A WKBJ INVERSE FOR THE ACOUSTIC WAVE EQUATION IN A LAYERED MEDIUM

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In SEP-24 Clayton and I presented an approximate inversion scheme for acoustic data. Though the method had the promise of greater speed and efficiency its realism was somewhat dubious. Among other things a constant background velocity was assumed. An extension of the method to the case of a vertically varying background velocity is given below.

The Unperturbed (WKBJ) World

The fundamental assumption in this approach is that wave propagation is governed by a slowly varying "background" velocity $v(z)$, and that reflections are caused by rapid fluctuations of velocity and density about their background values.

The background wave equation will be

$$\left[\nabla^2 + \frac{\omega^2}{v^2(z)}\right] \varphi_0(\vec{x},\omega) = 0 \tag{1}$$

The fundamental solutions to this equation will be taken to be the WKB wave functions $\varphi_0^\pm$. These functions are most easily expressed in the (p,z) representation, so, taking a Fourier transform over the horizontal coordinate(s) x, we have

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{v^2(z)} - k_x^2\right] \varphi_0^\pm(\omega,k_x,z) = 0 \tag{2}$$

with the approximate solutions

$$\varphi_0^\pm(\omega,k_x,z) = \sqrt{\frac{\omega}{q(z)}}\, e^{\pm i \int_0^z dz' q(z')} \tag{3}$$

$$q(z) = \frac{\omega}{v(z)} \sqrt{1 - k_x^2 v(z)^2/\omega^2} \tag{4}$$

The WKB solutions (3) of (2) are just Gazdag's phase-shift propagators with a depth-variable normalization tacked on. They are good approximations provided $v(z)$ is slowly varying over a wavelength of φ_0, and also provided $|p| < |\omega|/v$ (i.e., provided one doesn't get too close to critical angle. Actually they can be patched up to work at critical angle and beyond, with a little extra effort).

A WKBJ Green's operator is easily constructed from $\varphi_0^{\pm}$. Write

$$G_{\pm}^{0}[k_x,\omega;z\,|\,z'] = \frac{\varphi_0^{\pm}(z_>)\,\varphi_0^{\mp}(z_<)}{\mp 2i\omega}$$

$$= \frac{e^{\pm i \int_{z<}^{z>} dz'\, q(z')}}{\mp 2i[q(z)q(z')]^{1/2}} \tag{5}$$

where $z_>$ ($z_<$) is the greater (lesser) of z and z'. It is easily confirmed that

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{v^2(z)} - k_x^2\right] G_{\pm}^{0}(k_x,\omega;z\,|\,z') = -\,\delta(z - z') \tag{6}$$

From the definition (5) it is clear that G_+^0 is outgoing (exploding) while G_-^0 is incoming (imploding). For what it's worth, it is easy to see that G_+^0 (G_-^0) is nonzero only for positive (negative) time.

The Real (Acoustic) World

We will assume that $\varphi_0^{\pm}$ and $G_{\pm}^{0}$ can adequately model point-to-point propagation in the real world. To model reflections, however, we will need to look at the real wave equation, whose form we will take to be

$$\left[\vec{\nabla} \cdot \frac{\rho_0}{\rho}\vec{\nabla} + \omega^2 \frac{\rho_0}{K}\right]\varphi = 0\,; \quad \left[\vec{\nabla} \cdot \frac{\rho_0}{\rho}\vec{\nabla} + \omega^2 \frac{\rho_0}{K}\right] G_{\pm} = -\,\delta(\vec{x}) \tag{7}$$

where ρ and k are the "real" density and bulk modulus, and ρ_0 is a constant "background" or reference density. The equation for φ can be rewritten as

$$\left[\nabla^2 + \frac{\omega^2}{v^2(z)} + V(\omega,x)\right]\varphi(\omega,\vec{x}) = 0 \tag{8}$$

where the potential term $V(\omega,\vec{x})$ has two components

$$V(\omega,\vec{x}) = \frac{\omega^2}{v^2(z)}\, a_1(\vec{x}) + \vec{\nabla} \,.\, a_2(\vec{x})\, \vec{\nabla} \tag{9}$$

with

$$a_1(\vec{x}) = \frac{\rho_0 v^2(z)}{K(x,z)} - 1\,; \quad a_2(x) = \frac{\rho_0}{\rho(x,z)} - 1 \tag{10}$$

The Born Approximation

According to the Born approximation, a "real" impulse response can be taken to be

$$G_{\pm} \simeq G_{\pm}^{0} + G_{\pm}^{0}\, V\, G_{\pm}^{0} \tag{11}$$

The measured reflection response at the earth's surface is then $G_{\pm} - G_{\pm}^{0} \equiv D$, which is expressed in the $(p\,,z)$ representation as

$$D(\omega;k_g,z_g=0\,|\,k_s,z_s=0) = \int_0^\infty dz\; G_{+}^{0}(k_g,\omega;0\,|\,z)\; V(\omega,z;k_g\,|\,k_s)\; G_{+}^{0}(k_s,\omega;z\,|\,0)$$

$$\cong \frac{1}{4}\, \frac{1}{[q_g(0)\; q_s(0)]^{1/2}} \int_0^\infty dz\; \frac{e^{\,i\int_0^\infty dz'[q_g(z') + q_s(z')]}}{[q_g(z)\; q_s(z)]^{1/2}} \,\cdot$$

$$\cdot \left\{ \frac{\omega^2}{v^2(z)}\, a_1(k_g - k_s, z) + [q_r(z) q_s(z) - k_g k_s]\; a_2(k_g - k_s, z) \right\} \tag{12}$$

where $q_g = (\omega/v)(1 - k_g^2 v^2/\omega^2)^{1/2}$, $q_s = (\omega/v)(1 - k_s^2 v^2/\omega^2)^{1/2}$ are the vertical spatial frequencies associated with source and receiver, respectively.

The Inversion

Given D, we wish to use equation (12) to recover the "potentials" a_1 and a_2. Back in SEP-24, when the background v was constant, this was almost trivial since the integral in equation (12) turned out to be a simple Fourier transform over z.

Here, it isn't. It would appear that a rather messy integral equation must be inverted to get to a_2 and a_2.

Things, however, are not so bad as they would appear. It turns out that equation (12) can be inverted in a very straightforward manner. Here's the trick:

We first "migrate" the data D by downward continuation of sources and receivers followed by an integration over frequency to recover the $t=0$ component. That is, we define the migration M of D to be

$$M(k_m,k_h,z) \;=\; \int d\omega \, C(k_m,k_h,\omega,z)\, \varphi_0^-(\omega,k_g,z)\, \varphi_0^-(\omega,k_s,z)\cdot$$
$$\cdot\, D(\omega;k_g,0\,|\,k_s,0) \tag{13}$$

where $k_m = k_g - k_s$, $k_h = k_g + k_s$ are midpoint and offset spatial frequency, the two φ_0^- are just the WKB wave functions (equation 3) travelling in the desired directions, and $C(k_m,k_h,\omega,z)$ is just some slowly varying real function thrown in to massage the data, to be specified later. A sum over k_h of M would essentially be a phase-shift migration of the data.

We now argue that $M(k_m,k_h,z)$ can be a function of earth parameters only in the immediate neighborhood of z. (If downward continuation has really propagated us to the depth z, then this claim is just an expression of causality.) We can strengthen the argument by substituting (12) into (13):

$$M(k_m,k_h,z) \;=\; \int_0^\infty dz'\,[a_1(k_m,z')\,A_1(k_m,k_h,z,z') +$$
$$+\, a_2(k_m,z')\,A_2(k_m,k_h,z,z') \tag{14}$$

where

$$\begin{bmatrix} A_1(k_m,k_h,z,z') \\ A_2(k_m,k_h,z,z') \end{bmatrix} \;=\; \frac{1}{4}\int d\omega\, \frac{e^{\,i\int_z^{z'} dz''[q_g(z'')+q_s(z'')]}}{[q_g(0)q_s(0)q_g(z)q_s(z)q_g(z')q_s(z')]^{1/2}}\cdot$$
$$|\omega|\cdot C(k_m,k_h,\omega,z)\cdot \begin{bmatrix} \dfrac{\omega^2}{v^2(z')} \\ q_g(z')q_s(z') - k_g k_s \end{bmatrix} \tag{15}$$

Only at the point $z = z'$ do the phases in (15) line up. We are justified in claiming, therefore, that A is nonzero only for a very narrow range about $z' = z$. Within this range, q_g, and q_s can be considered constant, allowing a drastic simplification of (15).

Define $k_z = -q_g - q_s$. It is then easy to show that

$$\omega \;=\; \frac{-k_z v}{2}\sqrt{\left[1+\frac{k_m^2}{k_z^2}\right]\left[1+\frac{k_h^2}{k_z^2}\right]} \tag{16}$$

$$q_g q_s \;=\; \frac{k_z^2}{4}\left[1 - \frac{k_m^2 k_h^2}{k_z^2}\right] \tag{17}$$

$$q_g q_s - k_g k_s \;=\; \frac{\omega^2}{v^2}\,\frac{k_z^2 - k_h^2}{k_z^2 + k_h^2} \tag{18}$$

and

$$d\omega \;=\; dk_z\,\frac{v^2}{\omega^2}\,q_g q_s\,\frac{\omega}{k_z} \tag{19}$$

Substituting these tidbits into (15) gives

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}(k_m, k_h, z, z') \;=\; -\frac{1}{4}\int dk_z\; e^{ik_z(z-z')}\left|\frac{\omega^2}{k_z}\right| \frac{C(k_m, k_h, \omega, z)}{[q_r(0)q_s(0)]^{1/2}} \cdot \begin{bmatrix} 1 \\ \dfrac{k_z^2 - k_h^2}{k_z^2 + k_h^2} \end{bmatrix} \tag{20}$$

We are now ready to choose C. A convenient choice would be

$$C(k_m, k_h, \omega, z) \;=\; \frac{8[q_g(z)q_s(z)q_g(0)q_s(0)]^{1/2}}{\omega^2} \tag{21}$$

because then

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \;=\; \int dk_z\; e^{ik_z(z-z')}\left[1 - \frac{k_m^2 k_h^2}{k_z^4}\right]^{1/2} \cdot \begin{bmatrix} 1 \\ \dfrac{k_z^2 - k_h^2}{k_z^2 + k_h^2} \end{bmatrix} \tag{22}$$

With this choice for C, A_1 and A_2 depend only on the difference between z and z', and in fact are just Fourier transforms of very simple expressions.

Thus a Fourier transform over z of M yields

$$\frac{M(k_m, k_h, k_z)}{\sqrt{1 - \dfrac{k_m^2 k_h^2}{k_z^4}}} \;=\; a_1(k_m, k_z) + \frac{k_z^2 - k_h^2}{k_z^2 + k_h^2}\,a_2(k_m, k_z) \tag{23}$$

An expression which, given two or more k_h values, is easily inverted to obtain a_1 and a_2.

To summarize, then, the complete inversion algorithm is as follows.

1) The surface data field D is Fourier *transformed*[1] over time, source, and receiver coordinates to obtain $D(\omega;k_g,0\,|\,k_s,0)$.

2) D is migrated according to equation (13). Using expression (21) for C, and the defining equations for q_g, q_s, k_z, and φ_0^-,

$$M(k_m,k_h,z) \;=\; 8\int d\omega\, D(\omega;k_g,0\,|\,k_s,0)\, e^{-i\int_0^z dz' k_z} \left[\frac{q_g(0)q_s(0)}{\omega^2}\right]^{1/2} \tag{24}$$

That is, D is simply phase shifted and summed over ω. The multiplicative factor $[q_g(0)q_s(0)]^{1/2}/|\omega|$ is *not* depth-dependent, so the algorithm, if not exactly cheap, is at least simple.

3) M is Fourier transformed over z.

4) The result is inverted for a_1 and a_2 via equation (23), probably by least squares, since more than two k_h values should be available.

5) Double inverse Fourier transforms of a_1 and a_2 yield their spatial representations.

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[1]The "best" way to effect these transforms is moot. Some of the options are: (a) FFT's over x_s, x_r, time; (b) FFT's over midpoint, offset, and time; (c) FFT's over midpoint and time, but a Radon transform (slant stack) over offset. Take your pick.

APPENDIX A: EXTENSION TO 2-1/2 DIMENSIONS

The algorithm developed above will work in either a 2-D (line sources and receivers in a medium which changes in only one horizontal dimension) or a 3-D world. The seismic experiment is usually ~ 21/2-D (point sources and receivers in an otherwise 2-D environment) in which case some modifications are in order.

Suppose that a_1 and a_2 are functions of x and z only, and that D is measured along the plane $y_g = y_s = 0$. Then equation (12) becomes (because V does not depend on y)

$$D(\omega;k_g,0,0\,|\,k_s,0,0) = \int dz \int dk_y\, G_+^0(k_g,k_y,\omega;0\,|\,z)\, V G_+^0(k_s,k_y,\omega;z\,|\,0)$$

$$= \frac{1}{4}\int dz \int dk_y \frac{e^{i\int_0^z dz'[q_g(z')+q_s(z')]}}{[q_g(0)q_s(0)q_g(z)q_s(z)]^{1/2}}$$

$$* \left\{\frac{\omega^2}{v^2(z)}\, a_1(k_g-k_s,z) + [q_g(z)q_s(z) - k_g k_s - k_y^2]\, a_2(k_g-k_s,z)\right\} \qquad \text{(A1)}$$

where

$$q_{\substack{g\\s}}^2 = \frac{\omega^2}{v^2} - k_{\substack{g\\s}}^2 - k_y^2$$

This expression can be simplified by doing a stationary phase approximation to the k_y integral, yielding

$$D(\omega;k_g,0,0\,|\,k_s,0,0) = \frac{1}{4}\int dz \frac{e^{i\int_0^z dz'[q_g(z')+q_s(z')]}}{\left[q_g(0)q_s(0)q_g(z)q_s(z)\right]^{1/2}\sqrt{ih(z)}}\cdot$$

$$\cdot\left\{\frac{\omega^2}{v^2(z)}\, a_1(k_g-k_s,z) + [q_g(z)q_s(z)-k_g k_s]\, a_2(k_g-k_s,z)\right\} \qquad \text{(A2)}$$

where q_g and q_s are understood to be evaluated at $k_y = 0$, and

$$h(z) = \int_0^z dz'\,\frac{\omega^2}{v^2}\left[\frac{1}{q_g^3(z')} + \frac{1}{q_s^3(z')}\right] \qquad \text{(A3)}$$

Except for the new factor $[ih(z)]^{1/2}$ appearing in the denominator of (A2), this equation is identical to the 2-D equation (12). Thus a 2-1/2-D inversion will work just like the 2-D, except the multiplier C in the migration step should include the factor $[ih(z)]^{1/2}$:

$$C_{21/2} = C_2 \cdot \sqrt{ih(z)} \tag{A4}$$

Equation (24) (the migration equation) thus becomes

$$M_{21/2}\left[k_m, k_h, z\right] = 8\,(i)^{1/2} \int d\,\omega\, D(\omega; k_g, 0, 0 \,|\, k_s, 0, 0)\, e^{-i \int_0^z dz'\, k_z} \cdot \left\{ q_g(0) q_s(0) \int_0^z dz' \frac{1}{v^2} \left[\frac{1}{q_g^3(z')} + \frac{1}{q_s^3(z')} \right] \right\}^{1/2} \tag{A5}$$

The other steps in the algorithm are unchanged.