Recommendations Concerning Maximum Entropy Spectral Estimation

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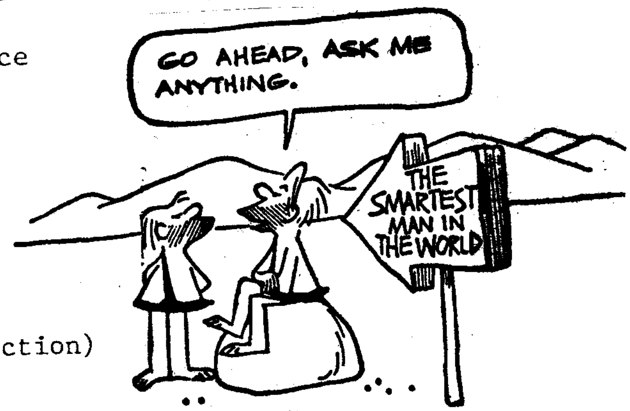
These notes are written in terms of a complex stationary series. Since most people are concerned with real stationary series, such as time series, some conversion is necessary. The notation used here arises from wavenumber spectral estimation in which a propagating plane wave noise field as seen by a linear array of sensors is analyzed at a selected frequency. The analysis gives the distribution of power in the stationary noise field as a function of apparent wavenumber in the direction of the linear array.

To change to time series analysis, replace



k (wavenumber)	by	f (frequency)
Δx	by	Δt (time sampling distance)
$k = 1/(2\Delta x)$	by	$w = 1/2\Delta t$ (Nyquist foldover)
$\Phi(0)$	by	$\psi(0)$ (real autocorrelation function)

The real time series spectra are symmetric about $f = 0$, so they can be multiplied by two and only the part from $f = 0$ to $f = w$ plotted. The reflection coefficients, C_n, will be real for real time series.

With regard to using maximum entropy spectral estimates, it should be emphasized that these are spectral density estimates and that the amount of power in a small bandwidth is not the peak value of the estimate but is the integral of the spectral density function over the bandwidth of interest. Following these thoughts, it should be stated that the peak value and bandwidth of a spectral line have considerable variance in the maximum entropy estimates from real data, but that their product, which is proportional to the total power, is estimated quite accurately. Partly because of this, I recommend that one also plot the integrated power spectra, i.e.

$$\int_{-K}^{k} P(k)\ dk\ .$$

This function, which should be plotted on a linear vertical scale (not in db), goes from zero to $\Phi(0)$. Of course, if one has not sampled the spectrum densely enough, the numerical integration may not be very close to $\Phi(0)$. This is a powerful clue that the spectrum as plotted is not a good representation of the true estimated spectrum and that a denser set of points is needed, especially at the peaks in the spectrum.

How densely should the spectral estimate be sampled? We can estimate this from (5) in the section on upper and lower bounds of the spectrum if we assume the following.

1) Let the phases of the C_n be such that the upper bound is achieved. I.e. if all C_n are real and negative, the upper bound is achieved at $k = 0$.

2) Almost all the power of the spectrum is in this peak. Then, the bandwidth of the peak is about

$$\text{B.W.} \cong \Phi(0)\ /\ \text{peak value}$$

or from (5) and (6)

$$\frac{\text{B.W.}}{2K} \cong \prod_{n=1}^{N} \left(\frac{1 - |C_n|}{1 + |C_n|} \right) \equiv Q$$

If we want to have at least two points per this bandwidth, then we need at least 2Q points from $-K$ to $+K$.

If one has a sharp peak in the spectrum and the sampling in wavenumber was not dense enough to accurately plot the peak, then a curve fitting technique can be applied to the peak. If there are three points on the spectral peak, which are well above the background level, then the curve of the functional form

$$\frac{A}{1 + \frac{(k-k_0)^2}{(B.W/2)^2}}$$

can be fitted to the three points.

A	=	peak value
k_0	=	center wavenumber
B.W.	=	bandwidth (± 3 db down points)

An estimate of the total power in the peak is reached by noting that

$$\int_{-\infty}^{+\infty} \frac{A}{1 + \frac{(k-k_0)^2}{(B.W./2)^2}} \, dk = \int_{-\infty}^{+\infty} \frac{A \, (B.W./2)^2}{(B.W./2)^2 + k^2} \, dk$$

$$= \frac{\pi}{2} \, A \cdot (B.W.) \, .$$

There are some observations to be made concerning numerical accuracy in calculating maximum entropy spectra. It should be noticed that when there is a sharp, high peak in the spectrum, this means that the fourier transform of the prediction error filter is very small in magnitude at the peak frequency. Thus in the fourier transform calculation, we are summing together a set of N numbers which almost cancel themselves out. This is of course bad from a numerical accuracy point of view. One way of deciding if the number of decimal places in the arithmetic is sufficient is to note that the spectrum can have a dynamic range of

$$\prod_{n=1}^{N} \left(\frac{1 + |C_n|}{1 - |C_n|} \right)^2 ,$$

which is the ratio of the upper bound to the lower bound of the spectrum.

Obtaining the Maximum Entropy Spectrum From Measurements of the Cross-Powers Between Sensors in an Equally Spaced Linear Array

We wish to maximize the integral of the logarithm of the wave number power density spectrum, i.e.,

$$\int_{-\frac{1}{2\Delta x}}^{\frac{1}{2\Delta x}} \log P\,(k)\;\; dk \;=\; \int_{-K}^{+K} \log P\,(k)\;\; dk,$$

under the constraints that

$$\int_{-K}^{+K} P\,(k)\; z^n \; dk \;=\; \Phi\,(n) \quad , \quad -N \leq n \leq N \tag{1}$$

where $z = e^{i2\pi k\Delta x}$, Δx is the spacing between sensors in an equally spaced linear array and $\Phi(n)$ is the cross-power between sensors $n\Delta x$ apart.

Using Lagrange multipliers, λ_n, we want the variation to be zero, i.e.,

$$\delta \int_{-K}^{+K} \left[\log P(k) - \sum_{n=-N}^{+N} \lambda_n (P(k)\;\; z^n - \Phi(n)/2K)\right] dk$$

$$= \int_{-K}^{+K} \left[\frac{1}{P(k)} - \sum_{n=-N}^{+N} \lambda_n \; z^n\right] \delta P(k)\; dk = 0$$

This gives

$$P\,(k) \;=\; \frac{1}{\sum_{n=-N}^{+N} \lambda_n \; z^n} \tag{2}$$

where the Lagrange multipliers need to be determined by satisfying the constraint equations (1). These constraint equations are equivalent to saying that P(k) has the z transform form of

$$P(k) = \left[\ldots + \Phi(N)\, z^{-N} + \ldots + \Phi(o) + \ldots + \Phi(-N)\, z^{N} + \ldots \right] / 2K \qquad (3)$$

where the coefficients of z^{-N} through z^{N} are the known cross-power values.

Looking at (2), since P(k) must be real and non-negative, it must be possible to rewrite (2) as

$$P(k) = \frac{P_N/2K}{(1+a_1 z+a_2 z^2+\ldots+a_N z^N)\,(1+a_1^* z^{-1}+a_2^* z^{-2}+\ldots+a_N^* z^{-N})}\ , \qquad (4)$$

where $1+a_1 z+a_2 z^2+\ldots+a_N z^N$ is minimum phase. Setting (3) equal to (4) and multiplying through by the minimum phase factor, we get

$$(\ldots+\Phi(N)\, z^{-N}+\ldots+\Phi(0)+\ldots\ \Phi(-N) z^{N}+\ldots)\ (1+a_1 z+a_2 z^2+\ldots+a_N z^N)$$

$$= \frac{P_N}{1+a_1^* z^{-1}+a_2^* z^{-2}+\ldots+a_N^* z^{-N}} = \ldots +b_{-2} z^{-2} + b_{-1} z^{-1} + P_N . \qquad (5)$$

The last equality shows that all the coefficients of positive powers of z are zero and that the coefficient of z^0 is P_N. Performing the convolution indicated by the first expression in (5) and equating the coefficients of z^0 through z^N, we have

$$\begin{bmatrix} \Phi(0) & \Phi(1) & \ldots & \Phi(N) \\ \Phi(-1) & \Phi(0) & \ldots & \Phi(N-1) \\ & & & \\ \Phi(-N) & \Phi(-N+1) & \ldots & \Phi(0) \end{bmatrix} \begin{Bmatrix} 1 \\ a_1 \\ \vdots \\ a_N \end{Bmatrix} = \begin{Bmatrix} P_N \\ 0 \\ \\ 0 \end{Bmatrix} . \qquad (6)$$

Thus the coefficients of the minimum phase filter and the value of P_N are determined by solving (6).

If the equally spaced linear array is thought of as a sampling of a stationary complex space series, then the minimum phase filter is the corresponding Nth prediction error filter and P_N is the mean square error in doing the prediction.

The extremal value for the integral of the logarithm of $P(k)$ is given by

$$\int_{-K}^{+K} \log P(k)\, dk = \int_{-K}^{+K} \log(P_N/2K)dk - \int_{-K}^{+K} \log\,(1+a_1z+\ldots+a_Nz^N)\, dk$$

$$- \int_{-K}^{+K} \log\,(1+a_1^*z^{-1}+\ldots +a_N^*z^{-N})\, dk = 2K \log(P_N/2K)$$

$$- \frac{1}{2\pi i\Delta x} \oint \log\,(1+a_1z+\ldots+a_Nz^N)\, z^{-1}\, dz - \frac{1}{2\pi i\Delta x} \oint \log\,(1+a_1^*z^{-1}+\ldots+a_N^*z^{-N})\, z^{-1}dz$$

$$= 2K \log(P_N/2K) = \frac{1}{\Delta x} \log\,(P_N/2K)$$

Here we have used the fact that the two contour integrals around the unit circle are both zero as shown in the appendix. This result can be rewritten in a slightly different form as

$$\frac{1}{2K} \int_{-K}^{+K} \log P(k)\, dk = \log(P_N/2K)$$

which says that the average value of the logarithm of the spectrum is the logarithm of the mean square error divided by 2K.

Upper and Lower Bounds on the Maximum Entropy Spectrum

In performing the Levinson recursion in solving for the prediction error filters, we have the following equation relating the Nth prediction error filter to the N-1 th prediction error filter.

$$\begin{pmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_{N-1} \\ 0 \end{pmatrix} + C_N \begin{pmatrix} 0 \\ b^*_{N-1} \\ \vdots \\ b^*_1 \\ 1 \end{pmatrix} . \tag{1}$$

Here C_N is the Nth reflection coefficient (or partial correlation coefficient if you prefer). In z-transform terms, where $z = e^{i2\pi k\Delta x}$, we can write

$$1 + a_1 z + \ldots + a_N z^N = 1 + b_1 z + \ldots + b_{N-1} z^{N-1} + C_N\, z^N (1 + b^*_1 z^{-1} + \ldots + b^*_{N-1} z^{-N+1}) . \tag{2}$$

If $H_N(k)$ is the fourier transform of the Nth prediction error filter, we discover that

$$H_N(k) = H_{N-1}(k) + C_N\, e^{i2\pi Nk\Delta x}\, H^*_{N-1}(k) . \tag{3}$$

Taking absolute values, we have

$$|H_{N-1}(k)|\, (1+|C_N|) \geq |H_N(k)| \geq |H_{N-1}(k)|\, (1 - |C_N|) . \tag{4}$$

One might notice that there is at least one value of k at which the upper bound is reached and likewise for the lower bound. This is proved by the fact that the net phase shifts in going from -K to +K of $H_{N-1}(k)$ and $e^{i2\pi Nk\Delta x}\, H^*_{N-1}(k)$ differ by 2π.

Starting with $H_0(k) = 1$, we derive that

$$\prod_{n=1}^{N} (1 + |C_n|) \geq |H_N(k)| \geq \prod_{n=1}^{N} (1 - |C_n|).$$

For $N > 1$, one can note that the bounds may not be achieved. Since

$$P_N = \Phi(0) \prod_{n=1}^{N} (1-|C_n|^2),$$

we can write

$$\frac{\Phi(0) \prod_{n=1}^{N} (1-|C_n|^2)}{2K \prod_{n=1}^{N} (1+|C_n|)^2} \leq P_N(k) \leq \frac{\Phi(0) \prod_{n=1}^{N} (1-|C_n|^2)}{2K \prod_{n=1}^{N} (1-|C_n|)^2}, \text{ or}$$

$$\prod_{n=1}^{N} \left(\frac{1-|C_n|}{1+|C_n|}\right) = \frac{P_N(k)}{\Phi(0)/2K} = \prod_{n=1}^{N} \left(\frac{1+|C_n|}{1-|C_n|}\right) \tag{5}$$

If $Q = \prod_{n=1}^{N} \left(\frac{1+|C_n|}{1-|C_n|}\right)$, we have (6)

$$\log \left[\Phi(0)/2K\right] - \log Q \leq \log P(k) \leq \log \left[\Phi(0)/2K\right] + \log Q.$$

Appendix

Proof that $\oint \log (1 + a_1 z + a_2 z^2 + \ldots) \; z^{-1} \, dz = 0$,

where $1 + a_1 z + a_2 z^2 + \ldots$ is minimum phase, i.e., is analytic and has no zeros on or inside the unit circle and where the contour of integration is the unit circle.

From Cauchy's integral formula, we have that

$$f(0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z} \, dz$$

if $f(z)$ is analytic on and inside the contour of integration, where the contour encloses the origin. Because of the minimum phase condition, we see that $\log (1 + a_1 z + a_2 z^2 + \ldots)$ is an analytic function on and inside the unit circle and thus

$$\oint \frac{\log (1+a_1 z+a_2 z^2+\ldots)}{z} \, dz = 2\pi i \log (1) = 0. \quad \text{Q.E.D.}$$

Using this result it easily follows that it will also be true that

$$\oint \log (1 + a_1 z^{-1} + a_2 z^{-2} + \ldots) \; z^{-1} \, dz = 0 .$$

This is done by letting $y = z^{-1}$, $dy = -z^{-2} \, dz = -y^2 \, dz$, so that the above integral is changed to

$$\oint_{\text{clockwise}} \log (1 + a_1 y + a_2 y^2 + \ldots) \; y \; \left(\frac{-1}{y^2}\right) dy$$

$$= \oint_{\substack{\text{counter}\\ \text{clockwise}}} \log (1 + a_1 y + a_2 y^2 + \ldots) \; y^{-1} \, dy = 0.$$