High Order Difference Approximations to the Differential Equation

$$P_{tz} = \frac{v(x,z)}{2} P_{xx} + f(x,z)$$

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1. Introduction

Consider the following partial differential equation, $t \geq 0$, $0 \leq x \leq 1$, $z \geq 0$,

$$P_{tz}(t,x,z) = \frac{v(x,z)}{2} P_{xx}(t,x,z) + f(x,z), \quad v(x,t) > 0 \tag{1.1}$$

with initial condition

$$P(t, x, 0) = P_0(t, x)$$

Different boundary conditions can be used like

$$\begin{aligned} P(0, x, z) &= 0 \\ P_x(t, 0, z) &= P_x(t, 1, z) = 0 \end{aligned} \tag{1.2}$$

or

$$\begin{aligned} P(0, x, z) &= 0 \\ P(t, 0, z) &= P(t, 1, z) = 0 \end{aligned} \tag{1.3}$$

The z-coordinate should be thought of as the evolution direction for the mixed initial boundary value problem.

The differential equation (1.1) arises when analyzing seismograms, see Claerbout and coworkers [1], [2] and [3]. It describes a restricted class of solutions to the scalar wave equation.

Let us present a simplified derivation to see which types of arguments are used. For more details, see e.g. [2].

The wave equation

$$-\frac{1}{v^2} P_{tt} + P_{xx} + P_{zz} = 0$$

can be transformed using

$$t' = t \ , \ x' = x \ , \ z' = z + v\,t$$

to

$$-\frac{1}{v^2} P'_{t't'} - \frac{2}{v} P'_{t'z'} + P'_{x'x'} = 0$$

Dropping the $P'_{t't'}$ term has little effect on waves traveling close to the -z-direction, but eliminates all waves traveling in the positive z-direction. Hence (1.1) is well suited to describe upgoing waves whose fronts are close to perpendicular to the z-axis.

The problem (1.1) has been approximated by difference schemes which use second order approximation in t, z, and x, see [2].

With methods which are higher order in the nonevolution direction it is possible to get the same accuracy on a coarser grid. That reduces the memory requirement and sparser sampled data can be used. When high accuracy is needed it also gives much faster programs.

The purpose of this paper is to present and analyze schemes which are fourth order in t, fourth order (or higher) in x and second order in z.

The stability conditions are studied with the use of normal mode analysis. Energy conserving properties and dissipative terms are discussed. The accuracy is checked through the local truncation error,

with numerical experiments and also with the Kreiss and Oliger [4] type of analysis. The latter consists of determining the number of mesh points per wavelength which gives a certain error.

The approximations which are second order in t can be solved by marching in t for each fixed z-level. There is no need for extra numerical boundary conditions at $t = t_{max}$ ($0 \le t \le t_{max}$) . This makes it easy to get explicit schemes or schemes which are implicit only in the x-direction.

A boundary condition for $t = t_{max}$ is needed for schemes which are higher order in t and still have a simple structure. This is shown by using the analysis developed by Dahlquist [5] regarding A - stable ordinary difference approximations.

In the schemes we will use, this boundary condition does not enter in the latest z-level. The difference equations can still be explicit or implicit only in the x-direction.

2. The Difference Schemes

Let us introduce a uniform rectangular mesh

$$\{t_j, x_k, z^n\} \; ; \; j = 0, \ldots, J, \quad k = 0, \ldots, K,$$

$$n = 0, 1, \ldots \; ; \quad t_j = j\,\Delta t, \quad x_k = k\,\Delta x, \quad z^n = n\,\Delta z$$

and a mesh function $P^n_{j,k}$ approximating $P(t_j, x_k, z^n)$.

Denote with a the ratio $\dfrac{v\,\Delta t\,\Delta z}{8\,\Delta x^2}$.

We will also define some difference operators. Indices are omitted when they do not change in formula.

$$D_+^x P_k = \frac{1}{\Delta x} (P_{k+1} - P_k) \qquad (\sim \frac{\partial}{\partial x} P)$$

$$D_-^x P_k = \frac{1}{\Delta x} (P_k - P_{k-1}) \qquad (\sim \frac{\partial}{\partial x} P)$$

D_+^t, D_-^t, D_+^z and D_-^z are defined analogously.

The following average operators will be used

$$Q_1 P_j^n = \frac{1}{2} (P_j^{n+1} + P_{j+1}^n)$$

$$Q_2 P_j^n = \frac{1}{4} (P_{j+1}^{n+1} + P_j^{n+1} + P_{j+1}^n + P_j^n)$$

In this notation we can now write the second order approximations to (1.1) which have been used for some time.

$$(D_+^t D_+^z - \frac{v (x_k, z^{n+1/2})}{2} Q_1 D_+^x D_-^x) P_{j,k}^n = f (x_k, z^{n+1/2}) \quad (2.1)$$

$$(D_+^t D_+^z - \frac{v (x_k, z^{n+1/2})}{2} Q_2 D_+^x D_-^x) P_{j,k}^n = f (x_k, z^{n+1/2}) \quad (2.2)$$

$$j = 0,\ldots,J-1 \; ; \; k = 1,\ldots,K-1 \; ; \; n = 0, 1, \ldots$$

The initial and boundary conditions corresponding to (1.2) are

$$P_{j,k}^0 = P_0 (t_j, x_k)$$

$$P_{0,k}^n = 0 \qquad (2.3)$$

$$P_{j,0}^n = P_{j,1}^n, \; P_{j,K}^n = P_{j,K-1}^n$$

Formula (2.1) is explicit since $P_{j+1,k}^{n+1}$ can be directly computed. The equation (2.2) is implicit in the x-direction, but $P_{j+1,k}^{n+1}$ can be evaluated by inverting a tri-diagonal matrix for each fixed n and $j = 0, 1, 2, \ldots$. For (2.1) to be stable the condition is $a \leq \frac{1}{8}$.

The local truncation error is defined by the error in the difference approximation when it is applied to the solution of the corresponding differential equation, see [6]. It is here of the order $O(\Delta t^2 + \Delta x^2 + \Delta z^2)$, when the x-boundaries are $\Delta x/2$ and $1 - \Delta x/2$.

We want to improve the accuracy of (2.1) and (2.2) but still have stable formulas which for each n can be solved marching in j (the t-direction).

We will not use a higher order approximation of P_{tz} but instead compensate for the errors in the approximation of $\frac{v(x,z)}{2} P_{xx}$.

Let us introduce two more difference operators.

$$Q_3 P_j^n = \frac{1}{2\Delta t^2}(P_{j-1}^{n+1} - P_j^{n+1} - P_{j+1}^n + P_{j+2}^n) \qquad (\sim \frac{\partial^2}{\partial t^2} P)$$

$$Q_4 P_j^n = \frac{1}{2\Delta t^2}(P_{j-1}^{n+1} - P_{j+1}^{n+1} - P_j^n + P_{j+2}^n) \qquad (\sim \frac{\partial^2}{\partial t^2} P)$$

The higher order explicit approximation has the form

$$(D_+^t D_+^z - \frac{v(x_k, z^{n+1/2})}{2}(Q_1 - \frac{\Delta t^2}{12} Q_3)(1 - \tag{2.4}$$

$$- \frac{\Delta x^2}{12} D_+^x D_-^x + \frac{5\Delta z \Delta t}{48} D_+^x D_-^x v(x_k, z^{n+1/2})) D_+^x D_-^x) P_{j,k}^n =$$

$$= (1 + \frac{5 v(x_k, z^{n+1/2})}{48} D_+^x D_-^x) f(x_k, z^{n+1/2})$$

and the implicit approximation

$$(D_+^t D_+^z - \frac{v(x_k, z^{n+1/2})}{2}(Q_2 - \frac{\Delta t^2}{12} Q_4) D_+^x D_-^x (1 - \tag{2.5}$$

$$- (\frac{\Delta x^2}{12} + \frac{\Delta z\ \Delta t\ v(x_k, z^{n+1/2})}{24}) D_+^x D_-^x)) P_{j,k}^n =$$

$$= (1 - \frac{v(x_k, z^{n+1/2})}{24} D_+^x D_-^x) f(x_k, z^{n+1/2})$$

$$j = 1, \ldots, J-2 \; ; \; k = 2, \ldots, K-2 \; ; \; n = 0, 1, \ldots$$

As initial conditions we use

$$P^0_{j,k} = P_0(t_j, x_k) \tag{2.6}$$

The boundary conditions are not so important in practice and we can think of the following low order approximations.

$$P^n_{0,k} = P^n_{1,k} = 0 \; , \; P_{J,k} = 0$$

Periodic in x (Period = 1) or

$$P^n_{j,0} = P^n_{j,3} \; , \; P^n_{j,1} = P^n_{j,2} \; , \; P^n_{j,K} = P^n_{j,K-3} \; , \; P^n_{j,K-1} = P^n_{j,K-2}$$

The latter corresponds to the condition (1.2) with x-boundaries $1.5\,\Delta x$ and $1 - 1.5\,\Delta x$. Other boundary conditions are commented on in the next section.

For both difference approximations the local truncation error is $0(\Delta t^4 + \Delta x^4 + \Delta z^2)$.

The stability condition for (2.4) is $a < 0.40$ and (2.5) is stable for all values of a.

The schemes can be modified in many ways to adapt to special situations and they can also be simplified, e.g., when v is constant. Let us note some variants.

1. If a is small the $\Delta z\,\Delta t$ terms can be dropped in the formulas. The stability limit for (2.4) will then be $a < 0.10$ and the error $0(\Delta t^4 + \Delta x^4 + \Delta z \Delta t + \Delta z^2)$.
2. If also the Δx^2 terms are small and dropped in (2.4) (or (2.5)), the stability limit is $a < 0.12$ and the error $0(\Delta t^4 + \Delta x^2 + \Delta z \Delta t + \Delta z^2)$.

3. $D_+^x D_-^x a(x,z)$ and $D_+^x D_-^x f(x,z)$ can be changed to $a_{xx}(x,z)$ and $f_{xx}(x,z)$ if possible.

4. The compensation for the P_{xxxx} error terms in (2.4) can be concentrated to P_j^{n+1} and P_{j+1}^n. When v is constant the formula will be

$$\ldots - \frac{v}{2}\left(Q_1 - \frac{\Delta t^2}{12} Q_3 - \left(\frac{\Delta x^2}{12} - \frac{5\Delta t \Delta z\, v}{48}\right) Q_1 D_+^x D_-^x \right) D_+ D_- \Big) P_{j,k}^n \ldots$$

5. Higher order difference formulas in the x-direction can easily be used, i.e., for small a we can modify (2.4) in the following way

$$\ldots - \frac{v}{2}\left(Q_1 - \frac{\Delta t^2}{12} Q_3 - \left(\frac{\Delta x^2}{12} Q_1 D_+^x D_-^x - \frac{\Delta x^4}{90} Q_1 \left(D_+^x D_-^x \right)^2 \right) D_+^x D_-^x \right) P_{j,k}^n \ldots$$

The local truncation error is then $O\left(\Delta t^4 + \Delta x^6 + \Delta t \Delta z + \Delta z^2\right)$.

6. Even higher order approximations can be reached with the quasi-spectral method [7]. Then the FFT (Fast Fourier Transform) is used to determine the derivatives in the x-direction. The equations could also be Fourier transformed in the t-direction.

7. If a is small, the implicit scheme can be simplified to

$$\left(D_+^t D_-^z - \frac{v(x_k, z^{n+1/2})}{2} \left(Q_1 + Q_5 \right) D_+^x D_-^x \left(1 + \frac{\Delta x^2}{12} D_+^x D_-^x \right)^{-1} \right) P_{j,k}^n =$$

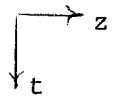
$$= f\left(x_k, z^{n+1/2} \right)$$

where $Q_5 P_j^n = \frac{1}{2\Delta t^2}\left(P_{j-1}^{n+1} - 2P_j^{n+1} + P_{j+1}^{n+1} + P_j^n - 2P_{j+1}^n + P_{j+2}^n \right)$

It is no longer unconditionally stable but has local truncation error $O\left(\Delta t^4 + \Delta x^4 + \Delta z^2\right)$. The stability condition is $a < 0.10$.

8. Dissipative terms can be entered, see Section 5.

Let us write the left hand side of (2.4) and (2.5) in the form of difference molecules for constant v. The axes are



$$\begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array} - \frac{a}{6} \begin{array}{|c|c|}\hline & -1 \\ \hline & 13 \\ \hline 13 & \\ \hline -1 & \\ \hline \end{array} \quad \left(\Delta x^2 \, D_+^x \, D_-^x - \frac{\Delta x^4}{12} \left(1 - 10\,a \right) \left(D_+^x \, D_-^x \right)^2 \right) \qquad (2.4')$$

$$\begin{array}{|c|c|}\hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array} - \frac{a}{6} \begin{array}{|c|c|}\hline & -1 \\ \hline 7 & 6 \\ \hline 6 & 7 \\ \hline -1 & \\ \hline \end{array} \quad \left(\Delta x^2 \, D_+^x \, D_-^x \left(1 + \left(\frac{1}{12} + \frac{a}{3} \right) \Delta x^2 \, D_+^x \, D_-^x \right) \right) \qquad (2.5')$$

$$\left(\Delta x^2 \, D_+^x \, D_-^x \, P_k \;=\; P_{k-1} - 2\,P_k + P_{k+1} \right)$$

$$\left(\Delta x^4 (D_+^x D_-^x)^2 P_k \;=\; P_{k-2} - 4\,P_{k-1} + 6\,P_k - 4\,P_{k+1} + P_{k+2} \right)$$

Note that the fourth difference in (2.4) vanishes if $a = 0.1$.

We have and we also will consider the explicit form more intensively than the implicit one. The value of a is already often chosen small from an accuracy point of view.

The difference formula (2.1) includes 8 points of the mesh and formula (2.4) contains 22. The corresponding values for the implicit schemes are 12 and 30. By considering point 4 on page 139 the number 22 is reduced to 18. If the explicit formulas are used in a straightforward way, these values are quite proportional to the number of multiplications in the inner loop.

The method (2.4) becomes much faster if the value of

$$\left(\frac{12}{(1-10a)} \Delta x^2 D_+^x D_-^x - \Delta x^4 (D_+^x D_-^x)^2 \right) P_{j,k}^n \qquad (2.6)$$

is stored for one j-loop (or one j-step). The expression (2.6) corresponds to the case when v is constant. The number of multiplications in the inner loop is reduced to 7 (or 10).

When using (2.4) or (2.5), only one n-level needs to be stored. We are, however, forced to keep some values like the vectors P_{j-1}^n and P_j^n in the memory when computing P_{j+1}^{n+1}.

3. Stability

In this section we will consider constant v. This means the schemes (2.4) and (2.5) with constant coefficients. In general, it is a good approach, even for variable coefficients, to analyze the stability for the corresponding constant coefficient problems, see [6].

We will first study the pure initial value problem or a problem with periodic boundary conditions. We look for an estimate of the following type

$$\| P^{n+1} \| < K_0 \| P^0 \| \quad , \quad \text{for} \quad f = 0, \; z^n \leq z_{max} \tag{3.1}$$

The array P^n is the solution to the pure initial value problem for (2.4) or (2.5) and the norm is defined by

$$\| P^n \|^2 = \Delta t \Delta x \sum_j \sum_k (P^n_{j,k})^2 \quad \text{(sum over all indices } j \text{ and } k \text{)}$$

It is easy to extend the estimate (3.1) to include inhomogeneous terms ($f \neq 0$), see [6].

Theorem 3.1: The estimate (3.1) with $K_0=1$ is valid for the solution to the initial value problems (2.4) and (2.5) when $f=0$.

Proof: We multiply (2.4) and (2.5) with $\Delta t \Delta z$ and Fourier transform in t and x . Let us denote the dual variables corresponding to t and x respectively ω and k_x . We will also use the following notation:

$$s(\omega) = \sin (\pi \omega \Delta t) , \; c(\omega) = \cos (\pi \omega \Delta t)$$

$$s(k_x) = \sin (2\pi k_x \Delta x), c(k_x) = \cos (2\pi k_x \Delta x)$$

Formula (2.4) is transformed to

$$2\,i\,s(\omega)(\hat{P}^{n+1} - \hat{P}^{n}) - \frac{a}{12}((13(c(\omega) - i\,s(\omega)) - (c(3\omega) - i\,s(3\omega)))\hat{P}^{n+1} \tag{3.2}$$

$$+ (13(c(\omega) + i\,s(\omega)) - (c(3\omega) + i\,s(3\omega)))\hat{P}^{n})(2(c(k_x) - 1) - \frac{1}{3}(1 - 10a)$$

$$(1 - c(k_x))^2) = 0$$

and formula (2.5) becomes

$$2\,i\,s(\omega)(\hat{P}^{n+1} - \hat{P}^{n}) - \frac{a}{6}((6c(\omega) + i\,s(\omega)) - (c(3\omega) - i\,s(3\omega)))\hat{P}^{n+1} +$$

$$+ ((6c(\omega) - i\,s(\omega)) - (c(3\omega) + i\,s(3\omega)))\hat{P}^{n})(2(c(k_x) - 1) - \tag{3.3}$$

$$- \frac{1}{3}(1+4a)(c(k_x) - 1)^2) = 0$$

A few comments on the derivation: The Fourier transform of a translation is given by

$$\hat{P}_{j+1} = e^{2\pi i\omega\Delta t}\,\hat{P}_j$$

$$\hat{P}_{k+1} = e^{2\pi i\,k_x\,\Delta x}\,\hat{P}_k$$

After these transformations trigonometric formulas were used to simplify the expressions.

Both formulas (3.2) and (3.3) can be reduced to the form

$$\hat{P}^{n+1} = \frac{A + i\,B}{A - i\,B}\,\hat{P}^{n} = \alpha(\omega, k_x)\,\hat{P}^{n} \tag{3.4}$$

where A and B are real. This gives $|\alpha| = 1$, so that $|\hat{P}^{n+1}| = |\hat{P}^{n}|$ and hence no Fourier mode increases. Parseval's relation guarantees (3.1) with $K_0 = 1$.

The same analysis applies to the periodic boundary problem, and it also has this energy preserving property that the L_2-norm is constant. The amplification factor $\alpha(\omega,k_x)$ is also studied in section 5.

Theorem 3.1 gives only necessary conditions for the stability of the full problem containing boundary conditions. The effect of the boundary conditions and the way the scheme is solved may still cause exponential growth of the error. This will for example occur for the scheme (2.4) if a is too large.

In the following analysis we concentrate our interest on the fact that we are solving the problem on a finite range in t. We assume the boundary conditions in x to be periodic.

Consider the difference equations (2.4) and (2.5) with constant v and $f=0$.

The initial and boundary conditions are

$$\begin{aligned} P^0_{j,k} &= 0 \\ P^n_{1,k} &= g^n_{1,k}\,,\quad P^n_{2,k} = g^n_{2,k}\,,\quad P^n_{J,k} = g^n_{J,k} \end{aligned} \tag{3.5}$$

Periodic boundary in k.

The estimate we are interested in has the form

$$\sum_{n=0}^{N} \| \hat{P}_j^{\,n} \|_x^2 \le K \sum_{n=0}^{N} \left(\| g_1^{\,n} \|_x^2 + \| g_2^{\,n} \|_x^2 + \| g_J^{\,n} \|_x^2 \right), \qquad j = 0, 1, \ldots, J \tag{3.6}$$

where $\| P \|_x^2 = \Delta x \sum_k |P_k|^2$, and K is a constant.

We will use the Kreiss type of normal mode analysis, see e.g. [8]. It can also be applied to inhomogeneous difference equations. Taking $f=0$ does not change the result but simplifies the argument.

Our problem differs from the hyperbolic problems treated in [8] in the way that the signal speed for (1.1) is infinite and hence the two boundaries must be treated simultaneously.

Theorem: There is an estimate of type (3.6) for the equations (2.4) and (2.5) with the initial and boundary values (3.5) if $a \leq 2/5$ in (2.4)

Proof: We Fourier transform (2.4) in j and Laplace transform in n.

$$(\lambda-1)(\hat{P}_{j+1} - \hat{P}_j) + C(k_x)(\lambda(-\hat{P}_{j-1} + 13\hat{P}_j) + 13\hat{P}_{j+1} - \hat{P}_{j+2}) = 0, \quad |\lambda| \geq 1 \tag{3.7}$$

$$\hat{P}_1 = \hat{g}_1, \quad \hat{P}_2 = \hat{g}_2, \quad \hat{P}_J = \hat{g}_J$$

$$C(k_x) = -\frac{a}{6}(2(c(k_x) - 1) - \frac{1}{3}(1-10a)(c(k_x) - 1)^2)$$

We have $0 \leq C(k_x) \leq 4/45$ if $a \leq 2/5$.

With the use of Parseval's relation it is sufficient to show (3.8) to get (3.6).

$$|\hat{P}_j|^2 \leq K(|\hat{g}_1|^2 + |\hat{g}_2|^2 + |\hat{g}_3|^2) \tag{3.8}$$

The characteristic equation corresponding to (3.7) is the following

$$(\lambda-1)(\kappa^2-\kappa) + C(k_x)(\lambda(-1 + 13\kappa) + 13\kappa^2 - \kappa^3) = 0 \tag{3.9}$$

If (3.9) has the roots κ_1, κ_2, and κ_3 with $|\kappa_1|<1-\varepsilon$, $|\kappa_2|\leq 1$ and $|\kappa_3|>1+\varepsilon$ $(\varepsilon>0)$, then (3.8) is valid. Let us first consider separated roots. The general solution of (3.7) is then

$$\hat{P}_j = A_1 \kappa_1^j + A_2 \kappa_2^j + A_3 \kappa_3^{j-J}$$

If this enters in the boundary we get the linear system

$$\begin{aligned} A_1 + A_2 + A_3 \kappa_3^{-J} &= \hat{g}_1 \\ A_1\kappa_1 + A_2\kappa_2 + A_3\kappa_3^{1-J} &= \hat{g}_2 \\ A_1\kappa_1^J + A_2\kappa_2^J + A_3 &= \hat{g}_3 \end{aligned}$$

The determinant of the system is bounded away from zero and (3.8) is valid. If two roots coincide the use of $\hat{P}_j = (A_1 + j A_2)\kappa_1^j + + A_3\kappa_3^{j-J}$ gives the same result. ($|\kappa_1| \leq 1 - \varepsilon$)

Let us first check the roots for $C(k_x)$ bounded away from zero and for large λ . One root will be large and the others tend to the solution of (3.10) when $\lambda \to \infty$.

$$\kappa^2 - \kappa + C(k_x)(-1 + 13\ \kappa) = 0 \qquad (3.10)$$

For $0 < C \leq \frac{4}{45}$ we have $|\kappa_1| \leq 1-\varepsilon$, $|\kappa_2| < 1$.

There cannot be any roots with $|\kappa| = 1$ for $|\lambda| > 1$ since that would violate the von Newman condition which was checked in theorem 3.1. In that case λ corresponds to the amplification factor. We need only to check for $|\lambda| = 1$.

Let us rewrite (3.9)

$$\kappa^3 - \kappa^2 \left(13 + \frac{\lambda-1}{C(k_x)} \right) + \kappa\left(-13\lambda + \frac{\lambda-1}{C(k_x)} \right) - \lambda = 0 \qquad (3.11)$$

Since $1 = |\lambda| = |\kappa_1\ \kappa_2\ \kappa_3|$ either all roots have absolute value 1 or else we have the situation we want. It is straightforward to check that $|\kappa_1| = |\kappa_2| = |\kappa_3| = 1$ cannot be valid when

$$\kappa_1 + \kappa_2 + \kappa_3 = 13 + \frac{\kappa-1}{C(k_x)}$$

$$\kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_1\kappa_3 = -13\lambda + \frac{\lambda-1}{C(k_x)}$$

$$0 < C(k_x) \leq 0.1 \ , \ |\lambda| = 1 \ .$$

Finally, there are the two cases $C(k_x) \to 0$ ($|\lambda| > 1$) where all roots separate nicely and $C(k_x) \to 0$, $|\lambda| \to 1$ where the arguments following (3.11) can be used.

The proof of stability for (2.5) is analogous and is omitted.

This analysis is essentially a check that there is no growing solution $P^n_{j,k} = \lambda^n e^{2\pi i k_x \Delta x\, k} \phi_j$ with $|\lambda| > 1$ for any mesh function ϕ_j .

The case $\lambda \to \infty$ gives the bound for one z-step when marching in t . The stability limits for the other schemes in section 2 are only checked for $\lambda \to \infty$.

4. Accuracy

In this section we will study how many mesh points per wavelength are needed to get a certain prescribed accuracy in some simple cases. However, first we consider the order of the local truncation errors of the difference approximations. This may not describe the errors for large step sizes (Δt, Δx and Δz) but gives the rate of decay of the error when the step sizes decrease.

We do not analyze all variants of schemes, but we carry out the proof in some detail so that it can be used for related difference approximations.

Theorem 4.1: The local truncation errors of (2.4) and (2.5) are are of the order $O(\Delta t^4 + \Delta x^4 + \Delta z^2)$.

Proof: Straightforward use of Taylor expansions gives the following results when the difference operators are applied to a smooth P. (The arguments are (t_j, x_k, z^n) to the left and $(t_{j+1/2}, x_k, z^{n+1/2})$ to the right of the equal sign if no arguments are written out.)

$$D_+^t D_+^z P = P_{tz} + \frac{\Delta t^2}{24} P_{tttz} + O(\Delta t^4 + \Delta z^2)$$

$$Q_1 P = P + \frac{\Delta t^2}{8} P_{tt} - \frac{\Delta t \Delta z}{4} P_{tz} + O(\Delta t^4 + \Delta z^2)$$

$$Q_2 P = P + \frac{\Delta t^2}{8} P_{tt} + O(\Delta t^4 + \Delta z^2)$$

$$Q_3 P = P_{tt} - \frac{\Delta z}{2\,\Delta t} P_{tz} + O\left(\frac{\Delta z^2}{\Delta t} + \Delta t^2\right)$$

$$Q_4 P = P_{tt} - \frac{\Delta z}{\Delta t} P_{tz} + O\left(\frac{\Delta z^2}{\Delta t} + \Delta t^2\right)$$

$$D_+^x D_-^x P(t_{j+1/2}, x_k, z^{n+1/2}) = P_{xx} + \frac{\Delta x^2}{12} P_{xxxx} + O(\Delta x^4)$$

If these formulas are entered into the difference equation (2.4) and (2.5) we get respectively (the arguments are $(t_{j+1/2}, x_k, z^{n+1/2})$):

$$P_{tz} + \frac{\Delta t^2}{24} P_{tttz} - \frac{v}{2} (P_{xx} + \frac{\Delta t^2}{24} P_{ttxx} - \frac{5\,\Delta t\,\Delta x}{24} P_{tzxx}$$

$$+ (\frac{5\,\Delta t \Delta z\, v}{48} P_{xxxx} + \frac{5\,\Delta t \Delta z\, v_{xx}}{48} P_{xx})) =$$

$$= f + \frac{5\,\Delta t\,\Delta z\, v}{48} f$$

$$P_{tz} + \frac{\Delta t^2}{24} P_{tttz} - \frac{v}{2} (P_{xx} + \frac{\Delta t^2}{24} P_{ttxx} + \frac{\Delta t \Delta z}{12} P_{tzxx} -$$

$$- (\frac{\Delta t \Delta z\, v}{24} P_{xxxx} + \frac{\Delta t \Delta z\, v_{xx}}{24} P_{xx})) =$$

$$= f - \frac{\Delta t\,\Delta z\, v}{24} f$$

If we use the differential equation (1.1) all terms but $O (\Delta t^4 + \Delta x^4 + \Delta z^2)$ cancel, which proves the theorem.

Let us briefly comment on point 8 on page 139. The operators Q_2 and Q_5 do not produce any P_{tz} term. That is why the formula is so simple.

As was mentioned in the beginning of this section, another way to analyze the error is to see which step sizes are necessary to follow a plane wave with a certain accuracy. This has been carried out for difference approximations to first order hyperbolic problems [4] and we follow that approach.

Let us concentrate on the error from the discretization in the t-direction, and hence, let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial x}$ stay as derivatives when approximating the following problem.

$$P_{tz} = P_{xx}$$

$$P(t, x, 0) = e^{2\pi i\omega(\alpha x + t)} \quad , \quad \alpha > 0 \tag{4.1}$$

The second order and fourth order approximations will then become

$$D_+^t P_z^j = \frac{1}{2}\left(P_{xx}^j + P_{xx}^{j+1}\right) \tag{4.2}$$

$$D_+^t P_z^j = \frac{1}{24}\left(-P_{xx}^{j-1} + 13\,P_{xx}^j + 13\,P_{xx}^{j+1} - P_{xx}^{j+2}\right) \tag{4.3}$$

(The time index is here written as a superscript.)

The differential equation has the solution

$$P(t, x, z) = e^{2\pi i\omega(\alpha x + t + \alpha^2 z)}$$

Both schemes preserve the amplitude but they will give phase errors.

Solution to (4.2): $e^{2\pi i\omega(\alpha x + t + \alpha_2(\omega)z)}$

Solution to (4.3): $e^{2\pi i\omega(\alpha x + t + \alpha_4(\omega)z)}$

By substituting these trial solutions into (4.2) and (4.3) we get

$$\alpha_2(\omega) = \alpha^2 \frac{\pi\omega\Delta t \cos(\pi\omega\Delta t)}{\sin(\pi\omega\Delta t)}$$

$$\alpha_4(\omega) = \alpha^2 \frac{\pi\omega\Delta t\,(13\cos(\pi\omega\Delta t) - \cos(3\pi\omega\Delta t))}{12\sin(\pi\omega\Delta t)}$$

The phase errors for the two methods are

$$e_2 = 2\pi\omega z\,|\alpha^2 - \alpha_2(\omega)|$$

$$e_4 = 2\pi\omega z\,|\alpha^2 - \alpha_4(\omega)|$$

Let us with m denote the number of periods in z ($0 \le z \le m / (\omega\alpha^2)$), and let N be the number of points per wave, $N = (\omega \Delta t)^{-1}$. We consider phase error less than π so the maximum error is reached for $z = \frac{m}{\omega \alpha^2}$.

This gives us

$$e_2(N) = 2\pi m \left| 1 - \frac{(\frac{\pi}{N}) \cos (\frac{\pi}{N})}{\sin (\frac{\pi}{N})} \right| \tag{4.4}$$

$$e_4(N) = 2\pi m \left| 1 - \frac{(\frac{\pi}{N})(13 \cos (\frac{\pi}{N}) - \cos(\frac{3\pi}{N}))}{12 \sin (\frac{\pi}{N})} \right| \tag{4.5}$$

If the errors are developed in powers of N^{-1} and higher order terms neglected we have

$$e_2(N) \sim \frac{2\pi^3}{3} N^{-2} m$$

$$e_4(N) \sim \frac{22\pi^5}{45} N^{-4} m$$

Denote with $N_2(e)$ and $N_4(e)$ the N values for the second and fourth order methods respectively corresponding to the error e .

$$N_2(e) = (\frac{2\pi^3 m}{3 e})^{1/2}$$

$$N_4(e) = (\frac{22\pi^5 m}{45 e})^{1/4}$$

For 10% error (e=0.1) and 1% error (e=0.01) we get the following numerical values:

$$N_2(0.1) = 14.4\ m^{1/2}$$

$$N_2(0.01) = 45.5\ m^{1/2}$$

$$N_4(0.1) = 6.2\ m^{1/4}$$

$$N_4(0.01) = 11.1\ m^{1/4}$$

Using (4.4) and (4.5) instead of the truncated series expansion does not change the result. The difference in N is less than 0.05 (for $m=1$).

Another way to check the error is of course numerical experiments. There are some presented in the last section.

5. Dissipation

Let us consider the initial value problem for the differential equation (1.1) with constant v and $f=0$. Each frequency (ω,k_x) in the solution preserves its amplitude for all z. In the proof of theorem 3.1 we saw that the same is valid for the discrete approximations. The amplification factor had absolute value one for all frequencies.

In some cases it is desirable to damp the high frequencies which are not accurately treated by the difference scheme. This can be done by changing the method such that the amplification factor is strictly less than one for those frequencies. The scheme is then called dissipative or said to contain artificial or numerical viscosity, see [6]. Entering dissipative terms should, if possible, be done in such a way that the order of approximation does not decrease.

We consider the following terms which can be added to the left hand side of formula (2.4). ($\varepsilon > 0$)

$$\varepsilon \frac{\Delta x^4 \Delta t^2}{\Delta z} D_+^t D_-^t D_+^t \left(D_+^x D_-^x \right)^2 P_{j,k}^n \tag{5.1}$$

$$\varepsilon \frac{\Delta x^6}{\Delta t\, \Delta z} \left(D_+^x D_-^x \right)^3 \left(P_{j,k}^n + P_{j+1,k}^n \right) \tag{5.2}$$

Both terms can be added at the same time. The local truncation error of (2.4) will be amplified with $0\left(\frac{\varepsilon \Delta x^4 \Delta t^2}{\Delta z}\right)$ or $0\left(\frac{\varepsilon\, \Delta x^6}{\Delta t\, \Delta z}\right)$ respectively.

Stronger dissipative terms with larger truncation error like

$$-\varepsilon \frac{\Delta x^2 \Delta t^2}{\Delta z} D_+^t D_-^t D_+^t \left(D_+^x D_-^x \right) P_{j,k}^n$$

or nonlinear terms like

$$\varepsilon \frac{\Delta t^4}{\Delta z} \left| D_+ D_- P_{j,k}^n \right| D_+^t D_-^t D_+^t$$

can also be tried.

The best choice must be determined from practical tests.

Since the denominator in the amplification factor $\alpha(\omega,k_x)$ vanishes for certain frequencies, the coice of dissipative terms is limited in order not to violate the vonNeuman condition. Let us check for (5.1) and (5.2) and denote the corresponding amplification factors β and γ.

Theorem 5.1: $|\beta| \le 1$, $|\gamma| \le 1$ for all ω, k_x. $|\beta| < 1$ for $\sin(\pi\omega)\sin(2\pi k_x) \neq 0$, $|\gamma| < 1$ for $\cos(\pi\omega)\sin(2\pi k_x) \neq 0$.

Proof: Let us first label the different parts of α (the amplification factor of (2.4)):

$$\alpha = \frac{i\,\alpha_1(\omega) - \alpha_2(\omega)\,\alpha_3(k_x)}{i\,\alpha_1(\omega) + \alpha_2(\omega)\,\alpha_3(k_x)} = \frac{i\,A - B}{i\,A + B}$$

where $\alpha_1(\omega) = 2\,s(\omega) + \frac{a}{12}\,(13\,s(\omega) - s(3\omega))$

$$\alpha_2(\omega) = \frac{a}{12}\,(13\,c(\omega) - c(3\omega))$$

$$\alpha_3(\omega) = 2\,(c(k_x) - 1) - \frac{1}{3}\,(1 - 10a)\,(1 - c(k_x))^2$$

$$\beta = \frac{i\,A' - B}{i\,A + B}, \quad \gamma = \frac{i\,A - B'}{i\,A + B} \quad \text{where}$$

$$A' = \alpha_1(\omega) - 16\,\varepsilon\,s(\omega)\,(c(2\omega) - 1)\,(1 - c(k_x))^2$$

$$B' = \alpha_2(\omega)\,\alpha_3(k_x) - 16\,\varepsilon\,c(\omega)\,(c(k_x) - 1)^3$$

The stability part of the theorem is true if $|A'| \le |A|$ and $|B'| \le |B|$. This can be shown from the following relations:

$$\alpha_1(\omega) / s(\omega) > \delta$$

$$\alpha_2(\omega) / c(\omega) > \delta$$

$$\alpha_3(k_x) / (1 - c(k_x)) > \delta$$

for $a < 0.4$ and some positive number δ . Similarly $|A'| < |A|$ if $s(\omega)s(k_x) \neq 0$ and $|B'| < |B|$ if $c(\omega)s(k_x) \neq 0$, which proves the theorem.

For small $\Delta t\,\omega$ and $\Delta x\,k_x$ we have

$$|\beta|^2 \sim 1 - \varepsilon_1 \frac{\Delta t^4 \Delta x^4 \omega^4 k_x^4}{\Delta t^2 \omega^2 + c_1 \Delta x^4 k_x^4} \quad , \quad \varepsilon_1 , c_1 > 0$$

$$|\gamma|^2 \sim 1 - \varepsilon_2 \frac{\Delta x^8 k_x^8}{\Delta t^2 \omega^2 + c_1 \Delta x^4 k_x^4} \quad , \quad \varepsilon_2 > 0$$

6. Limitations on the choice of schemes

To increase the understanding of approximating (1.1) we will determine some limitations on the possible choice of schemes due to the requirements of accuracy and stability.

We consider linear schemes of the following form.

$$\sum_{m=0}^{M} \sum_{r=0}^{R} \sum_{s=0}^{S} \left(\alpha_{rs}^{m} - \frac{v(x,z)}{2} \beta_{rs}^{m} \right) P_{j+r,k+s}^{n+m} = 0 \qquad (6.1)$$

where (x,z) is some point in the mesh. (Assume $f=0$ in equation (1.1).) The formula (6.1) is normalized, i.e., when applied to a smooth P the sum containing only α factors gives $P_{tz} + O(\Delta t + \Delta x + \Delta z)$ and the sum with β factors $P_{xx} + O(\Delta t + \Delta x + \Delta z)$. The factors α and β are real.

One of the leading terms in the truncation error in (2.1) and (2.2) was proportional to P_{xxtt}. To approximate that term only three points in the t-direction are needed. In scheme (6.2), written in schematic form, we have made that compensation.

$$\frac{1}{\Delta t \Delta z} \left(\begin{array}{|c|c|} \hline 1 & -1 \\ \hline & \\ \hline -1 & 1 \\ \hline \end{array} - \frac{4a}{3} \begin{array}{|c|c|} \hline & 1 \\ \hline 2 & 2 \\ \hline 1 & \\ \hline \end{array} D_{+}^{x} D_{-}^{x} \right) P = 0 \qquad (6.2)$$

The approximation (6.2) is explicit and has the local truncation error $O(\Delta t^4 + \Delta x^2 + \Delta t \Delta z + \Delta z^2)$. It also fulfills the von Newman condition. It is, however, not stable as a mixed problem for any value of a, with boundary conditions at $P_{0,k}^{n}$ and $P_{1,k}^{n}$.

In fact, no $O(\Delta t^4)$ scheme can be stable with boundary conditions only at $t=0$ ($j = 0, 1, \ldots, R-1$). That follows from the following theorem.

Theorem 6.1: Difference approximations of the form (6.1) which are stable in the sense of (3.1) and (3.6) and have all boundary conditions in the t-direction given at $t=0$,

$$p^n_{j,k} = g^n_{j,k} \qquad j = 0, 1, \ldots, R-1 \tag{6.3}$$

have at most order of approximation equals 2 in the t-direction.
We postpone proof to the end of the section.

To get higher order of approximation extra conditions at $t = t_{max}$ must be added. That will in general give rise to large implicit systems which must be solved for each n-step. The trick we used was to enter this extra boundary condition on the n-1 level at $t = t_{max}$. For this to work, at least four points are needed in the t-direction.

Theorem 6.2: Difference approximations of the form (6.1) which are stable in the sense of (3.1) and (3.6) with $R = 2$ and $M = 1$ cannot have order of approximation equals 4 in the t-direction and still be explicitly solvable marching in t.

There is also the possibility of using other than linear schemes in the t-direction, like implicit Runge-Kutta methods. Probably higher order methods of this type with boundary conditions only at $t = 0$ can be derived. They will be quite slow.

We have so far consentrated on errors due to the discretization in t. Let us comment on the Δx and Δz errors.

As was mentioned in point 5 on page 139 , it is easy to increase the order of approximation in x by using wide symmetric formulas. The number of mesh points in the formula will however also increase. To get $O(\Delta x^{2p})$ we need $2p+1$ terms in the x-direction.

It is much harder to improve the accuracy in the z-direction. Here we have the same type of limit as in theorem 1, and we cannot in practice impose conditions at $z = z_{max}$ like we could at $t = t_{max}$ if we want z as evolution direction.

Finally, a short comment on hyperbolic problems: The argument in theorem 1 applies to linear and symmetric difference approximations like Crank-Nicolson or Leap-Frog. The result is that it is not possible to get more than second order accuracy in time for unconditionally stable methods.

Let us define a linear multistep scheme (6.5) approximating the ordinary differential equation (6.4). See [9] for more general cases.

$$\begin{aligned} x(t) &= q\, x(t) \quad , \quad t \geq 0 \\ x(0) &= x_0 \end{aligned} \tag{6.4}$$

$$\begin{aligned} &\sum_{r=0}^{R} \alpha_r x_{r+j} = \Delta t\, q \sum_{r=0}^{R} \beta_r x_{r+j} \quad , \quad j = 0, 1, \ldots \\ &x_0 , x_1 , \ldots , x_{R-1} \quad \text{are given} \end{aligned} \tag{6.5}$$

The scheme is said to be A-stable if $x_j \to 0$ as $j \to \infty$ for fixed Δt and $\mathrm{Re}(q) < 0$, see [5] (Re denotes "the real part of"). We also introduce the polynomials

$$\rho(\zeta) = \sum_{r=0}^{R} \alpha_r \zeta^r \quad \text{and} \quad \sigma(\zeta) = \sum_{r=0}^{R} \beta_r \zeta^r$$

and assume that they have no common factor, and that α_r and β_r are real, see [9]. Consistency between (6.4) and (6.5) implies

$$\rho(1) = 0 \quad , \quad \rho'(1) = \sigma(1)$$

Lemma: If a solution x_v to a consistent linear multistep scheme is uniformly bounded for all v with $Re(q) = 0$ then the scheme is A-stable. Proof: We want to show that $\rho(\zeta) / \sigma(\zeta)$ is regular and has nonnegative real part for $|\zeta|>1$. Then lemma 2.1 in [5] guarantees A-stability.

The characteristic equation corresponding to (6.5) is

$$P(\zeta) = q\ \Delta t\ \sigma(\zeta)$$

Stability for all $Re\,(q\ \Delta t) = 0$ gives the relation

$$Re\,(q\ \Delta t) = 0 \Rightarrow |\zeta| \leq 1$$

or

$$|\zeta| > 1 \Rightarrow \rho \quad Re\,(q\ \Delta t) \neq 0 \tag{6.6}$$

Following the arguments in [5] we assume that $\rho(\zeta) / \sigma(\zeta)$ is not regular at ζ_0 , $|\zeta_0| > 1$. That is $\sigma(\zeta_0) = 0$ and $\rho(\zeta) \neq 0$ since ρ and σ have no common factors.

$$\rho(\zeta) / \sigma(\zeta) = a\,(\zeta - \zeta_0)^{-p} + O\,(|\zeta-\zeta_0|^{-p+1})$$

Here $a\neq 0$, and p is a positive integer. This gives that $\rho(\zeta) / \sigma(\zeta)$ takes on every argument close to ζ . In other words, there exists ζ_1 , $|\zeta_1| > 1$ such that $Re\,(\rho(\zeta_1) / \sigma(\zeta_1) = R\,(q\ \Delta t) = 0$ which contradicts (6.6). Hence, $\rho(\zeta) / \sigma(\zeta)$ is regular for $|\zeta| > 1$ and its real part is different from zero. That the real part is positive

follows from consistency:

$$\frac{\rho(1+\varepsilon)}{\sigma(1+\varepsilon)} = \frac{\rho(1) + \varepsilon\,\rho'(1)}{\sigma(1)} + O(\varepsilon) = 1 + \varepsilon + O(\varepsilon)\ ,\ \varepsilon > 0$$

Proof of theorem 6.1: When all boundary conditions in the t-direction are given at $t=0$, the difference equation (6.1) must be stable as an initial value problem in t. After Fourier transforming in x and z, (6.1) becomes

$$\sum_{r=0}^{R} \hat{\alpha}_r(k_z,k_x)\,\hat{P}_{r+j} = \frac{v}{2} \sum_{r=0}^{R} \hat{\beta}_r(k_z,k_x)\,\hat{P}_{r+j} \tag{6.7}$$

The consistency requirement implies

$$i\,k_z \sum_{r=0}^{R} (\hat{\alpha}_r + O(k_z+k_x))\hat{P}_{r+j} = -\frac{v\Delta x^2 k_x^2}{2\Delta t\Delta z} \sum_{r=0}^{R} (\hat{\beta} + O(k_z+k_x))\,\hat{P}_{r+j}$$

where the constant coefficients α_j, β_j are real. For the original difference approximation (6.1) to be stable, (6.7) must be stable for all k_x, k_z. For example, it must be stable when k_x, $k_z \to 0$, $\frac{\Delta x^2 k_x^2}{\Delta t\Delta z} \to C$ and hence (6.8) must be stable for all $\mathrm{Re}(C) = 0$.

$$\sum_{r=0}^{R} \hat{\alpha}_r \hat{P}_{r+j} = \frac{i\,\Delta t\, v\, C}{2} \sum_{r=0}^{R} \hat{\beta}_r \hat{P}_{r+j}$$

$$\hat{P}_j = \hat{g}_j \quad , \quad j = 0, 1, \ldots, R-1 \tag{6.8}$$

The formula (6.8) is a linear multistep method approximating $P_t = \frac{iv\,C}{2} P$. The lemma implies that (6.8) must be A-stable, and theorem 2.2 [5] then gives that the maximal order of approximation is two.

Proof of theorem 6.2: After Fourier transforming in x and z we get a scheme of the form (6.7). The only possible fourth order scheme will have the following coefficients, modulo higher order terms in k_x and k_z.

$$\hat{\alpha}_0 = -\frac{1}{2}\,,\ \hat{\alpha}_1 = 0\,,\ \hat{\alpha}_2 = \frac{1}{2}$$

$$\hat{\beta}_0 = \frac{1}{6}\,,\ \hat{\beta}_1 = \frac{4}{6}\,,\ \hat{\beta}_2 = \frac{1}{6}$$

Compare with [9]. The coefficients $\hat{\alpha}_0 = -1$ and $\hat{\alpha}_2 = 1$ implies that before the transformation in the z-direction $\hat{\alpha}_0' \neq 0$ and $\hat{\alpha}_2' \neq 0$. Theorem 6.1 gives that conditions must be posed at both boundaries. In order not to link the boundaries together, and hence get an implicit scheme, either $\hat{\alpha}_0'$ or $\hat{\alpha}_2'$ must be zero.

7. Numerical experiments

1. The initial values and boundary conditions for the analytic solution $P = \sin(2\pi\omega x) \sin 2\pi\omega(t+z)$, of $P_{tz} = P_{xx}$, were given. The mesh contained 10 interior positions in x and t, and 20 z-steps were calculated. We give the discrete L_2-norm of the error in % for $\Delta x = \Delta t = 1$, $\Delta z = 0.15$.

$\frac{1}{2\pi\omega}$	scheme (2.4)	scheme (2.1)
20	0.012	0.44
12	0.14	2.3
6	3.5	18
4	24	62

Note that the results correspond to different z-levels for different ω. The ratio between the errors are quite close to the one given by the local truncation error, even for a few points per wave.
Eg. for $\frac{1}{2\pi\omega} = 6$, and $\frac{1}{2\pi\omega} = 12$ we have:

$$\frac{3.5}{0.14} \sim \frac{1}{2^5}, \quad \frac{18}{23} \sim \frac{1}{2^3}$$

2. We have also solved $P_{tz} = P_{xx}$ with δ-function like intital values:

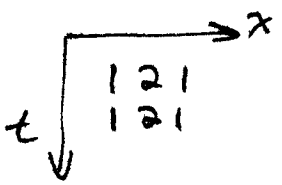
P^0_{jk} is given by $\begin{matrix} \rightarrow x \\ \downarrow t \end{matrix}$ $\begin{matrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{matrix}$

and is zero elsewhere (zero boundary values).

We used $\Delta t = \Delta x = 1$, $\Delta z = 0.15$ with $(J,K) = (80, 121)$.

```
      DIMENSION P(100,121),Q(100,121)
      COMMON NT,NX,DT,DX,DZ
      LOGICAL IA,IB,IC
      NT=100
      NX=121
      NZ=50
      DT=1
      DX=1
      DZ=.15
      AA=1
      NT1=NT-1
      NT2=NT-2
      NX1=NX-1
      NX2=NX-2
      K=1
      IZ=20
      A=DT*DZ*AA/(DX*DX)
      A2=A/2
      A24=A/24
      AC=A24*(AA*DT*DZ*2.5/(DX*DX)-1.)
C**       SET INITIAL VALUES
      DO 10 N=1,NT
      DO 10 J=1,NX
      P(N,J)=0.
   10 CONTINUE
      P(9,59)=5000
      P(9,60)=10000
      P(9,61)=5000
      P(10,59)=5000
      P(10,60)=10000
      P(10,61)=5000
C**
      CALL PLMAX(P,NT,NX)
C**       START Z-LOOP
   90 K=K+1
      DO 20 J=1,NX
      Q(1,J)=PP(1,J,K)
      Q(2,J)=PP(2,J,K)
      Q(NT,J)=PP(NT,J,K)
   20 CONTINUE
      DO 30 N=1,NT
      Q(N,1)=PP(N,1,K)
      Q(N,2)=PP(N,2,K)
      Q(N,NX1)=PP(N,NX1,K)
      Q(N,NX)=PP(N,NX,K)
   30 CONTINUE
      DO 40 N=2,NT2
      DO 40 J=3,NX2
C**
      S=Q(N,J)+P(N+1,J)-P(N,J)+
     1   A2*(P(N+1,J+1)-2*P(N+1,J)+P(N+1,J-1))
```

initialize



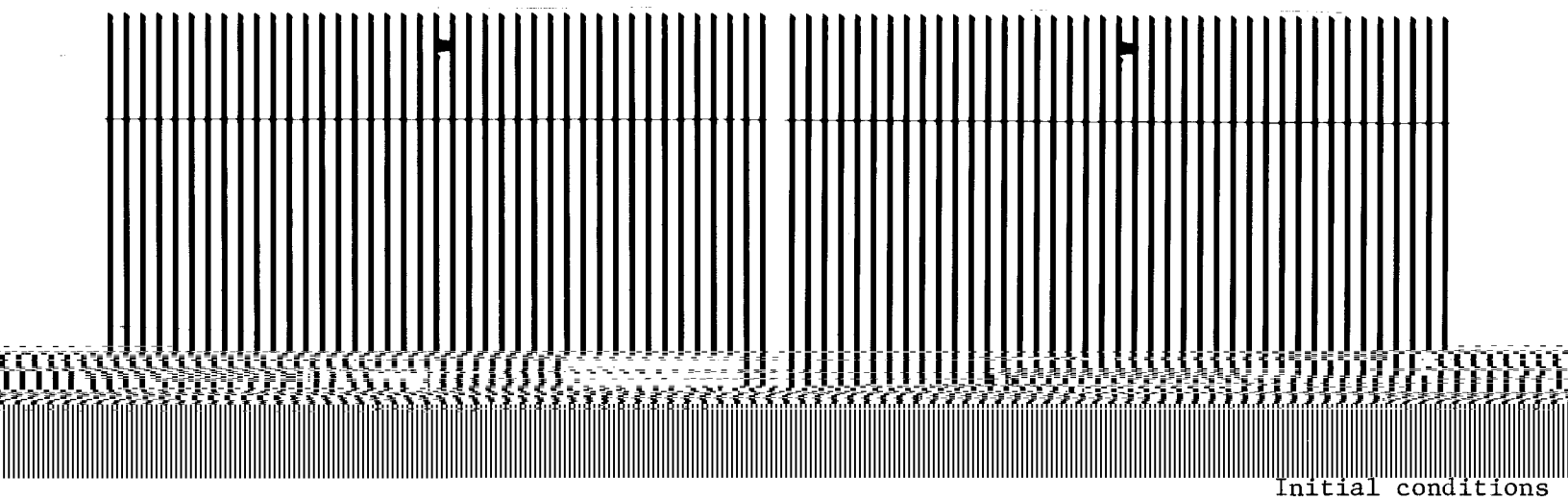
PLOT

4th order terms

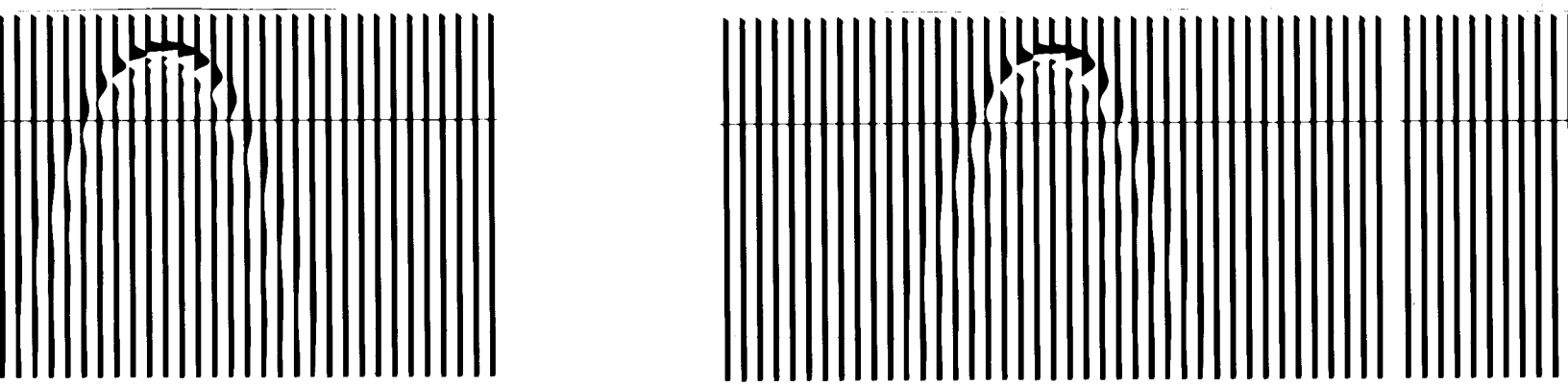
```
     2          +Q(N,J+1)-2*Q(N,J)+Q(N,J-1))
C**
      S=S+A24*(-(Q(N-1,J-1)-2*Q(N-1,J)+Q(N-1,J+1))+
     1           Q(N,J-1)-2*Q(N,J)+Q(N,J+1)+
     2           P(N+1,J-1)-2*P(N+1,J)+P(N+1,J+1)
     3         -(P(N+2,J-1)-2*P(N+2,J)+P(N+2,J+1)))
C**
      S=S+A60*(Q(N,J-2)-4*Q(N,J-1)+6*Q(N,J)-4*Q(N,J+1)+Q(N,J+2)+
     1         P(N+1,J-2)-4*P(N+1,J-1)+6*P(N+1,J)-4*P(N+1,J+1)+P(N+1,J+2)
     2                                                          )
C**
      P(N+1,J)=S
   40 CONTINUE
      DO 50 N=1,NT
      DO 50 J=1,NX
      P(N,J)=Q(N,J)
   50 CONTINUE
      IF(K/IZ*IZ.EQ.K) CALL PL4X(P,NT,NX)
      IF(K.LT.NZ) GOTO 99
C**       END OF Z-LOOP
      STOP
      END
```

PLOT EVERY 20th Z STEP

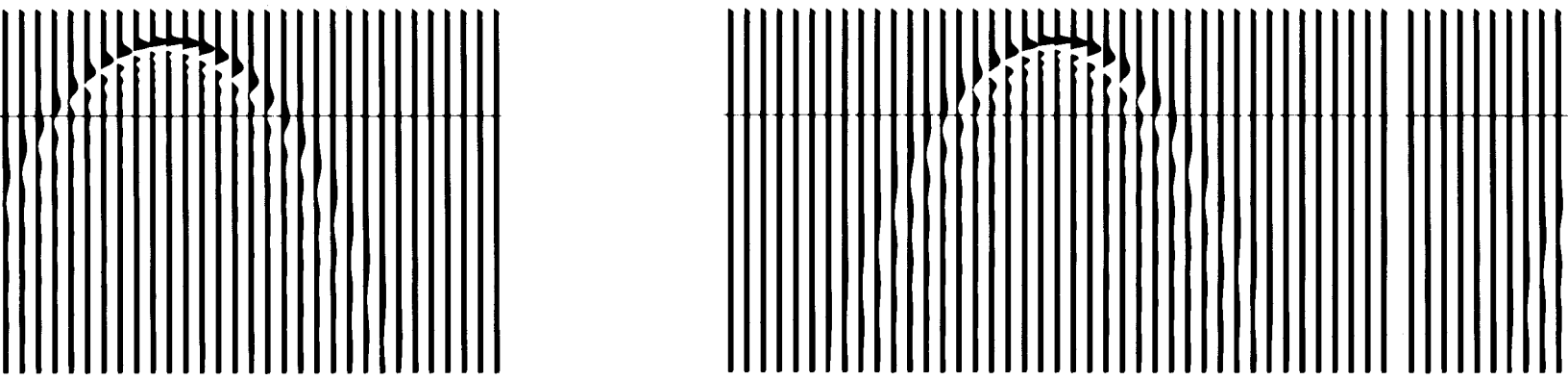
Fig. 1. Program for fourth order accuracy in x and t . It has not been "cleaned up" for presentation or efficiency. This program produced the results in Figure 2. The lines marked "fourth order terms" were removed to get the second order accuracy results.



Initial conditions



20 z steps



40 z steps

Fourth order

Second order

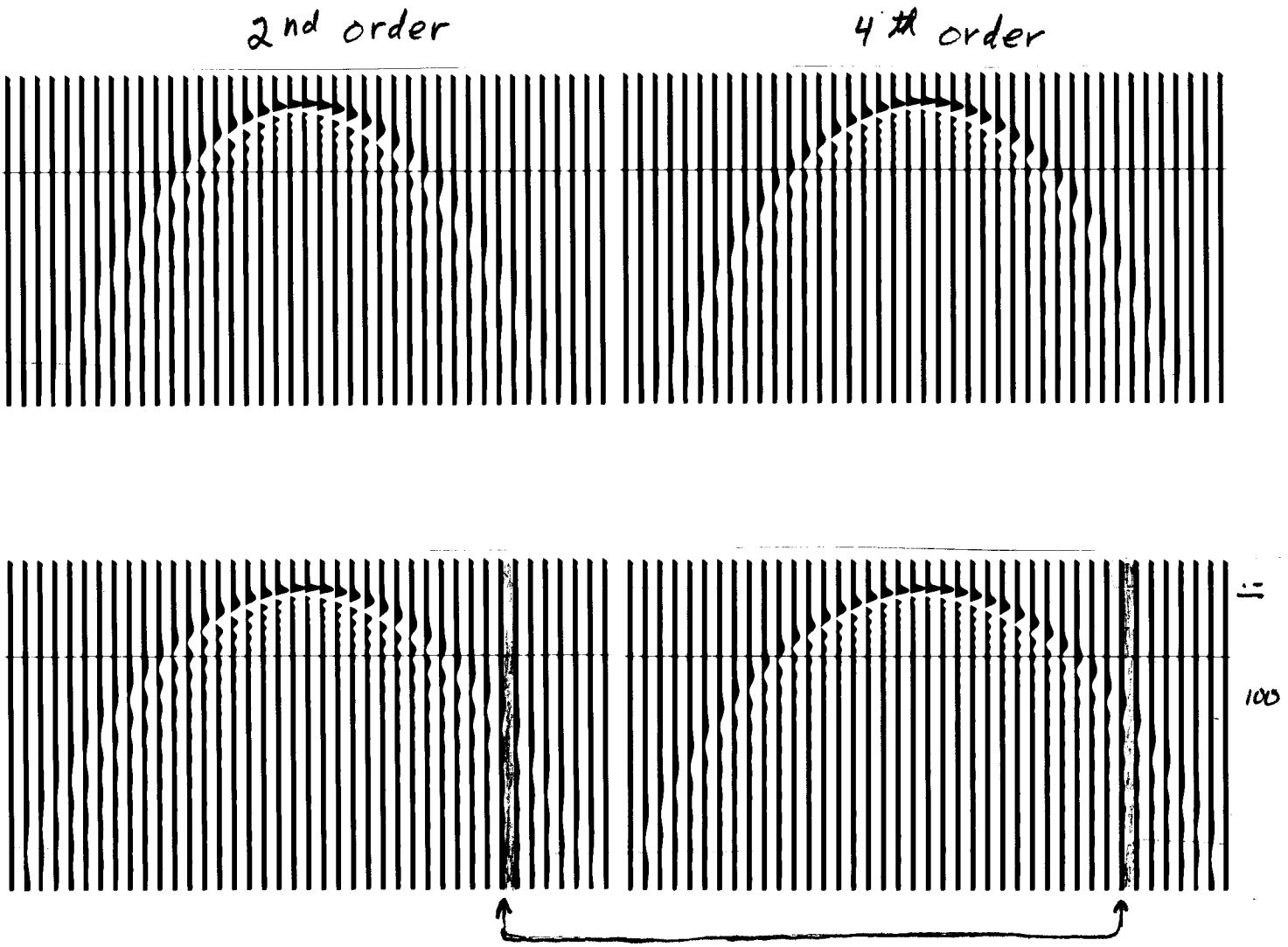


Fig. 2. Results of program of Figure 1 for second order accuracy (left) and fourth order (right). Note from the program that the x-axis was divided into 120 points but only every third one was plotted. The time axis has 100 points. The initial disturbance starts at the tenth time point. This initial disturbance was chosen to have a lot of high frequency energy which would not be easily modeled by difference equations. This short wavelength energy disperses much less for the fourth order. See for example the eighth channel from the right.

References:

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