Migration by Hartley Transform

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Abstract

A Hartley transform is a variation of the Fourier transform with an integration kernal $\cos \omega t + \sin \omega t$. Most of its properties are analogous to those of the Fourier transform. Computationally useful properties include that the forward and inverse transforms are identical and the transform of real numbers remain real numbers. Solutions to the wave equation can be derived in Hartley transform coordinates and used to image seismic data. These are cheaper to compute than in the Fourier domain.

Definition

Hartley (1942) defined the transform pair

$$HT: \qquad U(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) \mathrm{cas}(\omega \mathrm{t}) \mathrm{dt}, \tag{1}$$

$$HT^{-1}: \qquad u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\omega) \mathrm{cas}(\omega \mathrm{t}) \mathrm{d}\omega, \tag{2}$$

$$\mathrm{cas\,x} = \mathrm{cos\,x} + \mathrm{sin\,x}.$$

A proof that equation (2) is the inverse of equation (1) is found by decomposing these equations into sin and cos transforms and looking up proofs of these inverses in any advanced calculus book. Note that equations (1) and (2) are of the same form. There are no sign changes as in the traditional Fourier transform. Also the Hartley transform of a real function is a real function.

Properties of the Hartley Transform

Most Fourier transform identities and theorems have their Hartley transform counterparts. I will give those here which are useful for computing wave propagation. These are more exhaustively described in Bracewell (1983) and Hartley (194´ ˙han here. The derivation of these identities involves a line or two of simple trigonometry.

The mapping between data in the Hartley domain and the Fourier domain is

$$HT(\omega) = REAL[FT(\omega)] - IMAG[FT(\omega)], \tag{3}$$

$$REAL[FT(\omega)] = [HT(\omega) + HT(-\omega)]/2, \tag{4a}$$

$$IMAG[FT(\omega)] = [HT(-\omega) - HT(\omega)]/2. \tag{4b}$$

The *shift rule* useful for Hartley domain interpolation and Fast Hartley transforms.

$$HT(\omega + c) = HT(\omega)\cos(c) + HT(-\omega)\sin(c) \tag{5}$$

A *Fast Hartley Transform* is coded in analogous way to a FFT algorithm. However, instead of multiplying powers of complex exponentials, the shift rule is used. Stanford University is seeking to commercialize the FHT, which prohibits me listing a code in this paper.

The *derivative rule* is used in solving the wave equation.

$$HT[f(t)'] = \omega HT(-\omega) \tag{6a}$$

$$HT[f(t)''] = \omega^2 HT(\omega) \tag{6b}$$

Notice that the second derivative resembles the Fourier transform rule, but has a *positive* sign.

The *convolution rule* is used during migration.

$$2\,HT(\omega) = HT_1(\omega)\,[HT_2(\omega) + HT_2(-\omega)] + HT_1(-\omega)\,[HT_2(\omega) - HT_2(-\omega)] \tag{7}$$

Note that a convolution point uses two real multiplications, three real additions, and five memory accesses. Convolution in the Fourier domain is a complex multiplication which uses two more multiplications, one less addition, and one more memory access.

The *mean rule* is shortcut for computing one of the Hartley transforms during migration.

$$\sum_{\omega} HT(\omega) = u(0) \tag{8}$$

Migration by Downward Continuation

This migration method is analogous to the phase-shift method. Start with the two-dimensional wave equation.

$$P_{xx} + P_{zz} = \frac{1}{v^2} P_{tt} \tag{9}$$

The derivate rule is used to Hartley transform this equation over record coordinates (x, t)

$$k_x^2 \tilde{P} - \frac{\omega^2}{v^2} \tilde{P} = \frac{\partial^2}{\partial z^2} \tilde{P}, \tag{10}$$

where (k_x, ω) are the Hartley transform conjugates. A solution for this equation is

$$\tilde{P}(z, \omega, k_x) = \tilde{P}(z = 0, \omega, k_x) \mathrm{cas}\, \mathrm{z} \sqrt{\frac{\omega^2}{\mathrm{v}^2} - \mathrm{k}_\mathrm{x}^2}. \tag{11}$$

This equation downward continues the recorded data. An image is obtained by inverse Hartley transforming the data for $t = 0$ for each z. The mean rule (equation 8) can be used to apply this imaging step to equation (11).

$$\tilde{P}(z, t = 0, k_x) = \sum_{\omega} \tilde{P}(z = 0, \omega, k_x) \mathrm{cas}\, \mathrm{z} \sqrt{\frac{\omega^2}{\mathrm{v}^2} - \mathrm{k}_\mathrm{x}^2} \tag{12}$$

The downward continuation code for equation (12) is as follows. The input and output have been Hartley transformed outside of this code. The correct impulse response of a buried point scatterer is shown in Figure 1.

```
c CONSTANT VELOCITY MEDIA CODE
c OUTPUT IS IN MIGRATED TIME
c FREQUENCY SCALE FACTORS
  dimension in(nomega,nkx), out(ntau,nkx), sine(nomega), cose(nomega)
  dimension dsine(nomega), dcose(nomega)
  scalekx2 = .5 * pi2 * v * dt / (nkx * dx)
  scalekx2 = scalekx2 * scalekx2
  scalew2 = pi2 * pi2 / (nomega * nomega)
c COMPUTE EACH LATERAL FREQUENCY
  do 200 ikx=1,nkx/2+1
    k = (ikx-1) * (ikx-1) * scalekx2
    ikx1 = nkx - ikx + 1
c INITIALIZE SINES AND COSINES
c AVOID EVANESCENT FREQUENCIES
```

```
c AT THIS POINT ADD LEVIN'S (1983) ARTIFACT REDUCING ALGORITHM
     iomega = sqrt (k / scalew2) + 1
     do 100 iw=iomega,nomega/2+1
        iw1 = nomega - iw + 1
        kz = sqrt (scalesw * iw * iw - k)
        sine(iw) = 0.
        cose(iw) = 1.
        dsine(iw) = sin (kz)
        dcose(iw) = cos (kz)
100     continue
c DO EACH DEPTH
     do 200 itau=1,ntau
        out(itau,ikx) = 0.
        out(itau,ikx1) = 0.
c EACH NON-EVANESCENT FREQUENCY CONTRIBUTES TO EACH DEPTH
c FOUR-WAY HARTLEY COORDINATE SYMMETRY REDUCES CAS COMPUTATION
        do 200 iw=iomega,nomega/2+1
           casplus = cose(iw) + sine(iw)
           casminus = cose(iw) - sine(iw)
           out(itau,ikx)=out(itau,ikx)+in(iw,ikx)*casplus+in(iw1,ikx)*casminus
  if(ikx>1) out(itau,ikx1)=out(itau,ikx1)+in(iw,ikx1)*casplus+in(iw1,ikx1)*casminus
c UPDATE EACH CAS FREQUENCY FOR EACH DEPTH
           temp = sine(iw) * dcose(iw) + cose(iw) * dsine(iw)
           cose(iw) = cose(iw) * dcose(iw) - sine(iw) * dsine(iw)
           sine(iw) = tmp
200        continue
```

A cost analysis of the Hartley transform and Fourier transform methods is as follows:

- The FHT is two to three times faster than a FFT transform because no complex arithmetic is involved.
- All four quadrants of the Hartley domain data– plus and minus frequencies of (ω, k_x) must be used. Due to conjugate symmetry, only two Fourier transform quadrants are necessary.
- In both transformations, only one quadrant of sines and cosines need be calculated due to symmetry.
- The multiplication in equation (12) is real in Hartley transform coordinates but complex in Fourier transform coordinates.

Overall, the Hartley transform method is about twice as fast as the Fourier transform method.

Migration by Coordinate Transformation

This migration method is similar to the Stolt algorithm. Note equation (11) is *almost* an inverse Hartley transform. It can be turned into a Hartley transform with the definition

$$k_z = \sqrt{\omega^2/v^2 - k_x^2}.$$

$$\tilde{P}(k_z, k_x) = \tilde{P}(\omega = v\sqrt{k_x^2 + k_z^2}, k_x)\frac{vk_x}{\sqrt{k_x^2 + k_z^2}}\text{cas } zk_z \tag{13}$$

Equation (13) says that to migrate the Hartley domain seismic data, the data is interpolated from (ω, k_x) to (k_z, k_x) and scaled.

The migration code for equation (13) is as follows. The input and output have been Hartley transformed outside of this code. The correct impulse response of a buried point scatterer is shown in Figure 1.

```
c  CONSTANT VELOCITY MEDIA CODE
c  FREQUENCY SCALE FACTORS
   dimension in(nomega,nkx), out(nomega,nkx)
   scalekx2 = .5 * v * nt * dt / (nx * dx)
   nomega2 = nomega * nomega * .25
c  DO EACH WAVENUMBER
   do 200 ikx=1,nkx/2+1
      k = scalekx2 * (ikx-1) * (ikx-1)
      ikx1 = nkx - ikx + 1
c  DO EACH FREQUENCY
      do 200 iz=1,nomega/2+1
         iz1 = nomega - iz + 1
         t = iz * iz + k
         if ((t>0).and.(t<nt2)) goto 100
c  MAPPED FREQUENCIES IN SOURCE DATA DO NOT EXIST
            out(iz,ikx) = 0.
            out(iz1,ikx) = 0.
            if (ikx>1) out(iz,ikx1) = 0.
            if (ikx>1) out(iz1,ikx1) = 0.
            goto 200
```

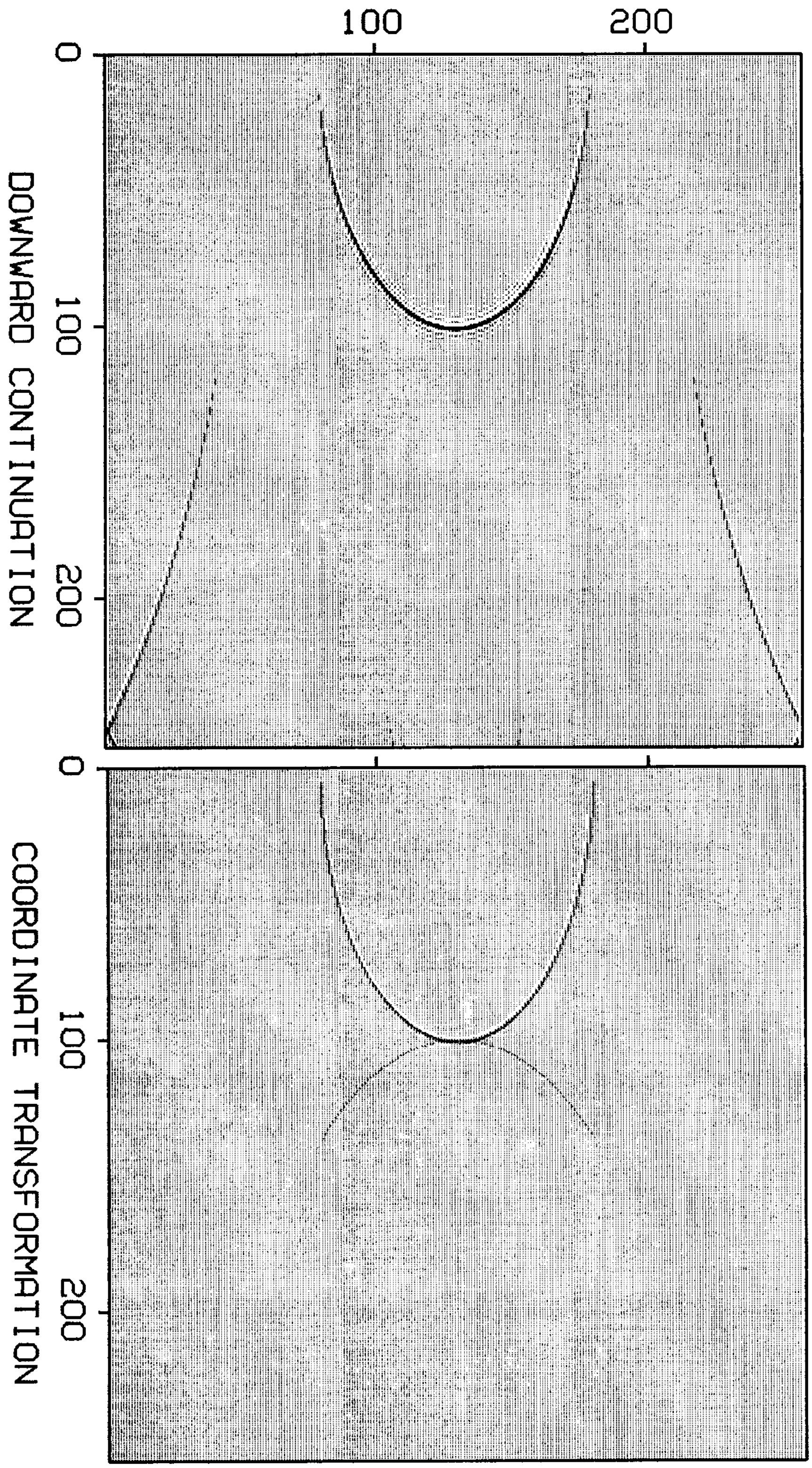

FIG. 1. Migration impulse response in Hartley transform space for downward continuation method (top) and coordinate transformation method (bottom). A point scatterer is located at (128, 100). The artifacts are not caused by the Hartley transform, but could be eliminated by the methods of Levin (1983) and Harlan (1982).

```
100t = sqrt (t)
c LINEAR INTERPOLATION, HARLAN'S TRUNCATED SINC (1982) WORKS BETTER
c USE FOUR QUADRANT SYMMETRY OF SQUARE ROOT TO SAVE COMPUTATION
      it = t
      it1 = nomega - it + 1
      weight1 = 1 - t + it
      weight2 = t - it
      out(iz,ikx) = weight1 * in(it,ikx) + weight2 * in(it+1,ikx)
      out(iz1,ikx) = weight1 * in(it1,ikx) + weight2 * in(it1-1,ikx)
      if (ikx>1) out(iz,ikx1) = weight1 * in(it,ikx1) + weight2 * in(it+1,ikx1)
      if (ikx>1) out(iz1,ikx1) = weight1 * in(it1,ikx1) + weight2 * in(it1-1,ikx1)
200   continue
```

A cost analysis of the Hartley transform and Fourier transform methods is as follows:

- The FHT is two to three times faster than a FFT transform because no complex arithmetic is involved.
- All four quadrants of the Hartley domain data– plus and minus frequencies of (ω, k_x) must be used. Due to conjugate symmetry, only two Fourier transform quadrants are necessary.
- In both transformations, only one quadrant of square roots need be calculated due to symmetry.
- The interpolation in equation (12) is real in Hartley transform coordinates but complex in Fourier transform coordinates.

The coordinate transformation part of the Hartley transform method is about the same cost as the Fourier transform method, though the Hartley transforms are faster.

REFERENCES

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CALCRUST

A CONSORTIUM APPROACH TO THE STUDY OF CRUSTAL STRUCTURE AND EVOLUTION

Earth scientists from the academic community in southern California, whose interests include crustal structure and evolution, have formed a regional consortium (CALCRUST) to apply the seismic reflection technique, in combination with geological and geochemical observations, toward an improved understanding of the subsurface in the southwestern United States. This consortium has received funding from NSF for an initial two-year program to carry out a research project entitled, "Deep-Seismic Profile from the San Andreas to the Colorado Plateau (Segment I): History of Continental Accretion Recorded in the Mojave-Sonoran Desert Region." A pilot project along the San Andreas fault to perform geophysical experiments with bore-hole and surface techniques will also be run. The Mojave-Sonoran profile will begin in the southwestern Whipple Mountains, cross the Riverside Mountains, and extend to the Big Maria-McCoy Mountains region as far as funding allows. The focus of this line will be on large-scale crustal extension as manifest in the Whipple-Riverside-Big Maria detachment faults and on major crustal compression as displayed by the stacks of Mesozoic thrusts in the area. Detailed geometries of detachment faulting, detachment-related crustal folding, and Tertiary basin formation will be analyzed. Imbricate thrusting and mylonitization of Mesozoic age will be studied in detail as will their interaction with subsequent deformation. Problems of terrane accretion will also be addressed if funding permits with the study of the McCoy Mountains Formation and its relationship to the Pelona-Orocopia Schist and Vincent-Chocolate Mountains thrust system. Besides solving problems of regional importance, this line should shed new light on many of the fundamental problems of continental tectonics as expressed in other orogenic belts.

CALCRUST will focus in large part on low-angle structures---their geometry and continuity in the subsurface---and in particular, will attempt to relate subsurface reflection data to surface geology and geochemistry. Outcrops are abundant in the southwestern United States, thereby providing excellent "ground-truth" control for subsurface interpretation and refinement of geophysical methods. Furthermore, the geology is rich in problems of current interest to the earth science community. CALCRUST will also seek to advance techniques by which subsurface reflection data can be related directly to subsurface rheology, petrology and/or geochemistry.

Because studies of the type proposed are leading toward work in increasingly complex structural terranes, and because we are attempting to tie subsurface reflectors to surface geology, this study will emphasize the acquisition of high-resolution data - principally from the upper crust. This will require procedures such as the use of high-density vibrator/sensor spacing, clustered points and geophone groups, non-standard vibrator/sensor geometries (offline, fan shoot, large-opening spreads, etc.) and advanced data processing methods (trace-by-trace analysis, "smart" stacking, migration before stack, travel-time curve analysis, etc.). This program will extend to the third dimension our reasonably complete understanding of the surface geology, and begin the process of establishing geological relationships and continuity between diverse and widely spaced terranes in the southwestern United States.