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Chebyshev Polynomials (Mathematica notebook: `http://math.lbl.gov/~fomel/128A/Chebyshev.nb`)

Polynomial Shape

Chebyshev polynomials can be defined by the explicit formula

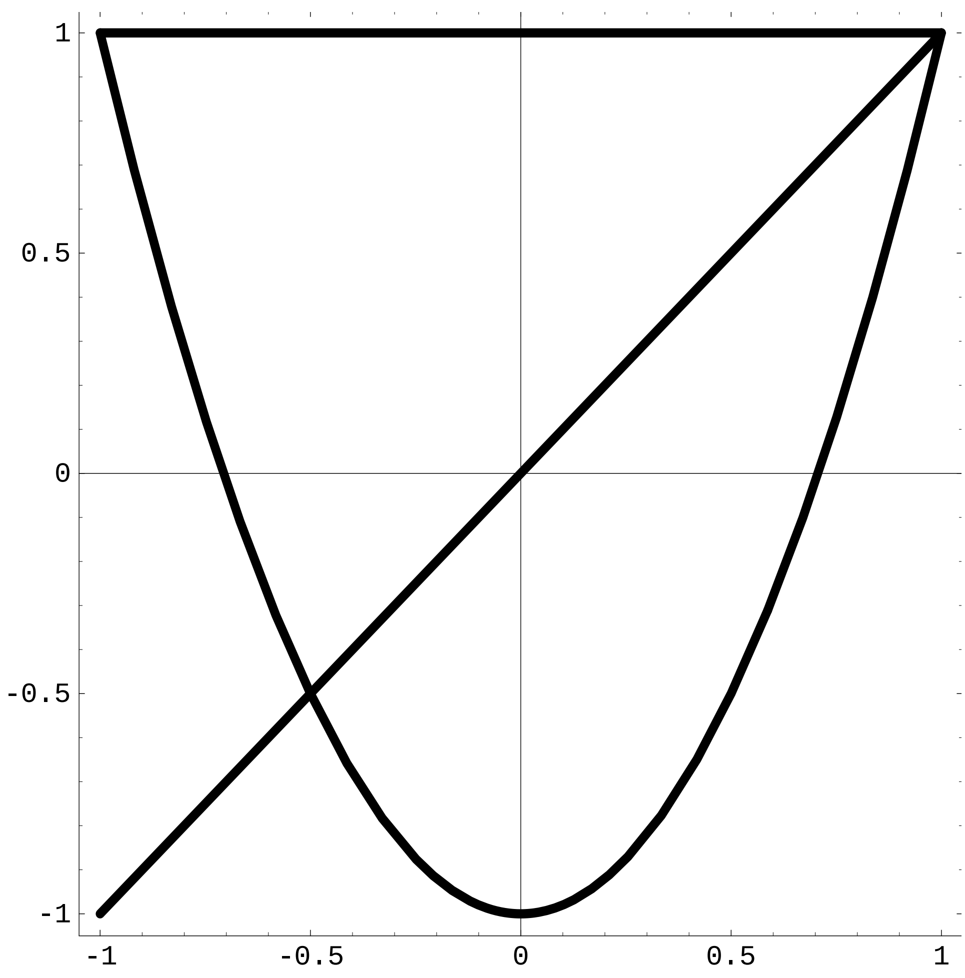
$$T_n(x) = \cos(n \arccos x)$$

or by the recursive relationship

$$T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x)\,.$$

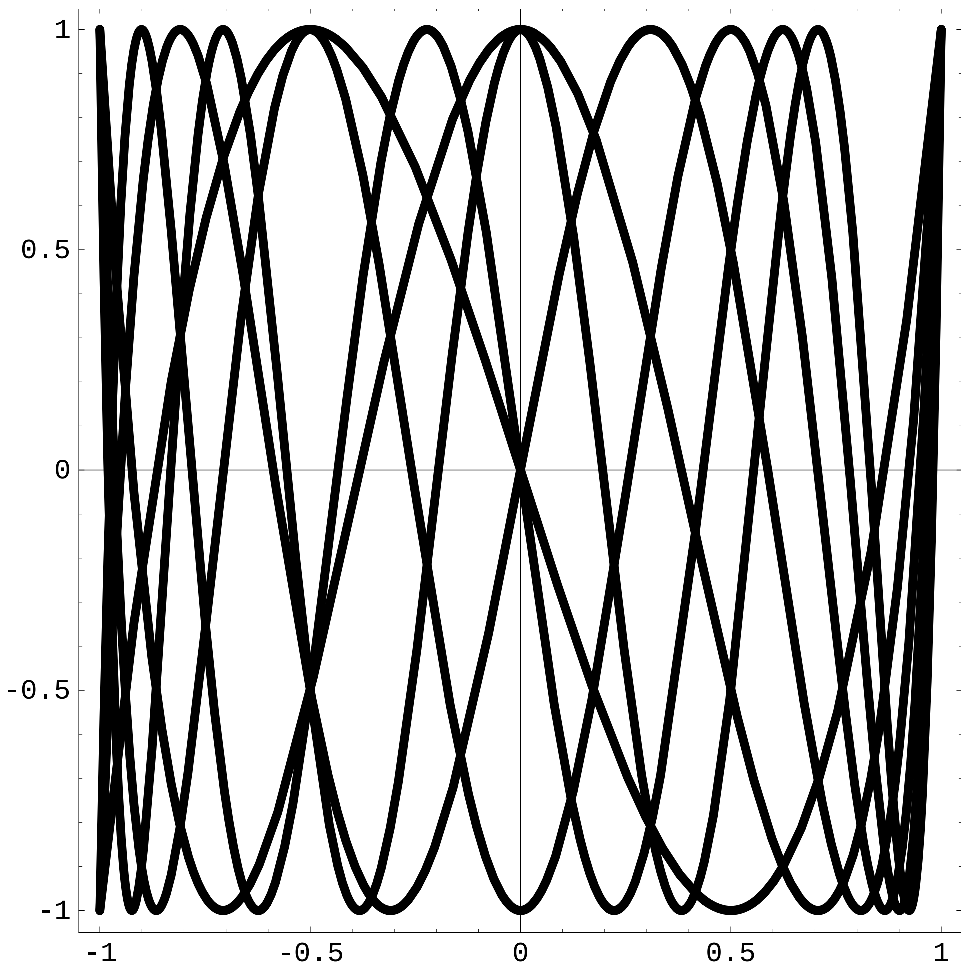
The first three polynomials are

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2\,x^2 - 1 \end{aligned}$$



The next six polynomials are

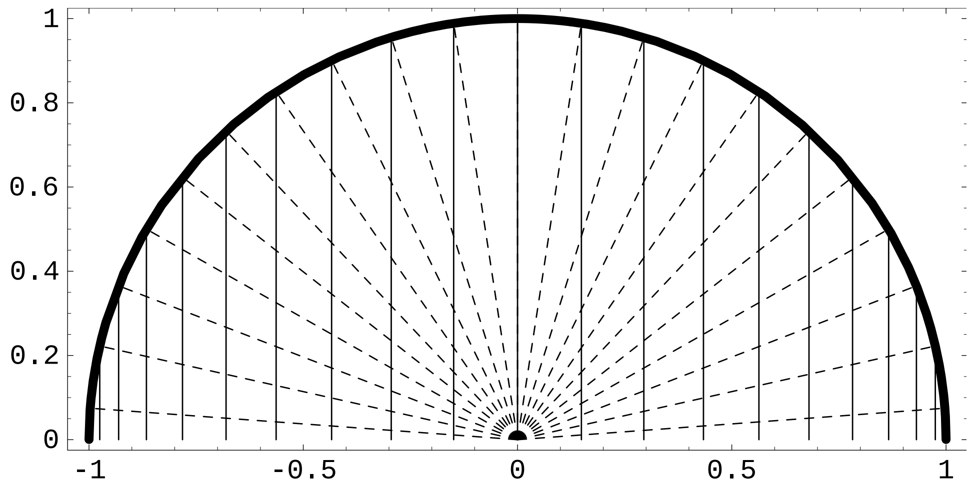
$$
\begin{aligned}
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{aligned}
$$

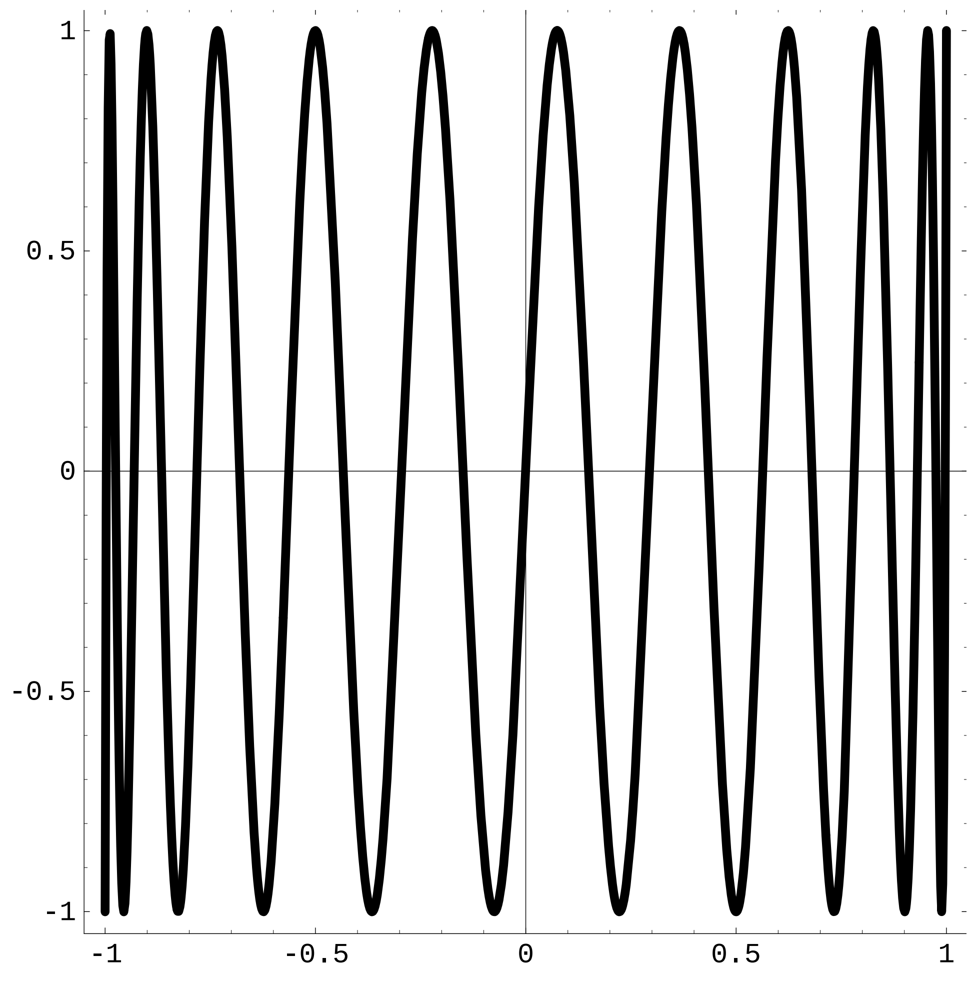


The extremum points of every Chebyshev polynomial alternate between -1 and 1.

Zeros of the Chebyshev polynomials

Polynomial of degree 21:





The zeros of $T_n(x)$ are distributed denser near the ends of the interval and sparser in the middle.

The explicit formula for k-th zero is

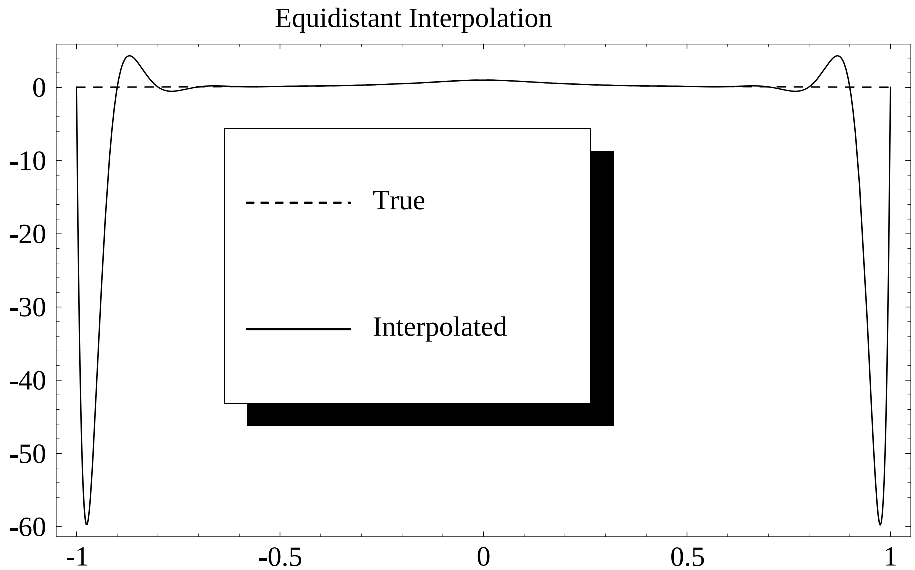
$$\hat{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, 2, \ldots, n\,.$$

Interpolation

In 1901, Runge demonstrated the pitfalls of equidistant polynomial interpolation using the function

$$f(x) = \frac{1}{1+25x^2}\,.$$

Interpolation with 21 equidistant (regularly spaced) nodes:



Interpolation with 21 Chebyshev nodes: