DISPERSION-RELATION DERIVATION OF WAVE EXTRAPOLATORS

[*condensed from SEP-16, p. 353-359*]

A wave-extrapolation equation is an expression for the derivative of a wave field (usually in the depth z-direction). With the wave field and its derivative known, extrapolation can proceed by various numerical representations of $P(z + \Delta z) = P(z) + \Delta z\, dP/dz$. Two methods for finding wave-extrapolation equations are the ***transformation*** method and the ***dispersion-relation*** method. In the transformation method a coordinate frame is found for the scalar wave equation in which the second depth derivative $\partial_{z'z'}$ may be neglected. Then the transformed equation is solved for the first-derivative term $\partial_{z'}$, giving the desired extrapolation form. In the dispersion-relation method one seeks various approximations to a square-root dispersion relation. Then the approximate dispersion relation is inverse transformed into a differential equation. Thanks largely to Francis Muir, the dispersion approach has evolved considerably since the writing of ***Fundamentals of Geophysical Data Processing***, and it is the subject of our present review.

Substitution of the plane wave $\exp(-i\omega t + ik_x x + ik_z z)$ into the two-dimensional scalar wave equation yields the dispersion relation

$$k_z^2 + k_x^2 = \frac{\omega^2}{v^2} \tag{1}$$

Solving for k_z we get a square root

$$k_z = \frac{\omega}{v}\left[1 - \left(\frac{vk_x}{\omega}\right)^2\right]^{\frac{1}{2}} \tag{2a}$$

To inverse transform the z-axis we only need to recognize that ik_z corresponds to ∂_z, which means that we have an expression for a wave-field extrapolator, namely

$$\frac{\partial P}{\partial z} = i \frac{\omega}{v}\left[1 - \left(\frac{v k_x}{\omega}\right)^2\right]^{\frac{1}{2}} P \tag{2b}$$

Muir Expansion

Regrettably, inverse transforming over x by $ik_x = \partial_x$ becomes practical only when the square root is regarded as some kind of truncated series expansion. It will be shown in a later chapter that the Taylor series is an unsatisfactory choice. Francis Muir showed that the original 15-degree and 45-degree methods were just truncations of a continued fraction expansion. To see this, let X and R be defined by writing (2a) as

$$k_z = \frac{\omega}{v}(1 - X^2)^{\frac{1}{2}} = \frac{\omega}{v} R \tag{3}$$

The desired polynomial ratio of order n will be denoted R_n, and it will be determined by the recurrence

$$R_{n+1} = 1 - \frac{X^2}{1 + R_n} \tag{4}$$

To see what this sequence converges to (if it converges) we set $n = \infty$ in (4) and solve

$$R_\infty = 1 - \frac{X^2}{1 + R_\infty}$$

$$R_\infty(1 + R_\infty) = 1 + R_\infty - X^2$$

$$R^2 = 1 - X^2 \tag{5}$$

The square root of (5) gives the required expression (3). Geometrically (5) says that the cosine squared of the incident angle equals one minus the sine squared. Truncating the expansion leads to angle errors.

Actually it is only the low-order terms in the expansion which are ever used. Beginning with $R_0 = 1$ we obtain

5^o	$R_0 = 1$
15^o	$R_1 = 1 - \frac{x^2}{2}$
45^o	$R_2 = 1 - \cfrac{x^2}{2 - \frac{x^2}{2}}$
65^o	$R_3 = 1 - \cfrac{x^2}{2 - \cfrac{x^2}{2 - \frac{x^2}{2}}}$

TABLE 1. First four truncations of Muir's continued fraction expansion.

For various historical reasons, the equations in table 1 are commonly referred to as the 5-degree, 15-degree, and 45-degree equations, respectively, the names giving a reasonable qualitative (but poor quantitative) guide to the range of angles that are adequately handled. A trade-off between complexity and accuracy frequently dictates choice of the 45-degree equation. It then turns out that a slightly wider range of angles can be accommodated if the recurrence is begun with something like $R_0 = \cos 45^o$. Accuracy enthusiasts might even have R_0 a function of velocity, space coordinates, or frequency.

Dispersion Relations

Performing the substitutions of Table 1 into equation (3) we get dispersion relationships for comparison to the exact expression (2a).

5^{o}	$k_z = \frac{\omega}{v}$
15^{o}	$k_z = \frac{\omega}{v} - \frac{v k_x^2}{2\omega}$
45^{o}	$k_z = \frac{\omega}{v} - \frac{k_x^2}{2\frac{\omega}{v} - \frac{v k_x^2}{2\omega}}$

TABLE 2. As displayed in figure 1 the dispersion relations of table 2 tend toward a semi-circle.

Depth-Variable Velocity

Identification of ik_z with ∂_z converts the dispersion relations of table 2 into the differential equations

5^{o}	$\frac{\partial P}{\partial z} = i\left(\frac{\omega}{v}\right) P$
15^{o}	$\frac{\partial P}{\partial z} = i\left(\frac{\omega}{v} - \frac{v k_x^2}{2\omega}\right) P$
45^{o}	$\frac{\partial P}{\partial z} = i\left(\frac{\omega}{v} - \frac{k_x^2}{2\frac{\omega}{v} - \frac{v k_x^2}{2\omega}}\right) P$

TABLE 3. Extrapolation equations when velocity depends only on depth.

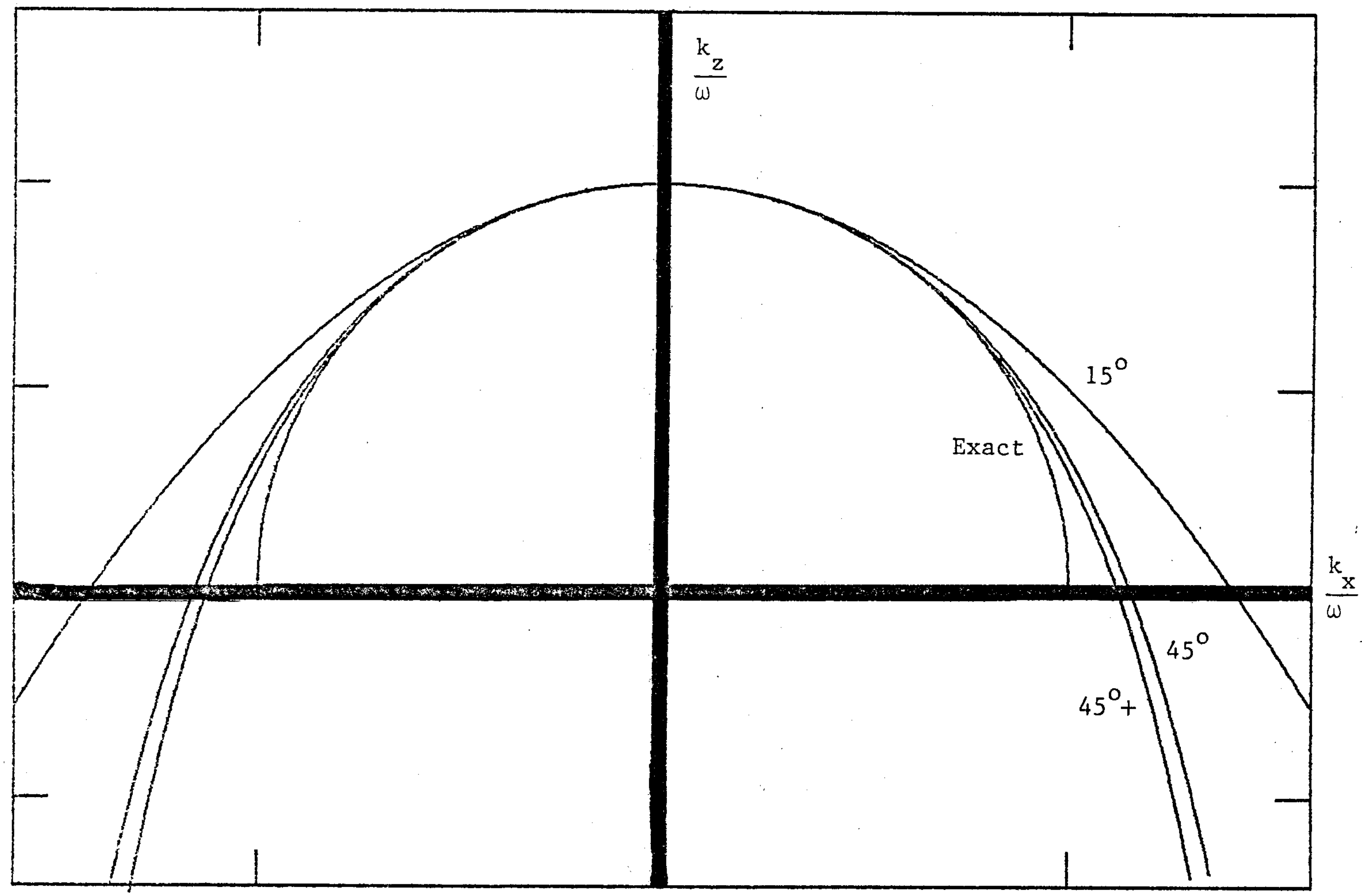

FIG. 1. Dispersion relation of equations (2a) and table 2. The curve labeled 45°+ was constructed with $R_0 = \cos 45^\circ$. It fits exactly at 0° and 45°.

The differential equations in table 3 were based on a dispersion relation that in turn was based on an assumption of constant velocity. So you might not anticipate that the equations have substantial validity and even great utility when the velocity is depth-variable, $v = v(z)$. The actual limitations are better characterized by their inability, by themselves, to describe reflection.

Migration methods based on equation (2b) or on table 3 are called ***phase-shift methods***.

Retardation (frequency domain)

Retardation is a reorganization of the wave equation so that a particular wave is handled theoretically, hence with no computational artifacts. Computational errors proportional to the grossness of the computational mesh increase with increasing departure from this particular wave, usually a normally incident plane wave in a medium of velocity $\bar{v}(z)$. A plane wave going straight down will be time-shifted from the surface by an amount t_0 determined from the velocity profile $\bar{v}(z)$ by

$$t_0 \quad = \quad \int_0^z \frac{dz}{\bar{v}(z)} \tag{6}$$

A time shift t_0 in the time domain corresponds to multiplication by $\exp(-i\omega t_0)$ in the ω-domain. Thus the actual wave field P is related to the time-shifted wave field Q by

$$P \quad = \quad Q(z) \; \exp\left[i\omega \int_0^z \frac{dz}{\bar{v}(z)}\right] \tag{7a}$$

Differentiating with respect to z we get

$$\frac{\partial P}{\partial z} \quad = \quad \frac{\partial Q}{\partial z} \exp\left[i\omega \int_0^z \frac{dz}{\bar{v}(z)}\right] + Q(z) \; \frac{i\omega}{\bar{v}(z)} \exp\left[i\omega \int_0^z \frac{dz}{\bar{v}(z)}\right]$$

or

$$P_z \quad = \quad \exp\left[i\omega \int_0^z \frac{dz}{\bar{v}(x)}\right] \; \left[\partial_z + \frac{i\omega}{\bar{v}}\right] Q \tag{7b}$$

Next we substitute (7) into table 3 to obtain the retarded equations:

5^o	$Q_z \;=\; \text{zero} \qquad + i\omega\left[\frac{1}{v} - \frac{1}{\bar{v}(z)}\right] Q$
15^o	$Q_z \;=\; -i\,\frac{v k_x^2}{2\omega}\,Q \qquad + i\omega\left[\frac{1}{v} - \frac{1}{\bar{v}(z)}\right] Q$
45^o	$Q_z \;=\; -i\,\frac{k_x^2}{2\frac{\omega}{v} - \frac{v k_x^2}{2\omega}}\,Q \qquad + i\omega\left[\frac{1}{v} - \frac{1}{\bar{v}(z)}\right] Q$
general	Q_z = diffraction + thin lens

TABLE 4. Retarded form of phase-shift equations.

Lateral Velocity Variation

Having approximated the square root by a polynomial ratio we can now inverse transform either table 3 or table 4 from the horizontal wavenumber domain k_x to the horizontal space domain x by substituting $(ik_x)^2 = \partial_{xx}$. As before, the result has a wide range of validity for $v = v(x,z)$ even though the derivation would not seem to permit it. Ordinarily $\bar{v}(z)$ will be chosen to be some kind of horizontal average of $v(x,z)$. Permitting $\bar{v}$ to become a function of x turns out to be hazardous and is rarely done.

Splitting

The customary numerical solution to the x-domain forms of the equations in Table 3 or 4 is done by splitting. That is, you march forward a small Δz-step alternately with the two extrapolators

$$\frac{\partial Q}{\partial z} \quad = \quad \text{lens term} \tag{8a}$$

$$\frac{\partial Q}{\partial z} \quad = \quad \text{diffraction term} \tag{8b}$$

Justification of the splitting process is found in a later chapter. The first equation, called the lens equation, is solved analytically, that is

$$Q(z + \Delta z) \quad = \quad Q(z) \exp\left\{ i\omega \left[\frac{1}{v(x,z)} - \frac{1}{\bar{v}(z)} \right] \right\} \tag{9}$$

Observe that the diffraction parts of tables 3 and 4 are the same. So we use them and equation (8b) to define a table of diffraction equations. Substitute ∂_x for ik_x and then clear ∂_x from the denominators to obtain

5^o	$\partial_z Q \quad = \quad \text{zero}$
15^o	$\partial_z Q \quad = \quad \frac{v(x,z)}{-2i\omega} \, \partial_{xx} Q$
45^o	$\left\{ 1 - \left[\frac{v(x,z)}{-2i\omega} \right]^2 \partial_{xx} \right\} \partial_z Q \quad = \quad \frac{v(x,z)}{-2i\omega} \, \partial_{xx} Q$

TABLE 5. Diffraction equations for laterally variable media.

Time Domain

To put the above equations in the time domain, it is necessary only to get ω into the numerator and then replace $-i\omega$ by ∂_t. For example, the 15-degree, retarded, $v = \bar{v}$ equation from table 5 becomes

$$\frac{\partial^2}{\partial z\ \partial t} Q = \frac{v}{2} \frac{\partial^2}{\partial x^2} Q \qquad (10)$$

Interpretation of time t for a retarded-time variable Q awaits further clarification in a later chapter.

Upcoming Waves

All the above equations are for ***downgoing*** waves. To get equations for ***upcoming*** waves you need only to change the sign of z and ∂_z. Letting D denote a ***downgoing*** wave field and U denote an ***upcoming*** wave field, equation (10), for example, takes the form

$D_{zt} = \frac{v}{2} D_{xx}$
$U_{zt} = -\frac{v}{2} U_{xx}$

TABLE 6. Time-domain equations for down- and upcoming wave diffraction with retardation and the 15-degree approximation.

It is the upcoming wave equation that always appears in migration problems. Migration is essentially the process of extrapolating waves backward along their actual path. Because of this and the sign difference in table 6, ***migration*** is said to be the inverse of ***diffraction***.