Final Exam: Sample Questions

Math 128A Spring 2002
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Your Name: ______________________________

- Time: 180 minutes.
- Answer ALL questions.
- Please read carefully every question before answering it.
- If you need extra space, use the other side of the page.

1. (X points) Consider the iteration

$$c_{k+1} = c_k + \alpha\, f(c_k)\,,$$

where α is a constant that does not change with k, and $f(x) \in C^{\infty}$.

a. What is the condition on α for this iteration to converge to a solution of $f(x) = 0$?

b. What is the convergence rate?

c. What value of α is required for the quadratic convergence rate?

2. (X points) Neville's interpolation algorithm

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NEVILLE(x, x_1, x_2, ..., x_n, f_1, f_2, ..., f_n)
for i ← 1, 2, ..., n
do
    N_{i,1} ← f_i
    d_i ← x − x_i
for k ← 2, 3, ..., n
do
    for i ← k, k+1, ..., n
    do
        N_{i,k} ← N_{i,k−1} + d_i (N_{i,k−1} − N_{i−1,k−1}) / (x_i − x_{i−k+1})
return N_{n,n}
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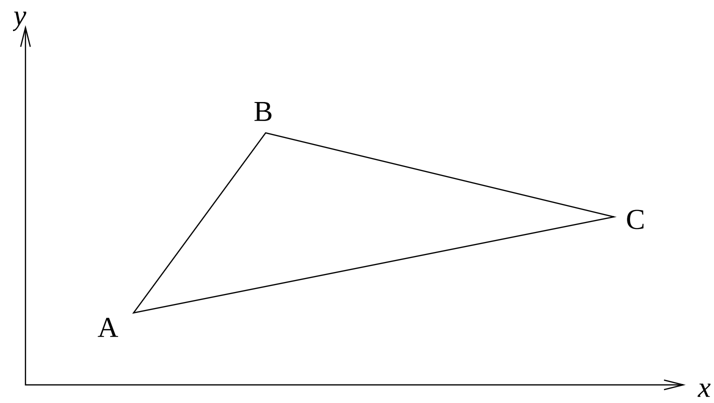
assumes that the data points $\{x_1, f_1\}, \{x_2, f_2\}, \ldots, \{x_n, f_n\}$ are known in advance. Modify the algorithm so that it processes the input data point by point.
Hint: loop by rows in the outer loop.

3. (X points) A two-dimensional function $f(x,y)$ is defined on a triangulated mesh.



a. Find an approximation of the form

$$f(x,y) \approx f(A)\phi_A(x,y) + f(B)\phi_B(x,y) + f(C)\phi_C(x,y)\,,$$

where A, B, C are the corners of a triangle, the point $\{x,y\}$ is inside the triangle, and the functions $\phi_A(x,y)$, $\phi_B(x,y)$, and $\phi_C(x,y)$ are linear in x and y.
Hint: The area of triangle ABC is equal to

$$S_{ABC} = 1/2\,(x_A\,y_B + x_B\,y_C + x_C\,y_A - x_B\,y_A - x_C\,y_B - x_A\,y_C)\,.$$

b. Find an approximation of the first partial derivatives of the form

$$\frac{\partial f}{\partial x} \approx \alpha_A\, f(A) + \alpha_B\, f(B) + \alpha_C\, f(C)\,.$$
$$\frac{\partial f}{\partial y} \approx \beta_A\, f(A) + \beta_B\, f(B) + \beta_C\, f(C)\,.$$

4. (X points) Find the first three polynomials orthogonal on the interval $[0,1]$ with respect to the inner product

$$<f,g>=\int_0^1 \frac{f(x)g(x)}{\sqrt{4-(x+1)^2}}\,dx\,.$$

5. (X points)

a. Derive a quadrature rule of the form

$$\int_a^b f(x)\,dx = \alpha\, f\left(\frac{2a+b}{3}\right) + \beta\, f\left(\frac{a+2b}{3}\right).$$

b. Determine its error assuming $f(x) \in C^2$.

6. (X points) What is the result of approximating the integral

$$\int_0^1 x^2\,dx$$

with the composite trapezoidal rule defined on n equal subintervals? Your answer should be in closed form and should not include the sum symbol.

8. (X points) Consider the initial-value problem

$$\begin{cases} y''(x) &= -\left[y'(x)\right]^2 x \\ y(-1) &= 0 \\ y'(-1) &= 1 \end{cases}$$

Using the step-size $h = 1$, find the output of one step of the midpoint method followed by one step of the second-order Adams-Bashforth method.

9. (X points)

a. How many floating-point operations are required to multiply $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$?

b. How many floating-point operations are required to compute the matrix $\mathbf{C} = \mathbf{u}\mathbf{u}^T$, where $\mathbf{u}$ is a column vector of length n?

c. How many floating-point operations are required to compute the product $\mathbf{A}\mathbf{C}$, where $\mathbf{A}$ is $n \times n$ matrix, and $\mathbf{C}$ is the matrix defined above?

10. (X points) Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix}$$

using Gaussian elimination. Show all steps of the computation.