Stable Extrapolation Of Scalar Wavefields

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A formulation for oneway wave equations is given that is both accurate and stable. The equations are generated by Muir's square root recursion with one additional term. Ignoring the term, results in a WKBJ-accurate solution in the extrapolation direction. The resulting operators are unconditionally stable. Including the term gives physical optics in the extrapolation direction, but it the medium variations are too rapid, the operators may be unstable. For acoustic wavefields the state variable of the system is pressure divided by the square root of impedance.

Introduction

Oneway wave extrapolators are being used for purposes (i.e. modelling and inversion) beyond the migration algorithms for which they were originally designed. These uses have put the additional requirements on the operators that they be both stable and accurate.

In SEP-16, Godfrey *et.al.* gave an *ad hoc* formulation that was stable, but it is unclear how solutions of that equation relate to solutions of the scalar wave equation. In this paper we arrive at almost the same formulation through a different approach. This time we are able to identify the physical variable of the system, and what approximations are present in the formulation.

The derivation starts with the scalar wave equation for pressure

$$\left[\frac{\omega^2}{K} + \nabla\cdot \frac{1}{\rho} \nabla \right] P = 0 \qquad (1)$$

where ρ is the density, and K is the bulk modulus. The first step is to isolate the ω^2 from the bulk modulus. The step is necessary because in the recurrence relation that will be discussed in a moment, we will require that z-derivatives and the ω^2 commute. This is accomplished by

$$\frac{1}{\sqrt{K}}\left[\omega^2 + \sqrt{K}\ \nabla\cdot\frac{1}{\rho}\nabla\cdot\sqrt{K}\right]\left[\frac{P}{\sqrt{K}}\right] = 0 \tag{2}$$

By cancelling the leading $1/\sqrt{K}$ term, and taking $P/\sqrt{K}$ to be the physical variable, we effectively isolate the ω^2 term.

To put equation (2) in the form of an extrapolator, the x-derivatives and the ω^2 term are moved to the right side of the equation. Hence,

$$(-D_z^H D_z)\left[\frac{P}{\sqrt{K}}\right] = \left[-\omega^2 + D_x^H D_x\right]\left[\frac{P}{\sqrt{K}}\right] \tag{3}$$

where

$$D_x = \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x}\sqrt{K}, \qquad D_x^H = -\sqrt{K}\frac{\partial}{\partial x}\frac{1}{\sqrt{\rho}}$$

and

$$D_z = \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial z}\sqrt{K}, \qquad D_z^H = -\sqrt{K}\frac{\partial}{\partial z}\frac{1}{\sqrt{\rho}}$$

In the equations above H denotes the Hermitian transpose. The minus sign in the definitions of D_x^H and D_z^H are included so that the operators $D_x^H D_x$ and $D_z^H D_z$ are positive definite.

To form oneway equations, we need to take the square root of both sides of equation (3). It is not clear how to directly take the square root of the operator $(-D_z^H D_z)$. One approach is to decompose the operator into the form

$$(-D_z^H D_z) = \tilde{D}_z^2 - C(x,z) \tag{4}$$

where

$$\tilde{D}_z = \sqrt{v}\frac{\partial}{\partial z}\sqrt{v}$$

and

$$C(x,z) = \frac{K}{4\rho}\left[\frac{3}{4}\left[\frac{K_z}{K}\right]^2 + \frac{7}{4}\left[\frac{\rho_z}{\rho}\right]^2 + \frac{1}{2}\left[\frac{\rho_z}{\rho}\right]\left[\frac{K_z}{K}\right] - \frac{K_{zz}}{K} - \frac{\rho_{zz}}{\rho}\right] \tag{5}$$

Note that since the correction term $C(x,z)$ contains no spatial derivatives that operate on the wavefield, it is simply a multiplicative operator. If the second derivatives of the medium parameters are small, then $C(x,z)$ will be a positive function. However, if the medium is varying rapidly such that the second derivatives become large (i.e. at interfaces), then $C(x,z)$ can become negative.

With the z-derivative operator defined by equation (4), the wave equation is now

$$\tilde{D}_z^2 \left[\frac{P}{\sqrt{K}}\right] = \left[-\omega^2 + D_x^H D_x + C(x,z)\right]\left[\frac{P}{\sqrt{K}}\right] \tag{6}$$

Formally taking the square root of (6) to obtain a oneway wave equation we have

$$\tilde{D}_z \left[\frac{P}{\sqrt{K}}\right] = \sqrt{v}\,\partial_z \sqrt{v}\left[\frac{P}{\sqrt{K}}\right] = \pm S_n \left[\frac{P}{\sqrt{K}}\right] \tag{7}$$

where S_n is the n^{th} approximate to the exact square root S_∞

$$S_\infty = \sqrt{-\omega^2 + D_x^H D_x + C(x,z)} \tag{8}$$

The approximations can be recursively generated by Muir's relation

$$S_n = -i\omega + \frac{D_x^H D_x + C(x,z)}{-i\omega + S_{n-1}}, \quad S_0 = -i\omega \tag{9}$$

As $n \to \infty$, the approximations generated by equation (9) converge to equation (8). The square root approximations generated by equation (9) will be unconditionally stable if $C(x,z)$ is strictly positive. If the second derivative variations in the medium parameters are small with respect to the first derivative variations then $C(x,z)$ is always greater than zero, and hence it may be included in the recursion relation without upsetting the stability properties. If the second derivatives are large then $C(x,z)$ should be modified or ignored before it is included in the expansion. The choice of sign in equation (7) corresponds to up and downgoing waves. In this paper we choose the minus sign which is the appropriate sign for downward wave extrapolation, but a similar result holds for the plus sign.

To actually solve equation (7) we will put it in the symmetric form

$$\frac{\partial}{\partial z}\tilde{P} = -\frac{1}{\sqrt{v}} S_n \frac{1}{\sqrt{v}} \tilde{P} \tag{10}$$

where $\tilde{P}$ is the state variable and is defined by

$$\tilde{P} = \frac{P}{\left[\rho K\right]^{1/4}} = \frac{P}{\sqrt{r}} \tag{11}$$

where r is the acoustic impedance.

To determine conditions on S_n for stable extrapolation, we use the condition that

$$\frac{\partial}{\partial z}\tilde{P}^H\tilde{P} = \frac{\partial}{\partial z}\left[\frac{P^H P}{r}\right] \leq 0 \tag{12}$$

which physically means that the net energy flux should not increase with z. Following Muir's rules for causal positive real operators, (i.e. $-i\omega$ and $D_x^H D_x$), we can show that S_n is causal positive real.

The role of $C(x,z)$ in equation (9) has several interesting aspects. The term involving $C(x,z)$ is $O(\omega^{-1})$, while the remaining terms in the relation are $O(\omega^1)$. This means that the correction is of a higher order (less significant at higher frequency) than the standard WKBJ correction. In this case the WKBJ amplitude correction would be $O(\omega^0)$. This means that if we ignore $C(x,z)$ altogether in the recurrence relation, we are producing an approximation that is geometric optics in the z-direction, and physical optics in the x-direction. To show this, consider the case of a normally incident plane wave traveling in a medium that contains no lateral variations. Thus all orders of equation (9) generate

$$S_n = -i\omega$$

Substituting this into equation (10) produces

$$\frac{\partial}{\partial z}\tilde{P} = i\frac{\omega}{v(z)}\tilde{P} \qquad or \qquad \tilde{P} = e^{i\omega\int_0^z \frac{dz}{v(z)}}\tilde{P}_0$$

In terms of the pressure field we have

$$P = \sqrt{\frac{r(z)}{r(0)}}\, e^{i\omega\int_0^z \frac{dz}{v(z)}} P_0$$

which is the WKBJ solution for P.

With the results obtained for the acoustic wave equation, it is easy to derive an analogous result for the scalar SH displacement equation

$$(\rho\omega^2+\nabla\cdot\mu\nabla)u=0 \tag{13}$$

Identifying in the above analysis ρ with $1/\mu$, and K with $1/\rho$ will produce the correct results. The state variable in this case is

$$\tilde{\mu} = (\rho\mu)^{1/4}\,u = \sqrt{r}\;u$$

The energy variable is power flux as before.