DIX REVISITED: A FORMALISM FOR RAYS IN LAYERED MEDIA

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ABSTRACT

Dix shows us how to calculate the moveout velocity of a stack of *isotropic* layers, but what about *anisotropic* layers requiring higher-order paraxial approximations? The usual derivation requires a great deal of algebra even for the standard hyperbolic-moveout case. The key is to realize that the Dix equations are an equivalent-medium theory: they provide a formula for replacing a heterogeneous layer stack with an equivalent homogeneous block. Another equivalent medium theory, the Schoenberg-Muir calculus, suggests a cleaner way of deriving Dix's result. Identify layer variables that are *constant* through the entire stack; these are the "knowns". Identify layer variables that *add* through the stack, and express these additive parameters in terms of the known stack constants and elastic parameters in each layer. The coefficients multiplying the stack constants in this formula are the *layer-group elements*. Map from layer parameters to layer-group elements, sum over all layers, and map back to find the *equivalent medium*.

For the Dix case, the stack constant is the ray parameter $p = dT/dx$, the additive variable is the traveltime through each layer T, and the "layer elastic constants" are given by a moveout equation $T(x)$. We are interested in paraxial (near-offset) traveltime behavior; this suggests finding a power series for T in terms of x, but T in terms of p is also paraxial and a far better choice because p is *constant through all layers*. To get this series, expand $p = dT(x)/dx$ as a power series in x, revert to obtain x as a series in p, and then substitute into the series for T in terms of x. Because the T's for each layer sum for any given p, *the coefficients of each power of p are thus the additive "layer-group" parameters.* For the standard case the first layer-group parameter is "vertical traveltime" and the second is "moveout velocity squared".

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The equivalent-medium algorithm similarly provides a direct method for calculating the analogous Dix layer-group parameters for arbitrary anisotropic systems.

INTRODUCTION

The Dix equations are usually written in the form

$$T(0)_{\text{total}} = \sum_i T(0)_i \tag{1}$$

and

$$V_{\text{RMS}} = \sqrt{\frac{\sum_i T(0)_i V_i^2}{\sum_i T(0)_i}}, \tag{2}$$

where $T(0)_i$ is the vertical traveltime and V_i is the moveout velocity for the ith layer, $T(0)_{\text{total}}$ is the total traveltime through the stack, and V_{RMS} is the near-offset moveout velocity for the stack. This form for the equations is favored because it corresponds directly to surface data measurements. It is also *consistent*: the stack as a whole is parameterized in the same way as the individual layers. This would not have been the case if we had used, for example, layer thickness and isotropic velocity; the vertical velocity for the stack as found from $T(0)_{\text{total}}$ is not generally the same as the near-offset moveout velocity V_{RMS}, so an isotropic parameterization for the stack as a whole is not possible.

The usual method of deriving equations (1) and (2) involves finding a power series for the stack traveltime in the form

$$T(x)^2 = C_0 + C_1 x^2 + C_2 x^4 + C_3 x^6 + \dots \tag{3}$$

and comparing this stack equation with the corresponding one for a single layer,

$$T(x)^2 = T(0)^2 + \frac{x^2}{V^2}; \tag{4}$$

see for example section 4.1 in Hubral and Krey (1980). This method is not particularly well suited to the task of finding anisotropic extensions of the Dix equations. The algebra involved is tedious even for the isotropic case, and a general expression for the C_i in equation (3) provides more information than we need. We propose instead a direct method that makes use of key concepts from the Schoenberg-Muir calculus (Schoenberg and Muir, 1989). Although the Dix equations and the Schoenberg-Muir calculus may appear to be unrelated, both are *equivalent-medium theories*; both show how to replace a stack of layers with a bulk homogeneous equivalent that is (in some sense) indistinguishable from the heterogeneous stack. The concepts underlying the Schoenberg-Muir derivation apply equally well to rays in layered media and provide a framework for an alternative derivation of the Dix equations more amenable to anisotropic extension.

We begin with the canonical problem of the type, "springs in series". We then show how the same concepts apply to the Schoenberg-Muir calculus, and finally recast Dix's model of paraxial rays in layered media.

SPRINGS IN SERIES

A standard high-school physics problem asks the student to find the effective "spring constant" of several springs in series. Although this problem is trivial it serves to illustrate in canonical form the fundamental features of all equivalent layered-medium problems.

Each spring obeys Hooke's law,

$$F_i = k_i \, \Delta x_i, \tag{5}$$

where i is the spring number, F_i is the tension, k_i is the "spring constant" (stiffness), and Δx_i is the displacement from the equilibrium position.

We get two more equations from the way the springs are connected. First, the tension is the same in all the springs:

$$F_i \equiv F. \tag{6}$$

Second, the individual displacements of all the springs add to give the total displacement:

$$\Delta x_{\text{total}} = \sum \Delta x_i. \tag{7}$$

To solve the problem we write the additive term Δx as a function of the globally constant tension F,

$$\Delta x_i = \frac{1}{k_i} \; F, \tag{8}$$

and sum over all the springs to find the total displacement

$$\Delta x_{\text{total}} = \sum \Delta x_i = \sum \left(\frac{1}{k_i} \; F \right) = \left(\sum \frac{1}{k_i} \right) \; F. \tag{9}$$

Comparing equation (9) with (8) we see that springs in series behave like a single spring with a stiffness k_{total} determined by the equation

$$\frac{1}{k_{\text{total}}} = \sum \frac{1}{k_i}. \tag{10}$$

The equivalence of form between equations (8) and (9) is clearly the key to this problem. Written this way it is clear that we can add springs in series by summing their *compliances* $1/k_i$. The coefficient on the constant term in equation (8), $1/k_i$, provides an alternative way of representing the spring properties, the "*spring-group*" representation. There is really no reason except convention which prevented us from starting this derivation by writing Hooke's law as

$$\Delta x = c \; F, \tag{11}$$

with the compliance $c = 1/k$ called the "spring constant". If Hooke's law were normally written this way, generations of beginning physics students could have answered "when springs are connected in series the spring constants add" and actually have been correct.

The spring group

Does the "spring group" form a group in the formal mathematical sense? A group is a set of elements and a binary operator "+" that satisfies four conditions (Herstein, 1975):

1) Closure: A and B in the group implies $A + B$ in the group.

2) Associativity: A, B, C in the group implies that $A + (B + C) = (A + B) + C$.

3) Identity: There exists an element E in the group such that $A + E = E + A = A$ for all A in the group.

4) Inverse: For every element A in the group there exists another element A_{inverse} in the group such that $A + A_{\text{inverse}} = A_{\text{inverse}} + A = E$.

If in addition a group also satisfies

5) Commutativity: For all A and B in the group $A + B = B + A$.

it is said to be an *Abelian* Group.

For our spring example the elements in the group are springs parameterized by their compliance $1/k$. The binary operator "+" represents connecting two springs in series and replacing the result with an equivalent single spring. Note that in terms of compliance (the "spring-group" representation) "+" behaves exactly like standard addition. Given this definition, closure, associativity, and commutativity are obvious enough. The identity element is just the infinitely stiff spring with $1/k = 0$. In order to have inverses as required we must allow $1/k < 0$; to get the inverse of a given spring just change the sign on the spring compliance $1/k$.

It appears that if we allow springs with negative k into our set the "spring group" is indeed a formal Abelian group. Is there any practical reason to do so? The group notation does not seem particularly enlightening for the trivial case of springs in series, but the group concept of inverse elements is quite a useful one for more complex equivalent-medium systems such as the Schoenberg-Muir calculus. Inverse elements form the theoretical basis for layer stripping and more generally allow us to decompose a bulk equivalent (what we can actually observe) back into candidate heterogeneous models that may represent the real earth.

THE CANONICAL METHOD

The method we have used to solve the simple spring problem is quite general and applies equally well to other layered-medium problems, for example the calculus of Schoenberg-Muir and the Dix equations. For this reason it is worthwhile to pause here and enumerate the steps we used explicitly:

1) Find the general equation describing layer behavior (e.g., equation (5)).

2) Divide the variables that occur in the general layer equation into three classes:

   2a) The **layer** parameters are assumed given (e.g., the k_i).

   2b) The **constant** parameters have the same value for all layers in the stack (e.g., F).

   2c) The **additive** parameters sum through the stack (e.g., the Δx_i).

3) Rewrite the equation from step 1 so it has the form

$$\textbf{additive}_i = \text{Function}(\textbf{layer}_i) \cdot \textbf{constant} \tag{12}$$

   (e.g., equation (8)).

4) Identify the term "Function($\textbf{layer}_i$)" as an alternative *layer-group* representation of the layer parameters (e.g., the $1/k_i$). To find the homogeneous equivalent of a stack, convert from the standard representation to the layer group, sum, and convert back.

THE SCHOENBERG-MUIR CALCULUS

We now apply these steps to the the layer-averaging techniques of Backus (1962), Helbig and Schoenberg (1987) and Schoenberg and Muir (1989). These methods consider the effect of a static stress on a welded stack of infinite elastic layers. Although derived for the static case, the results are also applicable to the case of propagating waves so long as the wavelength is much longer than the scale of the layering.

Step 1: Hooke's law (equation (5)) generalizes to

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}, \tag{13}$$

where

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} .$$

Note that equation (13) is not exactly the three-dimensional equivalent of equation (5). In equation (5) the "strain" Δx is measured in units of displacement, whereas in equation (13) the strain is a dimensionless proportionality of the form Δlength/length. Similarly, in equation (5) the left hand side is in units of force, whereas in equation (13) the left hand side is in units of force per unit area. We will have to take these length normalizations into account in step 2c.

Step 2a: The layer parameters are the stiffness matrix $\mathbf{C}$, the layer thickness h, and the layer density ρ.

Step 2b: Instead of "springs in series" we have a stack of infinite flat layers with welded interfaces. (We will assume the layering is normal to the z axis.) Instead of a constant scalar spring tension we have that the stress normal to the layering must be the same in all layers. Thus equation (6) becomes

$$\sigma_{3_i} \equiv \sigma_3 \ , \quad \sigma_{4_i} \equiv \sigma_4 \ , \quad \sigma_{5_i} \equiv \sigma_5 \ . \tag{14}$$

Because the layers are infinite and welded together and so cannot accommodate any differential horizontal expansion, we also find that the strain tangential to the layering must be the same in every layer:

$$\epsilon_{1_i} \equiv \epsilon_1 \ , \quad \epsilon_{2_i} \equiv \epsilon_2 \ , \quad \epsilon_{6_i} \equiv \epsilon_6 \ . \tag{15}$$

Step 2c: The first two additive parameters are obvious ones. The thickness h of each layer adds, as does the areal density $h\,\rho$; thus the first two layer-group elements are trivially h and $h\,\rho$.

To obtain vertical displacements, which sum through the stack, we must multiply the vertical components of strain by the layer thickness h. Thus equation (7) becomes

$$h_{\mathrm{total}}\,\epsilon_{3_{\mathrm{total}}} = \sum h_i\ \epsilon_{3_i} \ , \quad h_{\mathrm{total}}\,\epsilon_{4_{\mathrm{total}}} = \sum h_i\ \epsilon_{4_i} \ , \quad h_{\mathrm{total}}\,\epsilon_{5_{\mathrm{total}}} = \sum h_i\ \epsilon_{5_i} \ , \tag{16}$$

where h_i is layer thickness and $h_{\mathrm{total}} = \sum h_i$ is the total thickness of the stack of layers.

There is one more additive equation to find. Think of squeezing our layercake horizontally in a vise; all layers are forced to undergo the same horizontal distortion. The various layers will exert differing forces on the vise blades; to find the total traction, sum over the horizontal forces exerted by each of the layers. Stress is force per area, so force is area times stress. The (infinite) horizontal dimension tangential to the vice blades is the same for all layers and so can be factored out, leaving only the vertical thickness h multiplied by the horizontal components of stress. We thus have

$$h_{\mathrm{total}}\,\sigma_{1_{\mathrm{total}}} = \sum h_i\ \sigma_{1_i} \ , \quad h_{\mathrm{total}}\,\sigma_{2_{\mathrm{total}}} = \sum h_i\ \sigma_{2_i} \ , \quad h_{\mathrm{total}}\,\sigma_{6_{\mathrm{total}}} = \sum h_i\ \sigma_{6_i} \ . \tag{17}$$

Step 3: We can now rearrange equation (13) to segregate the additive and constant elastic terms, obtaining

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} & & C_{13} & C_{14} & C_{15} \\ C_{12} & C_{22} & C_{26} & & C_{23} & C_{24} & C_{25} \\ C_{16} & C_{26} & C_{66} & & C_{36} & C_{46} & C_{56} \\ \\ C_{13} & C_{23} & C_{36} & & C_{33} & C_{34} & C_{35} \\ C_{14} & C_{24} & C_{46} & & C_{34} & C_{44} & C_{45} \\ C_{15} & C_{25} & C_{56} & & C_{35} & C_{45} & C_{55} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \\ \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} . \tag{18}$$

Written explicitly as a block matrix equation in terms of the additive and constant parameters,

$$\sigma_{\text{add}} = h \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} , \quad \epsilon_{\text{add}} = h \begin{bmatrix} \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{bmatrix} , \quad \sigma_{\text{const}} = \begin{bmatrix} \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} , \quad \epsilon_{\text{const}} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{bmatrix} , \tag{19}$$

equation (18) becomes

$$\begin{bmatrix} \frac{1}{h}\sigma_{\text{add}} \\ \\ \sigma_{\text{const}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{TT} & \mathbf{C}_{TN} \\ \\ \mathbf{C}_{TN}^T & \mathbf{C}_{NN} \end{bmatrix} \begin{bmatrix} \epsilon_{\text{const}} \\ \\ \frac{1}{h}\epsilon_{\text{add}} \end{bmatrix} . \tag{20}$$

After a little matrix algebra we finally obtain Hooke's law in the desired form, the elastic equivalent of equation (11):

$$\begin{bmatrix} \sigma_{\text{add}} \\ \\ \epsilon_{\text{add}} \end{bmatrix} = h \begin{bmatrix} \left(\mathbf{C}_{TN}\mathbf{C}_{NN}^{-1}\right) & \mathbf{C}_{TT} - \mathbf{C}_{TN}\mathbf{C}_{NN}^{-1}\mathbf{C}_{TN}^T \\ \\ \mathbf{C}_{NN}^{-1} & -\left(\mathbf{C}_{TN}\mathbf{C}_{NN}^{-1}\right)^T \end{bmatrix} \begin{bmatrix} \sigma_{\text{const}} \\ \\ \epsilon_{\text{const}} \end{bmatrix} . \tag{21}$$

Equation (21) is really nothing more than another form of Hooke's law, one better suited to the geometry of layered media.

Step 4: The somewhat complicated matrix coefficient on the constant vector parameter in equation (21),

$$h \begin{bmatrix} \left(\mathbf{C}_{TN}\mathbf{C}_{NN}^{-1}\right) & \mathbf{C}_{TT} - \mathbf{C}_{TN}\mathbf{C}_{NN}^{-1}\mathbf{C}_{TN}^T \\ \\ \mathbf{C}_{NN}^{-1} & -\left(\mathbf{C}_{TN}\mathbf{C}_{NN}^{-1}\right)^T \end{bmatrix} , \tag{22}$$

defines the Schoenberg-Muir layer-group representation of the elastic stiffness matrix $\mathbf{C}$. Note this 6 by 6 block matrix has 21 independent elements, the same number as $\mathbf{C}$; the complete layer group also includes two more independent parameters, h and $h\,\rho$, for a total of 23 group elements.

Does the Schoenberg-Muir system form an Abelian group? Yes. Association and commutativity are obviously satisfied. The inverse can be formed simply by changing the sign of h, and the identity element is the layer of zero thickness. The group is closed in the layer-group domain, although it is possible to create representations in the layer-group domain by layer subtraction that correspond to physically disallowed elastic media (Schoenberg and Muir, 1989). The group concept of subtraction coupled with the Schoenberg-Muir calculus does provide a useful theoretical framework for decomposing observed elastic properties into geophysically meaningful components; for example observed orthorhombic anisotropy can be decomposed into a transversely isotropic background with fractures (Hood, 1991) and (Hood and Schoenberg, 1992).

THE DIX EQUATIONS

The familiar Dix equations, equations (1) and (2), form a paraxial (near-offset) equivalent-medium system accurate to second order in offset. Although the moveout recorded over a stack of even two isotropic layers is not exactly hyperbolic, it is well known that for near-vertical propagation a stack of isotropic layers produces a moveout that is close enough to hyperbolic for most practical purposes. We next show how to extend the Dix equations to encompass arbitrary anisotropic moveout using the same canonical equivalent-medium method. We base our extension here on the "first anelliptic anisotropic approximation" of Dellinger and Muir (1992), although the solution method itself is completely general.

Step 1: The first-anelliptic moveout equation is

$$T(x)^2 = \frac{T(0)^4 + (F_W + 1) T(0)^2 V_{\rm NMO}^{-2} x^2 + F_W^2 V_{\rm NMO}^{-4} x^4}{T(0)^2 + F_W V_{\rm NMO}^{-2} x^2}, \tag{23}$$

where x is the offset, $T(0)$ is the vertical traveltime, $V_{\rm NMO}$ is the near-offset NMO velocity, and F_W is a dimensionless anisotropy parameter. If $F_W \equiv 1$ the moveout described by equation (23) becomes exactly hyperbolic, reproducing the standard Dix moveout equation; the more F_W departs from unity the more nonhyperbolic the moveout becomes.

Step 2a: The layer parameters are vertical traveltime through the layer, $T_i(0)$, layer moveout velocity, ${V_{\rm NMO}}_i$, and layer anisotropy factor, F_{W_i}.

Step 2b: The constant parameter is the *ray parameter* p. Although p does not occur directly in equation (23) we can find it as a function of the other parameters by using the formula $p(x) = \frac{dT}{dx}(x)$.

Step 2c: The additive parameters are traveltime T and offset x.

Step 3: T and x cannot be written as linear functions of p because the Dix model is a paraxial equivalent-medium system good for "small" offsets. Instead we must express T and x as power series in p; each distinct power of p (p^0, p^2, p^4, etc) then functions as an independent constant parameter.

The required power series $T(p)$ can be conveniently calculated in the following way:

1. Expand $T(x)$ as a power series about $x = 0$.

2. Differentiate the power series $T(x)$ with respect to x, giving the power series $p(x) = \frac{dT}{dx}(x)$.

3. Revert this power series for $p(x)$ to obtain a series for $x(p)$ in terms of powers of p. (See Knuth (1981) for details.)

4. Compose the power series for $T(x)$ with the power series for $x(p)$, obtaining a series for $T(p)$ in powers of p.

5. The coefficients of the lowest powers of p define the layer-group representation.

Note that because

$$T(p) = T(0) + p\,x(p) - \int_0^p x(p')dp' \tag{24}$$

the series $x(p)$ provides a redundant subset of the group elements in $T(p)$ and can safely be ignored.

Although this algorithm does require somewhat less algebra than the standard method described in the introduction, the amount required can still be formidable. Fortunately, each step corresponds to a basic command in the program Mathematica (TM) (Wolfram, 1988), and the desired result is given *directly* in the form of coefficients in the power series $T(p)$. If we use this algorithm with equation (23) we obtain

$$\begin{gathered} T(p) = \Big[T(0)\Big]\,p^0 + \Big[T(0)V_{\mathrm{NMO}}^2\Big]\,\frac{1}{2}p^2 + \\ \Big[T(0)(1 + 4F_W - 4F_W^2)V_{\mathrm{NMO}}^4\Big]\,\frac{3}{8}p^4 + \\ \Big[T(0)(1 + 12F_W + 12F_W^2 - 56F_W^3 + 32F_W^4)V_{\mathrm{NMO}}^6\Big]\,\frac{5}{16}p^6 + \ldots . \end{gathered} \tag{25}$$

Step 4: From the p^0 term we find the first Dix layer-group element, $T(0)$. This group element corresponds to equation (1), "vertical traveltimes add". From the p^2 term we find the second Dix layer-group element, $T(0){V_{\mathrm{NMO}}}^2$. This group element corresponds to Dix's familiar RMS-velocity equation, equation (2).

In the standard case the first two powers of p exhaust the available free layer parameters $T(0)$ and V_{NMO}, and the coefficient of p^4 cannot also be made consistent. The Dix equations thus form a paraxial equivalent-medium theory exact up to second order in p. (Note that the speed of convergence of the series in equation (25) actually depends on the magnitude of a *dimensionless* stack parameter, $p\,V_{\mathrm{NMO}}$.)

For the first-anelliptic approximation the F_W layer parameter yet remains, and so the p^4 coefficient in equation (25) defines a third Dix layer-group element,

$$T(0)V_{\mathrm{NMO}}^4(1 + 4F_W - 4{F_W}^2) \ . \tag{26}$$

If $F_W \neq 1$ the calculated stack moveout is *non*hyperbolic. Nonhyperbolic moveout can be caused by intrinsically anisotropic layers within the stack ($F_{W_i} \neq 1$), but more generally it will also be caused by ray bending at layer boundaries not accounted for by the first two layer-group elements. The three-term first-anelliptic extension of the Dix equivalent-medium system is exact up to order p^4 and so should do a better job of accounting for ray bending than the first two terms alone can, at least paraxially. (Note we have been discussing the power series $T(p)$ here, not the series $T^2(x^2)$ which is more commonly found in the literature; see for example Hake et. al. (1984).)

Our Dix layer group also forms a true Abelian group. In terms of the layer-group parameters there is no difficulty satisfying closure, associativity, and commutativity. The identity is the zero-traveltime layer, and to obtain the inverse just change the sign of $T(0)$. As was the case for the Schoenberg-Muir layer group, it is possible to create representations of physically disallowed media by subtraction in the Dix layer-group domain. For example, the infamous layers with imaginary interval velocities that are the bane of layer strippers everywhere are the result of subtraction in the Dix layer-group domain without due regard for physical constraints. For our first-anelliptic example uncareful subtraction will similarly create layers with complex F_W.

CONCLUSIONS

We have shown that the Schoenberg-Muir calculus and the Dix equations share a common theoretical underpinning: both are equivalent-medium systems, and both can be derived in a similar way. The equivalent-medium formulation of the Dix model provides a convenient direct algorithm for finding the extension of the Dix equations to anisotropic layered-medium systems.

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