INTRODUCTION TO STABILITY

Experience shows that, as soon as you undertake an application which departs significantly from the textbook situations, stability becomes a more vital concern than accuracy. Stability, or its absence, determines whether the object is available at all, whereas accuracy merely determines the price. Here we will look at the stability of the heat-flow equation with real and with imaginary heat conductivity. The latter case corresponds to seismic migration so these two cases provide a useful background of experience in stability analysis.

The basic method of stability analysis is based on Fourier transformation. More simply, we just examine single sinusoidal or complex exponential trial solutions. If a method becomes unstable for any frequency, then it will be unstable for real solutions which are just conbinations of all frequencies. Begin with the sinusoidal function $P(x) = P_0 \exp(ikx)$. For its second derivative we have

$$\frac{\partial^2 P}{\partial x^2} = -k^2 P \tag{1}$$

We define $\hat{k}$ by an analogous expression with the second difference operator

$$\frac{\delta^2 P}{\delta x^2} = \frac{P(x + \Delta x) - 2P(x) + P(x - \Delta x)}{(\Delta x)^2} \tag{2a}$$

$$= -\hat{k}^2 P \tag{2b}$$

Ideally $\hat{k}$ should equal k. Inserting the complex exponential into (2a) we get an expression for $\hat{k}$:

$$-\hat{k}^2 P = \frac{P_0}{\Delta x^2}\left[e^{ik(x+\Delta x)} - 2e^{ikx} + e^{ik(x - \Delta x)}\right] \tag{3a}$$

$$(\hat{k}\Delta x)^2 = 2\left[1 - \cos(k\Delta x)\right] \tag{3b}$$

It is a straightforward matter to make plots of (3b) or its square root, which through the half-angle trig identity is

$$\hat{k}\Delta x \quad = \quad 2 \sin \frac{k\Delta x}{2} \tag{3c}$$

Series expansion shows that $\hat{k}$ matches k very well at low frequencies. At the Nyquist frequency, defined by $k\Delta x = \pi$, we have $\hat{k}\Delta x = 2$, a poor approximation to π. As with any Fourier transform on the discrete domain, $\hat{k}$ is a periodic function of k above the Nyquist frequency. Although k ranges from minus infinity to plus infinity, $\hat{k}^2$ is compressed into the range zero to four. The limits to the range are important since instability often starts at one end of the range.

Explicit Heat-Flow Equation

Begin with the heat-flow equation and Fourier transform over space. Thus $\partial^2/\partial x^2$ becomes simply $-k^2$. We have

$$\frac{\partial q}{\partial t} \quad = \quad -\frac{\sigma}{c}\, k^2 q \tag{4}$$

Finite differencing explicitly over time gives an equation which is identical in form to the inflation-of-money equation.

$$\frac{q_{t+1} - q_t}{\Delta t} \quad = \quad -\frac{\sigma}{c}\, k^2 q_t \tag{5a}$$

$$q_{t+1} \quad = \quad \left(1 - \frac{\sigma \Delta t}{c}\, k^2\right) q_t \tag{5b}$$

For stability the magnitude of q_{t+1} should be less than or equal to the magnitude of q_t. That requires the factor in parentheses to have a magnitude less than or equal to unity. So there is instability when $k^2 > 2c/(\sigma\Delta t)$. This means that the high frequencies are diverging with time. The explicit finite differencing on the time axis has caused disaster for short wavelengths on the space axis. Surprisingly, the disaster can be recouped if we finite difference the space axis coarsely enough! The second space derivative in the

Fourier-transform domain is $-k^2$. When the x-axis is discretized it becomes $-\hat{k}^2$. So to discretize (4) and (5) we just replace k by $\hat{k}$. From (1) we see that $\hat{k}^2$ has an upper limit of $\hat{k}^2 = 4/\Delta x^2$ at the Nyquist frequency $k\Delta x = \pi$. So the factor in (5b) will be less than unity and we will have stability provided that

$$\hat{k}^2 = \frac{4}{\Delta x^2} \le \frac{2c}{\sigma \Delta t} \tag{6}$$

Evidently instability can be averted by a sufficiently dense sampling of time compared to space. But such a solution becomes unbearably costly if the heat conductivity $\sigma(x)$ takes on a wide range of values $\sigma_{min} << \sigma(x) << \sigma_{max}$. Luckily, for problems in one space dimension we have an easy escape by turning to implicit methods. For problems in higher-dimensional spaces we will need to reconsider explicit methods.

Explicit 15-degree Migration Equation

It turns out that the retarded 15-degree wave-extrapolation equation is like the heat-flow equation except that the heat conductivity σ must be replaced by the purely imaginary number i. The amplification factor [the magnitude of the factor in parentheses in equation (5b)] is now the square root of the sum squared of real and imaginary parts. Since the real part is already one, the amplification factor exceeds unity for all non-zero values of k^2. The resulting instability is manifested by the growth of dipping plane waves. The more dip, the faster the growth. Further discretizing the x-axis does not solve the problem.

Implicit Equations

Recall that the inflation-of-money equation

$$q_{t+1} - q_t = r\, q_t \tag{7}$$

is a simple explicit finite differencing of the differential equation $dq/dt \simeq q$. And recall that a better approximation to the differential equation is given by the Crank-Nicolson form

$$q_{t+1} - q_t \;=\; r\,\frac{q_{t+1} + q_t}{2} \tag{8a}$$

which may be arranged to

$$\left(1 - \frac{r}{2}\right) q_{t+1} \;=\; \left(1 + \frac{r}{2}\right) q_t \tag{8b}$$

or

$$\frac{q_{t+1}}{q_t} \;=\; \frac{1 + r/2}{1 - r/2} \tag{8c}$$

The amplification factor (8c) has magnitude less than unity for all negative r values, even r equal to minus infinity. Recall that the heat-flow equation corresponds to

$$r \;=\; -\,\frac{\sigma \Delta t}{c}\,k^2 \tag{9}$$

where k is the spatial wavelength. Since (8c) is good for all negative r, the heat-flow equation, implicitly time-differenced, is good for all spatial frequencies k. It is stable regardless of whether the space axis is discretized as $k \rightarrow \hat{k}$ and regardless of the sizes of Δt and Δx. Furthermore, the 15-degree wave-extrapolation equation will also be unconditionally stable. This follows by letting r in (8c) be purely imaginary so the amplification factor (8c) takes the form of some complex number $1 + r/2$ divided by its complex conjugate. Expressing the complex number in polar form it is evident that such a number has a magnitude exactly equal to unity. Again we have unconditional stability.

At this point it is natural to add a historical footnote. When finite difference migration was first introduced many objections were raised on the basis of unfamiliar theoretical assumptions. Despite these it quickly became very popular. I think this was because, compared to other methods of the time, it was a gentle operation on the data. More specifically, since (8c) is of exactly unit magnitude the output has the same (ω,k)-spectrum as the input. There may be a wider lesson to be learned here. Any process acting on data should do as little to the data as possible.

Leapfrog Equations

The leapfrog method of finite differencing, it will be recalled, is to express the time derivative over two time steps. This keeps the centering of the differencing operators in the same place. For the heat-flow equation Fourier-transformed over space we have

$$\frac{q_{t+1} - q_{t-1}}{2\Delta t} \quad = \quad - \frac{\sigma}{c} k^2 \, q_t \tag{10}$$

It is a bit of a nuisance to analyze this equation because of the fact that it covers times t-1, t, and t+1 and requires slightly more difficult analytical techniques. Therefore, it seems worthwhile to state the results first, because we may lose some of our readers while covering the technique. The result for the heat-flow equation is that the solution always diverges. The result for the wave-extrapolation equation is much more useful: it is that there is stability provided certain mesh size restrictions are satisfied, namely, Δz must be less than some factor times Δx^2. This result is not awfully exciting in one space dimension where implicit methods seem ideal. But in higher-dimensional space, such as in the so-called 3-D prospecting surveys, we may be quite happy to have the leapfrog method.

The best way to analyze equations over three or more time levels like (10) is to use Z-transform filter analysis. Converted to a Z-transform filter problem the question about (10) becomes whether or not the filter has zeros inside (or outside) the unit circle. This kind of consideration is necessary for all possible numerical values of k^2. When it is done you find that you are always in trouble if k^2 ranges over 0 to infinity. But with the wave-extrapolation equation you find you can avoid instability with certain mesh size restrictions because $(\hat{k}\Delta x)^2$ lies between zero and four.