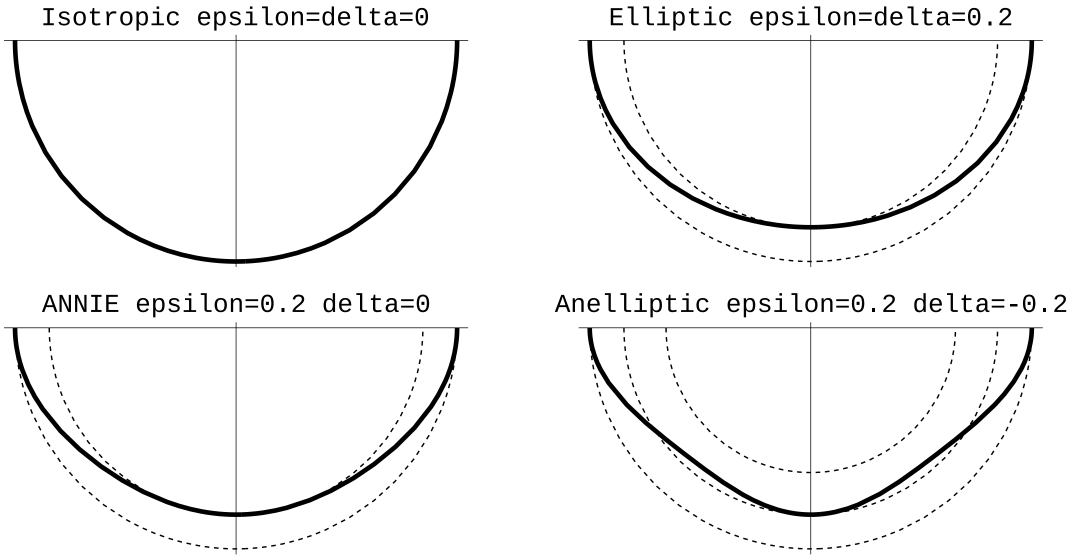

Figure 1: Wavefronts in weakly anisotropic media. Solid curves denote anisotropic wavefronts. Dashed curves denote isotropic wavefronts for the corresponding vertical, horizontal, and normal moveout velocities. Top left: isotropic case ($\epsilon = \delta = 0$); top right: elliptic case ($\epsilon = \delta = 0.2$); bottom left: ANNIE model ($\epsilon = 0.2, \delta = 0$); bottom right: strongly anelliptic case ($\epsilon = 0.2, \delta = -0.2$). aniso-nmofro [CR]

HORIZONTAL REFLECTOR IN A HOMOGENEOUS VTI MEDIUM

To exemplify the use of weak anisotropy, let us consider the simplest case of a homogeneous anisotropic medium with a horizontal reflector. In the isotropic case, the reflection traveltime curve is an exact hyperbola, which follows directly from Pythagoras's theorem (see Figure 2):

$$t^2(h) = \frac{z^2 + h^2}{V_z^2} = t_0^2 + \frac{h^2}{V_z^2} \,, \tag{13}$$

where z denotes the depth of the reflector, h is the half-offset, $t_0 = t(0)$ is the zero-offset traveltime, and V_z corresponds to half of the actual isotropic velocity. In the case of a homogeneous VTI medium, the velocity V_z in formula (13) is replaced by the angle-dependent group velocity V_g. This replacement leads to the exact traveltimes, if no approximation for the group velocity is used, since the ray trajectories in homogeneous VTI media remain straight, and the reflection point does not shift because of the vertical axis symmetry. We can also obtain an approximate traveltime using the approximate velocity V_g defined in equation (1) or (9). From the simple trigonometric considerations, the ray angle ψ in this case is defined by the equation

$$\sin^2\psi = \frac{h^2}{z^2 + h^2} \,. \tag{14}$$

Substituting equation (14) into (9) and linearizing the expression

$$t^2(h) = \frac{z^2 + h^2}{V_g^2(\psi)} \tag{15}$$

with respect to anisotropic parameters δ and η, we arrive at the three-parameter nonhyperbolic approximation (Tsvankin and Thomsen, 1994)

$$t^2(h) = t_0^2 + \frac{h^2}{V_n^2} - \frac{2\eta h^4}{V_n^2\left(V_n^2 t_0^2 + h^2\right)} \,, \tag{16}$$

where the normal moveout velocity V_n is defined by equation (6). At small offsets ($h \ll z$), the influence of the parameter η is negligible, and the traveltime curve is nearly hyperbolic. At large offsets ($h \gg z$), the third term of equation (16) has a clear influence on the behavior of the traveltime. The Taylor series expansion of formula (16) in the vicinity of the vertical zero-offset ray has the form

$$t^2(h) = t_0^2 + \frac{h^2}{V_n^2} - \frac{2\eta h^4}{V_n^4 t_0^2} + \frac{2\eta h^6}{V_n^6 t_0^4} - \ldots \,. \tag{17}$$

When the offset h approaches infinity, the traveltime approximately satisfies the intuitively reasonable relationship

$$\lim_{h\to\infty} t^2(h) = \frac{h^2}{V_x^2} \,, \tag{18}$$

where the horizontal velocity V_x is defined by (5). Approximation (16) is analogous, within the weak anisotropy assumption, to the "skewed hyperbola" formulas (Byun et al., 1989; Harlan, 1995), which use the three velocities V_z, V_n, and V_x as the parameters of the approximation, as follows:

$$t^2(h) = t_0^2 + \frac{h^2}{V_n^2} - \frac{h^4}{V_n^2 t_0^2 + h^2}\left(\frac{1}{V_n^2} - \frac{1}{V_x^2}\right) \,. \tag{19}$$

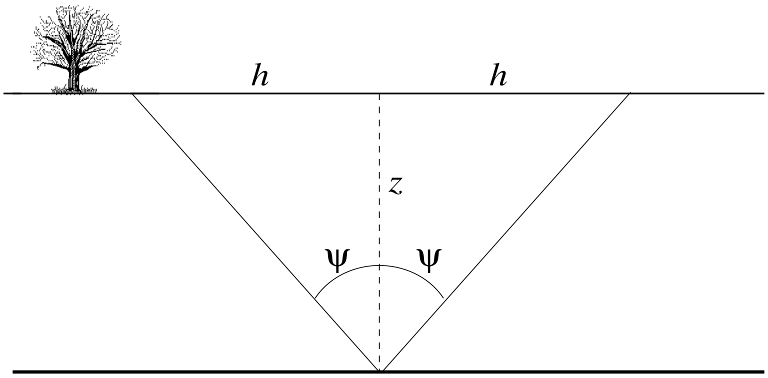

Figure 2: Reflected rays in a homogeneous layer with a horizontal reflector (a scheme). aniso-nmoone [NR]

The accuracy of formula (16) for many realistic situation lies within 1% error and can be further improved at a finite offset by modifying the denominator of the third term (Alkhalifah and Tsvankin, 1995; Grechka and Tsvankin, 1996).

The anelliptic moveout approximation suggested by Muir and Dellinger (1985) has the form

$$t^2(h) = \frac{t_0^4 + (1+f)\,\frac{h^2}{V_n^2} + f^2\,\frac{h^4}{V_n^4}}{t_0^2 + f\,\frac{h^2}{V_n^2}} = t_0^2 + \frac{h^2}{V_n^2} - \frac{f\,(1-f)\,h^4}{V_n^2\left(V_n^2 t_0^2 + f\,h^2\right)}\,, \tag{20}$$

where f is a dimensionless parameter of anellipticity. At large offsets, formula (20) approaches

$$\lim_{h\to\infty} t^2(h) = f\,\frac{h^2}{V_n^2}\,. \tag{21}$$

Comparing equations (18) and (21), we can establish the correspondence

$$f = \frac{V_n^2}{V_x^2} = \frac{1+2\,\delta}{1+2\,\epsilon} \approx 1 - 2\,\eta\,. \tag{22}$$

Taking this correspondence into account, we can see that formula (20) is approximately equivalent to formula (16) in the sense that their difference has the order of η squared.

VERTICAL HETEROGENEITY

Vertical heterogeneity is an important cause of nonhyperbolic moveout. We start this section with reviewing the well-known results of the isotropic theory. We show that these results can be interpreted in terms of an effective anisotropy, which has different properties than the transversally isotropic model. Then we extend the theory to the case of anisotropy coupled with vertical heterogeneity and perform a comparative analysis of different three-parameter nonhyperbolic approximations.

Isotropic Case

Nonhyperbolicity of reflection moveout in vertically heterogeneous isotropic media has been extensively studied with the help of the Taylor series expansion in the powers of the offset (Bolshykh, 1956; Taner and Koehler, 1969; Al-Chalabi, 1973). The most important property of vertically heterogeneous media is that the ray parameter $p = \frac{\sin\psi(z)}{V_z(z)}$ doesn't change with the depth along each ray (Snell's law). This fact leads to the explicit parametric relationships

$$t(p) = \int_0^z \frac{dz}{V_z(z)\cos\psi(z)} = \int_0^{t_z} \frac{dt_z}{\sqrt{1-p^2\,V_z^2(t_z)}}\,, \tag{23}$$

$$h(p) = \int_0^z dz\,\tan\psi(z) = \int_0^{t_z} \frac{p\,V_z^2(t_z)\,dt_z}{\sqrt{1-p^2\,V_z^2(t_z)}}\,, \tag{24}$$

where

$$t_z = t(0) = \int_0^z \frac{dz}{V_z(z)}\,. \tag{25}$$

Straightforward differentiation of parametric formulas (23) and (24) yields the first four coefficients of the Taylor series expansion

$$t^2(h) = a_0 + a_1\,h^2 + a_2\,h^4 + a_3\,h^6 + \ldots \tag{26}$$

in the vicinity of the vertical zero-offset ray. Series (26) contains only even powers of the offset h because of the reciprocity principle: the reflection traveltime is an even function of the offset. Taylor coefficients for the isotropic case are defined as follows:

$$a_0 = t_z^2\,, \tag{27}$$

$$a_1 = \frac{1}{V_{rms}^2}\,, \tag{28}$$

$$a_2 = \frac{1-S_2}{4\,t_z^2\,V_{rms}^4}\,, \tag{29}$$

$$a_3 = \frac{2\,S_2^2 - S_2 - S_3}{8\,t_z^4\,V_{rms}^6}\,, \tag{30}$$

where $V_{rms}^2 = M_1$,

$$M_k = \frac{1}{t_z}\int_0^{t_z} V_z^{2k}(t)\,dt \ \ (k=1,2,\ldots)\,, \tag{31}$$

$$S_k = \frac{M_k}{V_{rms}^{2k}} \ \ (k=2,3,\ldots)\,. \tag{32}$$

Equation (28) shows that, at small offsets, the reflection moveout has a hyperbolic form with the normal moveout velocity V_n equal to the root-mean-square velocity V_{rms}. At large offsets, however, the hyperbolic approximation is not accurate. Studying the Taylor series expansion

(26), Malovichko introduced a remarkable three-parameter approximation for the reflection traveltime in a vertically heterogeneous isotropic medium (Malovichko, 1978; Sword, 1987). Malovichko's formula has the form of a shifted hyperbola (Castle, 1988; de Bazelaire, 1988):

$$t(h) = \left(1 - \frac{1}{S}\right) t_0 + \frac{1}{S} \sqrt{t_0^2 + S \frac{h^2}{V_n^2}}\,. \tag{33}$$

If we set the zero-offset traveltime t_0 equal to the vertical traveltime t_z, the velocity V_n equal to V_{rms}, and the *parameter of heterogeneity* S equal to S_2, formula (33) guarantees the correct coefficients a_0, a_1, and a_2 in the Taylor series (26). Note that the parameter S_2 is related to the variance σ^2 of the squared velocity distribution, as follows:

$$\sigma^2 = M_4 - V_{rms}^4 = V_{rms}^4 \, (S_2 - 1)\,. \tag{34}$$

According to formula (34), this parameter is always greater than 1 (it equals 1 in homogeneous media). In the most common practical cases, the value of S_2 lies between 1 and 2. We can roughly estimate the accuracy of approximation (33) at large offsets by comparing the fourth term of its Taylor series with the fourth term of the exact traveltime expansion (26). According to this estimate, the error of Malovichko's approximation is

$$\frac{\Delta t^2(h)}{t^2(0)} = \frac{1}{8} (S_3 - S_2^2) \left(\frac{h}{t_0\, V_n}\right)^6 . \tag{35}$$

As follows from the definition of the parameters S_k (32) and the Schwarz (Cauchy-Bunyakovski) inequality from calculus, expression (35) is greater than zero for any non-uniform velocity distribution $V_z(t_z)$. This means that Malovichko's approximation tends to overestimate traveltimes at large offsets. As the offset approaches infinity, the limit of this approximation is

$$\lim_{h \to \infty} t^2(h) = \frac{1}{S} \frac{h^2}{V_n^2}\,. \tag{36}$$

Formula (36) indicates that the effective horizontal velocity for Malovichko's approximation (the slope of the shifted hyperbola asimptote) is different from the normal moveout velocity. We can interpret this difference as an evidence of the effective depth-variant anisotropy. However, the anisotropic effect implied in formula (33) is different from the effect of a homogeneous transversely isotropic medium described by Thomsen's formula (1). To reveal this difference, let us substitute the effective values $t(h) = \frac{\sqrt{z^2+h^2}}{V_g(\psi)}$, $t_0 = \frac{z}{V_z}$, $h = z \tan\psi$, and $S = \frac{V_x^2}{V_n^2}$ into (33). After we eliminate the variables z and h, the resultant expression takes the form

$$\frac{1}{V_g(\psi)} = \frac{1}{V_z} \left\{\cos\psi \left(1 - \frac{V_n^2}{V_x^2}\right) + \sqrt{\frac{V_z^2}{V_x^2} \sin^2\psi + \frac{V_n^4}{V_x^4} \cos^2\psi}\right\}. \tag{37}$$

If the anisotropic effect is induced by a vertical heterogeneity, V_x is greater than V_n, while V_n is greater than V_z. Both of these inequalities follow from the definitions of V_{rms}, t_v, and S_2

and the Schwartz inequality. They reduce to equalities only in the case of a constant velocity. Linearizing expression (37) with respect to Thomsen's anisotropic parameters δ and ϵ, we can transform it to a form analogous to that of equation (9), as follows:

$$V_g^2(\psi) = V_z^2 \left(1 + 2\,\delta\,\sin^2\psi + 2\,\eta\,(1-\cos\psi)^2\right)\;, \tag{38}$$

Figure 3 illustrates the difference between the weak transversally isotropic model and the effective anisotropy implied by Malovichko's approximation. The difference is noticeable in the shapes of both the effective wavefront (left plot) and the traveltime curve (right plot).

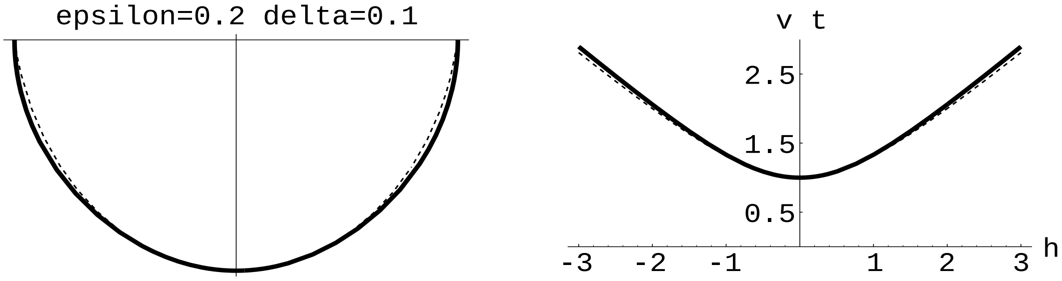

Figure 3: Comparing a weak transversally isotropic model and Malovichko's shifted hyperbola approximation. The left plot shows effective wavefronts; right: reflection moveouts. Solid lines correspond to the anisotropic model; dashed lines: Malovichko's approximation. The values of the effective vertical, horizontal, and moveout velocities are the same in both cases and correspond to Thomsen's parameters $\epsilon = 0.2$, $\delta = 0.1$. aniso-nmofrz [CR]

Deriving formula (38), we have assumed the correspondence

$$S = \frac{V_x^2}{V_n^2} = \frac{1+2\,\epsilon}{1+2\,\delta} \approx 1 + 2\,\eta\;. \tag{39}$$

We could also take the value of the parameter of heterogeneity S so as to match the coefficient a_2 given by formula (29) with the corresponding term in the Taylor series (17). In this case, the value of S would be (Alkhalifah, 1996)

$$S = 1 + 8\,\eta\;. \tag{40}$$

The difference between equations (39) and (40) is an additional indicator of the fundamental difference between the homogeneous VTI model and the vertically heterogeneous model. The three-parameter anisotropic approximation (16) can match the reflection moveout curve in the isotropic model up to and including the fourth-order term in the Taylor series expansion, if the value of η is chosen in accordance with formula (40). We can estimate the error of such an approximation with an equation analogous to (35). It takes the form

$$\frac{\Delta t^2(h)}{t^2(0)} = \frac{1}{8}(S_3 - 2 + 3\,S_2 - 2\,S_2^2)\left(\frac{h}{t_0\,V_n}\right)^6\;. \tag{41}$$

The difference between the error estimates (35) and (41) is

$$\frac{\Delta t^2(h)}{t^2(0)} = \frac{1}{8}(2-S_2)(S_2-1)\left(\frac{h}{t_0\,V_n}\right)^6 . \tag{42}$$

For the usual values of the parameter of heterogeneity S_2, which range from 1 to 2, expression (42) is greater than zero. This means that anisotropic approximation (16) overestimates the traveltimes in the isotropic heterogeneous model even more than the shifted hyperbola approximation (33) (as shown in the right plot of Figure 3). Which of the two approximations is more suitable if the model includes both vertical heterogeneity and anisotropy? We address this question in the following subsection.

Vertical Heterogeneity plus Anisotropy

In a model that includes vertical heterogeneity and anisotropy, both factors affect bending of the rays. However, the weak anisotropy approximation allows us to neglect the effect of anisotropy on ray trajectories and to consider its effect on traveltimes only. This assumption is analogous to the linearization concept, conventional for tomographic inversion. Its application to weak anisotropy has been discussed by Grechka and McMechan (1996). According to the linearization assumption, we can retain isotropic formula (24) as describing the ray trajectories and rewrite formula (23) in the form

$$t(p) = \int_0^z \frac{dz}{V_g(z,\psi(z))\cos\psi(z)} , \tag{43}$$

where V_g is the anisotropic group velocity, which varies both with depth and with the ray angle ψ and has the expression (1). Differentiation of the parametric traveltime formulas (43) and (24) and linearization with respect to Thomsen's anisotropic parameters shows that the general form of equations (27) through (30) remains valid if we change the definition of the root-mean-square velocity V_{rms} and the parameters S_2 and S_3, as follows:

$$V_{rms}^2 = \frac{1}{t_z}\int_0^{t_z} V_z^{2k}(t)\,(1+2\,\delta(t))\,dt\;, \tag{44}$$

$$M_k = \frac{1}{t_z}\int_0^{t_z} V_z^{2k}(t)\,(1+2\,\delta(t))^{2k}\,(1+8\,\eta(t))\,dt\;\;(k=2,3,\ldots)\,, \tag{45}$$

$$S_k = \frac{M_k}{V_{rms}^{2k}}\;\;(k=2,3,\ldots)\,. \tag{46}$$

It is easy to verify that in the homogeneous case, expressions (44) through (46) transform series (26) with coefficients (27) through (30) to the form equivalent to series (17). Two important conclusions follow from the mathematical form of equations (44) and (45). First, we see that if the mean value of the anisotropic coefficient δ is less than zero, the presence of anisotropy can reduce the difference between the effective root-mean-square velocity and the effective vertical velocity $V_z = z/t_z$. In this case, the effects of anisotropy and heterogeneity partially cancel each other, and the moveout curve behaves at small offsets so as if the medium were homogeneous and isotropic. This behavior has been noticed by Larner and Cohen (1993). On the other hand, if the anelliptic parameter η is positive and different from zero, it can significantly increase the values of the heterogeneity parameters S_k. In this case, the nonhyperbolicity of reflection moveouts at large offsets is stronger than in the isotropic case.

To exemplify the general theory, let us consider a simple analytic model with constant anisotropy parameters and a vertical velocity linearly increasing with depth according to the equation

$$V_z(z) = V_z(0)\,(1+\alpha\,z) = V_z(0)\,e^{\kappa(z)}\,, \tag{47}$$

where κ is the logarithm of the velocity change. In this case, the analytic expression for the RMS velocity V_{rms} is found according to formula (44) to be

$$V_{rms}^2 = V_z^2(0)\,(1+2\,\delta)\,\frac{e^{2\kappa}-1}{2\,\kappa}\,, \tag{48}$$

while the mean vertical velocity is

$$\widehat{V_z} = \frac{z}{t_z} = V_z(0)\,\frac{e^{\kappa}-1}{\kappa}\,, \tag{49}$$

where $\kappa = \kappa(z)$ is evaluated at the reflector depth. Comparing equations (48) and (49), we can see that the squared RMS velocity V_{rms}^2 equals the squared mean velocity $\widehat{V}_z^2$ if

$$1+2\,\delta = \frac{2\,(e^{\kappa}-1)}{\kappa\,(e^{\kappa}+1)}\,. \tag{50}$$

For small κ, the estimate of δ from equation (50) is

$$\delta \approx -\frac{\kappa^2}{24}\,. \tag{51}$$

For example, if the vertical velocity near the reflector is four times higher than the velocity at the surface, having the anisotropic parameter $\delta \approx -0.067$ is sufficient to cancel out the effect of heterogeneity on the normal moveout velocity. The values of the parameters S_2 and S_3 are found from formula (46) to be

$$S_2 = (1+8\,\eta)\,\kappa\,\frac{e^{2\kappa}+1}{e^{2\kappa}-1}\,, \tag{52}$$

$$S_3 = \frac{4}{3}\,(1+8\,\eta)\,\kappa^2\,\frac{e^{4\kappa}+e^{2\kappa}+1}{\left(e^{2\kappa}-1\right)^2}\,. \tag{53}$$

Substituting (52) and (53) into formulas (35) and (41) and linearizing both in η and in κ, we find that the error of anisotropic traveltime approximation (16) in the linear velocity model is approximately

$$\frac{\Delta t^2(h)}{t^2(0)} = \frac{\kappa^2\,(1-8\,\eta)}{12}\,\left(\frac{h}{t_0\,V_n}\right)^6\,, \tag{54}$$

while the error of the shifted hyperbola approximation (33) is

$$\frac{\Delta t^2(h)}{t^2(0)} = \left(\frac{\kappa^2\,(1-8\,\eta)}{24} - \eta\right)\left(\frac{h}{t_0\,V_n}\right)^6 . \tag{55}$$

Comparing formulas (54) and (55), we conclude that if the medium is isotropic ($\eta = 0$), the shifted hyperbola can be twice as accurate as the anisotropic formula (assuming the optimal choice of parameters). It is, however, less accurate if the coefficient η is positive and satisfies the approximate inequality

$$\eta \geq \frac{\kappa^2}{8\,(1+\kappa^2)} . \tag{56}$$

Stolt Stretch

Stolt stretch (Stolt, 1978; Levin, 1985; Claerbout, 1985) is a method of extending constant-velocity frequency-domain migration to the case of a vertically variable velocity. The method consists of stretching the time axis according to the formula

$$\tau(t_z) = \left(\frac{2}{V_0^2}\int_0^{t_z} t\,V_{rms}^2(t)\,dt\right)^{1/2} , \tag{57}$$

double Fourier transform, and migration according to the dispersion relation

$$\omega_m(k,\omega_0) = \left(1-\frac{1}{W}\right)\omega_0 + \frac{\mathrm{sign}(\omega_0)}{W}\sqrt{\omega_0^2 - W\,V_0^2\,k_x^2} , \tag{58}$$

where V_0 is a constant frame velocity, ω_0 and ω_m are the frequencies before and after the migration, corresponding to the stretched time coordinate, k_x is the wavenumber, and W is a constant parameter ($W = 1$ in the constant velocity case). Fomel (1995) has shown that the optimal choice of the Stolt stretch parameter W for a particular traveltime depth t_z is given by the expression

$$W = 1 - \frac{V_0^2\,\tau^2(t_z)}{V_{rms}^2\,(t_z)\,t_z^2}\left(\frac{V^2\,(t_z)}{V_{rms}^2\,(t_z)} - S_2\,(t_z)\right) . \tag{59}$$

This expression remains valid in the case of a vertically heterogeneous VTI medium if the values of V_{rms} and S_2 are computed according to formulas (44) and (46). The method of cascaded migrations (Larner and Beasley, 1987) can improve the performance of Stolt migration in the case of variable velocity (Beasley et al., 1988). However, this method affects only the isotropic part of the model and cannot change the contribution of the anisotropic parameters. Therefore, in the anisotropic case, it is important to incorporate anisotropic parameters into the Stolt stretch correction.

CURVILINEAR REFLECTOR

Reflector curvature is yet another possible cause of nonhyperbolic moveout. In the isotropic case, the normal moveout velocity at small offsets is affected by the local dip of the reflector, while the curvature affects the nonhyperbolic part of the moveout. In the case of anisotropy, both effects are combined with the influence of anisotropic parameters.

Isotropic case

If the reflector has the shape of a dipping plane, and the medium is homogeneous and isotropic, the normal moveout curve is a hyperbola of the form (Levin, 1971)

$$t^2(h) = t_0^2 + \frac{h^2}{V_n^2}\,, \tag{60}$$

where

$$t_0 = \frac{2\,L}{V_z}\,, \tag{61}$$

$$V_n = \frac{V_z}{\cos\alpha}\,, \tag{62}$$

L is the length of the zero-offset ray, and α is the reflector dip angle. Formula (60) is not accurate if the reflector is both dipping and curved. The Taylor series expansion of the reflection moveout in this case has the form of equation (26) with the coefficients (Fomel, 1994)

$$a_2 = \frac{\cos^2\alpha\,\sin^2\alpha\,G}{4\,V_z^2\,L^2}\,, \tag{63}$$

$$a_3 = -\frac{\cos^2\alpha\,\sin^2\alpha\,G^2}{16\,V_z^2\,L^4}\left(\cos 2\alpha + \sin 2\alpha\,\frac{G\,K_3}{K_2^2\,L}\right)\,, \tag{64}$$

where

$$G = \frac{K_2\,L}{1 + K_2\,L}\,, \tag{65}$$

α and K_2 are the dip angle and curvature of the reflector at the reflection point of the central (zero-offset) ray, and K_3 is the third-order curvature. If the reflector has an explicit representation of the form $z = z(x)$, then the parameters in formulas (63) and (64) have the expressions

$$\tan\alpha = \frac{dz}{dx}\,, \tag{66}$$

$$L = \frac{z}{\cos\alpha}\,, \tag{67}$$

$$K_2 = \frac{d^2z}{dx^2}\cos^3\alpha\,, \tag{68}$$

$$K_3 = \frac{d^3z}{dx^3}\cos^4\alpha - 3\,K_2^2\,\tan\alpha\,. \tag{69}$$

Leaving only three terms in the Taylor series leads to the approximation

$$t^2(h) = t_0^2 + \frac{h^2}{V_n^2} + \frac{\tan^2\alpha\, G\, h^4}{V_n^2\left(V_n^2 t_0^2 + G\, h^2\right)}\,, \tag{70}$$

where we have included the denominator in the third term to stabilize the traveltime behavior at large offsets according to the obvious limit

$$\lim_{h\to\infty} t^2(h) = \frac{h^2}{V_z^2}\,. \tag{71}$$

As indicated by formula (68), the sign of the curvature K_2 is positive if the reflector surface is locally convex. The sign is negative if the reflector is concave. Therefore, the coefficient G expressed by formula (65) and, likewise, the nonhyperbolic term in (70) can take both positive and negative values. This means that only for concave reflectors in homogeneous media do nonhyperbolic moveouts resemble those in VTI and vertically heterogeneous media. Convex surfaces produce nonhyperbolic effects with the opposite sign. For obvious reasons, formula (70) is not accurate for strong negative curvatures $K_2 \approx 1/L$, which cause focusing of the reflected rays and triplications of the reflection traveltimes.

In order to evaluate the accuracy of approximation (70), we can compare it with the exact expression for the case of a point diffractor. A point diffractor is formally a convex reflector with an infinite curvature. The exact expression for normal moveout is written in the present notation as

$$t(h) = \frac{\sqrt{z^2+(z\,\tan\alpha-h)^2}+\sqrt{z^2+(z\,\tan\alpha+h)^2}}{2\,V_z}\,, \tag{72}$$

where z is the depth of the diffractor, and α is the central ray angle. Figure (**??**) shows the relative error of approximation (70) as a function of the ray angle for the half-offset h equal to the depth z. We can see that the maximum error occurs at $\alpha \approx 50°$ and is about 1%. We can expect formula (70) to be even more accurate for reflectors with smaller curvatures.

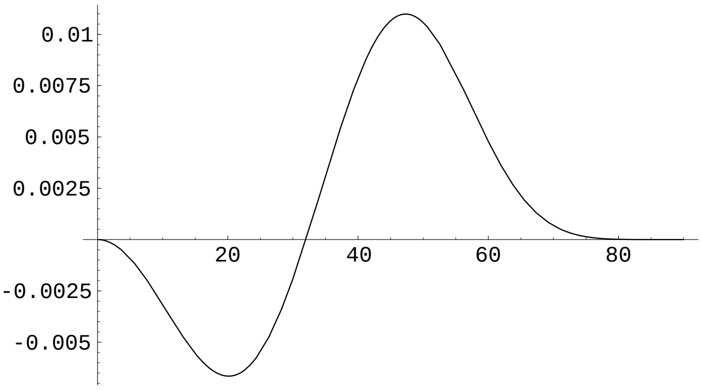

Figure 4: Relative error of the nonhyperbolic moveout approximation for a curved reflector in the case of a point diffractor. The relative error corresponds to the half-offset h equal to the diffractor depth z and is plotted against the central ray angle. $\boxed{\text{aniso-nmoerr}}$ [CR]

Curved reflector in a homogeneous VTI medium

In the case of a dipping curved reflector in a homogeneous VTI medium, the ray trajectories of the incident and reflected waves are straight, but the location of the reflection point is no longer controlled by the isotropic laws. In order to obtain analytic expressions for this case, we use the theorem, which connects the derivatives of the common-midpoint traveltime curves with the derivatives of the forward traveltimes for the imaginary wave originating at the reflection point of the central ray. This theorem has been introduced for the second-order derivatives by Chernjak and Gritsenko (1979) and is usually referred to as the normal incidence point (NIP) theorem (Hubral and Krey, 1980; Hubral, 1983). Though the original proof didn't address the case of anisotropy, it is applicable to this case as well, being based on the fundamental Fermat's principle. The normal incidence point in the anisotropic case should be replaced by the point of incidence for the central ray (which is in general not normal to the reflector surface). In the appendix, we review the NIP theorem as well as its extension for high-order traveltime derivatives (Fomel, 1994).

Two basis formulas derived in the appendix take the following form:

$$\left.\frac{\partial^2 t}{\partial h^2}\right|_{h=0} = 2\,\frac{\partial^2 T}{\partial y^2}\,, \tag{73}$$

$$\left.\frac{\partial^4 t}{\partial h^4}\right|_{h=0} = 2\,\frac{\partial^4 T}{\partial y^4} - 6\left(\frac{\partial^2 T}{\partial x^2}\right)^{-1}\left(\frac{\partial^3 T}{\partial y^2\,\partial x}\right)^2\,, \tag{74}$$

where $T(x,y)$ is the traveltime of the direct wave propagating from the reflector point x to the point y at the surface $z=0$. All the derivatives in formulas (73) and (74) are evaluated in the vicinity of the central (zero-offset) ray. Both formulas are based solely on Fermat's principle and therefore remain valid in any type of medium for reflectors of arbitrary shape, assuming that the traveltimes possess the required order of smoothness. It is especially convenient to use formulas (73) and (74) in the case of homogeneous media, because the forward traveltime in this case has an explicit expression.

In order to apply formulas (73) and (74) to the VTI case, we need to start by tracing the central ray. According to Fermat's principle, the ray trajectory must correspond to the extremum value of the traveltime. For the central ray, this simply means that in the vicinity of the central ray, the traveltime of the direct ray satisfies the equation

$$\frac{\partial T}{\partial x} = 0\,, \tag{75}$$

where

$$T(x,y) = \frac{\sqrt{z^2(x) + (x-y)^2}}{V_g(\psi(x,y))}\,, \tag{76}$$

$z(x)$ describes the reflector surface, and ψ is the ray angle, which satisfies the evident trigonometric relationship (see Figure 5)

$$\cos\psi(x,y) = \frac{z(x)}{\sqrt{z^2(x) + (x-y)^2}}\,. \tag{77}$$

Substituting approximate equation (9) for the group velocity V_g into formula (76) and linearizing with respect to the anisotropic parameters δ and η, we can solve equation (75) for y, obtaining

$$y = x + z\,\tan\alpha\,(1 + 2\,\delta + 4\,\eta\,\sin^2\alpha) \tag{78}$$

or, in other terms,

$$\tan\psi = \tan\alpha\,(1 + 2\,\delta + 4\,\eta\,\sin^2\alpha)\,, \tag{79}$$

where α is the local dip angle of the reflector at the reflection point x. Equation (79) clearly shows that in VTI media the central ray angle ψ differs from the dip angle α. As one can expect, the difference is approximately proportional to Thomsen's anisotropic parameters.

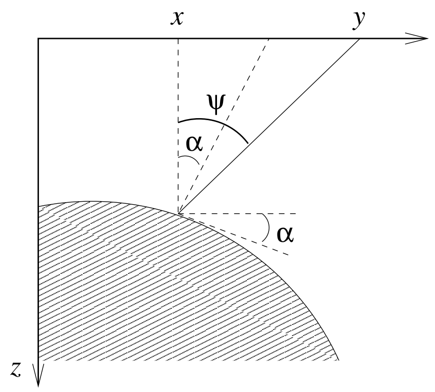

Figure 5: Zero-offset reflection from a curved reflector in a VTI medium (a scheme). Note that the ray angle is not equal to the local dip angle. aniso-nmoray [NR]

Now we can apply formula (73) to evaluate the second term of the Taylor series expansion (26) for the case of a curved reflector. The linearization in anisotropic parameters in this case leads to the expression

$$a_1 = \frac{1}{V_n^2} = \frac{\cos^2\alpha}{V_z^2\,\left(1 + 2\,\delta\,(1+\sin^2\alpha) + 6\,\eta\,\sin^2\alpha\,(1+\cos^2\alpha)\right)}\,, \tag{80}$$

which is equivalent to Tsvankin's result (Tsvankin, 1995). As in the isotropic case, the normal moveout velocity does not depend on the curvature. Its dip dependence is an important indicator of anisotropy, especially in areas of conflicting dips (Alkhalifah and Tsvankin, 1995).

Finally, we can apply formula (74) to determine the third coefficient of the Taylor series. After linearization in anisotropic parameters and lengthy algebra, the resulting expression takes the form

$$a_2 = \frac{A}{V_n^4\,t_0^2}\,, \tag{81}$$

where

$$\begin{aligned} A = {} & G\,\tan^2\alpha + 2\,\delta\,G\,\sin^2\alpha\,(2 + \tan^2\alpha - G) - 2\,\eta\,(1 - 4\,\sin^2\alpha) + \\ & + 4\,\eta\,G\,\sin^2\alpha\,\left(6\,\cos^2\alpha + \sin^2\alpha\,(\tan^2\alpha - 3\,G)\right)\,, \end{aligned} \tag{82}$$

and the coefficient G is defined by equation (65). In the case of a zero curvature (a plane reflector), G is also equal to zero, and the only term remaining in formula (82) is

$$A = -2\,\eta\,(1 - 4\,\sin^2\alpha)\,. \tag{83}$$

In the general case of a curved reflector, we can rewrite the isotropic formula (70) in the form

$$t^2(h) = t_0^2 + \frac{h^2}{V_n^2} + \frac{A\,h^4}{V_n^2\,\left(V_n^2 t_0^2 + G\,h^2\right)}\,, \tag{84}$$

where the normal moveout velocity V_n is given by (80). Equation (84) approximates nonhyperbolic moveouts in a VTI medium with a curved reflector. In the isotropic case, it reduces to formula (70). In the case of a small curvature, the accuracy of formula (84) at finite offsets can be increased by modifying the denominator term.

TI MOVEOUT IN TERMS OF LATERAL HETEROGENEITY

In a simple model with one horizontal reflector, the anisotropic effect of the group velocity changing with the angle of propagation is somewhat similar to the effect of lateral heterogeneity. In this section, we address the question of whether nonhyperbolic moveout in isotropic weakly heterogeneous model can mimic that in a homogeneous weakly anisotropic model. The analysis follows the results of Grechka (1996).

The angle dependence of the group velocity in formulas (1) and (9) is characterized by *small* anisotropic coefficients. Therefore, we can assume that an analogous effect of lateral heterogeneity might be cause by a *small* velocity perturbation. The appropriate model is a laterally heterogeneous (LH) medium with velocity

$$V(x) = V_0\,[1 + c(x)]\;, \tag{85}$$

where $|c(x)| \ll 1$ is a dimensionless function. The velocity function in formula (85) has the generic perturbation form that allows us to use the tomographic linearization assumption. That is, we neglect the ray bending caused by the small velocity perturbation c and compute the perturbation of traveltimes along straight rays in the constant velocity V_0. Thus, we can rewrite equation (13) for this case as

$$t(h) = \frac{\sqrt{z^2 + h^2}}{2\,h} \int_{y-h}^{y+h} \frac{d\xi}{V_z(\xi)}\,, \tag{86}$$

where y is the midpoint location, and the integral limits correspond to the source and receiver locations. For simplisity, and without loss of generality, we can set y to zero. Linearizing with respect to the small perturbation $c(x)$, we get

$$t(h) = \frac{\sqrt{z^2 + h^2}}{V_0} \left[1 - \frac{1}{2\,h} \int_{-h}^{h} c(\xi) d\xi\right]\,. \tag{87}$$

From the form of equation (87) it is clear that lateral heterogeneity can cause many different types of nonhyperbolic moveout shapes. In particular, comparing equations (87) and (15), we conclude that a pseudo-anisotropic behavior of traveltimes is caused by lateral heterogeneity of the form

$$c(h) = \frac{d}{dh}\left[\frac{h^3(h^2\epsilon + z^2\delta)}{(h^2+z^2)^2}\right] \tag{88}$$

or, in the linear approximation,

$$c(h) = \left[\delta\, t_0^2\, V_n^2\, h^2\,(3t_0^2 V_n^2 - h^2) + \epsilon\, h^4\,(5t_0^2 V_n^2 + h^2)\right] / \left(t_0^2 V_n^2 + h^2\right)^3 , \tag{89}$$

where δ and ϵ should be considered now as the parameters of the *isotropic* lateral heterogeneous velocity field. Equation (89) indicates that the velocity heterogeneity $c(x)$, reproducing moveout (16) in a homogeneous TI medium, is the symmetric function of the offset h. It is not surprising because the velocity function (1), corresponding to transverse isotropy, is symmetric as well.

For more details on the relation between lateral heterogeneity and transevse isotropy in interpreting P-wave reflection moveout, see (Grechka, 1996).

CONCLUSIONS

Nonhyperbolic reflection moveout of P-waves is an important indicator of anisotropy. However, its correct interpretation is impossible without taking other factors into account. In this paper, we have considered three other important factors: vertical heterogeneity, curvature of the reflector, and lateral heterogeneity. Each of these three factors can have an effect on nonhyperbolic behavior of the reflection moveout comparable with the effect of anisotropy. In particular, vertical heterogeneity produces a depth-variant anisotropic pattern, different from the pattern of VTI media. In the isotropic case, this pattern is reasonably well approximated by the shifted hyperbola formula. In the case of a VTI vertically heterogeneous medium, the parameters of anisotropy should be replaced with their effective values. For the case of a curved reflector in a homogeneous VTI medium, we have developed an approximation based on the Taylor series expansion of the traveltime with both the reflector curvature and the anisotropic parameters entering the nonhyperbolic term. In the case of a lateral heterogeneity, virtually any effectively anisotropic effect can be created.

The theoretical results of this paper are directly applicable for *modeling* nonhyperbolic moveouts. Particularly attractive in this context are the general formulas connecting the reflection traveltime derivatives with the traveltime derivatives of a direct wave. For smooth velocity models, these formulas may reduce the problem of tracing a family of reflected rays to the problem of tracing one central ray. Practical estimation and *inversion* of nonhyperbolic moveout is a different and more difficult problem. Nevertheless, the theoretical guidelines provided by the analytical theory are helpful for a correct formulation of the inversion problem. They show us explicitly what parameters of the medium we may hope to extract from the kinematics of P-wave seismic reflection data.

ACKNOWLEDGMENTS

This paper is the result of a year-long e-mail correspondence. Its final outline was created when the first author visited the Colorado School of Mines. We acknowledge the support of the Stanford Exploration Project and the Center for Wave Phenomena Consortium Project and their directors, Jon Claerbout and Norman Bleistein. We are thankful to Ilya Tsvankin, Ken Larner, and the members of CWP's "A-team" for insightful discussions.

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APPENDIX A

NORMAL MOVEOUT BEYOND THE NIP THEOREM

In this Appendix, we derive formulas that relate traveltime derivatives of the reflected wave, evaluated at the zero offset point, and traveltime derivatives of the direct wave, evaluated in the vicinity of the zero-offset (central) ray. Such a relationship for second-order derivatives is known as the NIP (normal incidence point) theorem (Chernjak and Gritsenko, 1979; Hubral and Krey, 1980; Hubral, 1983). Its extension to high-order derivatives is described by Fomel (1994). Reflection traveltime in any type of model can be considered as a function of the source and receiver locations s and r and the location of the reflection point x, as follows:

$$t(y,h) = F\left(y,h,x(y,h)\right) , \tag{A-1}$$

where y is the midpoint $\left(y = \frac{s+r}{2}\right)$, h is the half-offset $\left(h = \frac{r-s}{2}\right)$, and the function F has a natural decomposition into two parts corresponding to the incident and reflected rays:

$$F(y,h,x) = T(y-h,x) + T(y+h,x) , \tag{A-2}$$

where T is the traveltime of the direct wave. Clearly, at the zero-offset point,

$$t(y,0) = 2T(y,x) , \tag{A-3}$$

where $x = x(y,0)$ corresponds to the reflection point of the central ray. Differentiating formula (A-1) with respect to the half-offset h and applying the chain rule, we obtain

$$\frac{\partial t}{\partial h} = \frac{\partial F}{\partial h} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial h} . \tag{A-4}$$

According to Fermat's principle, one of the fundamental principles of ray theory, the ray trajectory of the reflected wave corresponds to an extremum value of the traveltime. Parameterizing the trajectory in terms of the reflection point location x and assuming that F is a smooth function of x, we can write Fermat's principle in the form

$$\frac{\partial F}{\partial x} = 0 . \tag{A-5}$$

Equation (A-5) must be satisfied for any values of x and h. Substituting this equation into formula (A-4) leads to the equation

$$\frac{\partial t}{\partial h} = \frac{\partial F}{\partial h} . \tag{A-6}$$

Differentiating (A-6) again with respect to h, we arrive at the formula

$$\frac{\partial^2 t}{\partial h^2} = \frac{\partial^2 F}{\partial h^2} + \frac{\partial^2 F}{\partial h\,\partial x}\,\frac{\partial x}{\partial h}\,. \tag{A-7}$$

Interchanging the source and receiver locations doesn't change the reflection point position (the principle of reciprocity). Therefore, x is an even function of the offset h, and we can simplify formula (A-7) at zero offset, as follows:

$$\left.\frac{\partial^2 t}{\partial h^2}\right|_{h=0} = \left.\frac{\partial^2 F}{\partial h^2}\right|_{h=0}\,. \tag{A-8}$$

Substituting the expression for the function F (A-2) into (A-8) leads to the equation

$$\left.\frac{\partial^2 t}{\partial h^2}\right|_{h=0} = 2\,\frac{\partial^2 T}{\partial y^2}\,, \tag{A-9}$$

which is the mathematical formulation of the NIP theorem. It proves that the second-order derivative of the reflection traveltime with respect to the offset is equal, at zero offset, to the second derivative of the direct wave traveltime for the wave propagating from the incidence point of the central ray. One immediate conclusion from the NIP theorem is that the short-spread normal moveout velocity, connected with the derivative in the left-hand-side of equation (A-9) can depend on the reflector dip but doesn't depend on the curvature of the reflector. Our derivation up to this point has followed the derivation suggested by Chernjak and Gritsenko (1979). Differentiating formula (A-7) twice with respect to h evaluates, with the help of the chain rule, the fourth-order derivative, as follows:

$$\frac{\partial^4 t}{\partial h^4} = \frac{\partial^4 F}{\partial h^4} + 3\,\frac{\partial^4 F}{\partial h^3\,\partial x}\,\frac{\partial x}{\partial h} + 3\,\frac{\partial^4 F}{\partial h^2\,\partial x^2}\left(\frac{\partial x}{\partial h}\right)^2 + 3\,\frac{\partial^4 F}{\partial h\,\partial x^3}\left(\frac{\partial x}{\partial h}\right)^3 +$$

$$+3\,\frac{\partial^3 F}{\partial h^2\,\partial x}\,\frac{\partial^2 x}{\partial h^2} + 3\,\frac{\partial^3 F}{\partial h\,\partial x^2}\,\frac{\partial^2 x}{\partial h^2}\,\frac{\partial x}{\partial h} + \frac{\partial^2 F}{\partial h\,\partial x}\,\frac{\partial^3 x}{\partial h^3}\,. \tag{A-10}$$

Again, we can apply the principle of reciprocity to eliminate the odd-order derivatives of x in formula (A-10) at the zero offset. The resultant expression has the form

$$\left.\frac{\partial^4 t}{\partial h^4}\right|_{h=0} = \left.\left(\frac{\partial^4 F}{\partial h^4} + 3\,\frac{\partial^3 F}{\partial h^2\,\partial x}\,\frac{\partial^2 x}{\partial h^2}\right)\right|_{h=0}\,. \tag{A-11}$$

In order to determine the unknown second derivative of the reflection point location $\frac{\partial^2 x}{\partial h^2}$, we differentiate Fermat's equation (A-5) twice, obtaining

$$\frac{\partial^3 F}{\partial^2 h\,\partial x} + 2\,\frac{\partial^3 F}{\partial h\,\partial x}\,\frac{\partial x}{\partial h} + \frac{\partial^3 F}{\partial^3 x}\left(\frac{\partial x}{\partial h}\right)^2 + \frac{\partial^2 F}{\partial^2 x}\,\frac{\partial^2 x}{\partial h^2} = 0\,. \tag{A-12}$$

Simplifying this equation at zero offset, we can solve it for the second derivative of x. The solution has the form

$$\left.\frac{\partial^2 x}{\partial h^2}\right|_{h=0} = -\left[\left(\frac{\partial^2 F}{\partial^2 x}\right)^{-1}\,\frac{\partial^3 F}{\partial^2 h\,\partial x}\right]_{h=0}\,. \tag{A-13}$$

Here we neglect the case of $\frac{\partial^2 F}{\partial^2 x} = 0$, which corresponds to a focusing of the reflected rays at the surface. Finally, substituting expression (A-13) into (A-11) and recalling the definition of the F function from (A-2), we obtain the equation

$$\left.\frac{\partial^4 t}{\partial h^4}\right|_{h=0} = 2\,\frac{\partial^4 T}{\partial y^4} - 6\left(\frac{\partial^2 T}{\partial x^2}\right)^{-1}\left(\frac{\partial^3 T}{\partial y^2\,\partial x}\right)^2 , \tag{A-14}$$

which is the same as equation (74) in the main text. Higher-order derivatives can be expressed in an analogous way with a set of recursive algebraic functions (Fomel, 1994). In the derivation of formulas (A-9) and (A-14), we have used Fermat's principle, the principle of reciprocity, and the rules of calculus. Both these formulas remain valid in anisotropic media as well as in heterogeneous media, providing that the traveltime function is smooth and that focusing of the reflected rays doesn't occur at the surface of observation.