Predictive Deconvolution Implies the Earthquake Geometry

by Don C. Riley

The results of least squares predictive deconvolution and the Burg Algorithm (Burg 1968) are identical for infinitely long time series. They differ essentially only in the treatment of estimating statistics from a truncated time series. We wish to show, through use of the Burg technique, that in the process of predictive deconvolution reflection coefficients of a layered media model are properly estimated. This is based on the assumption that, in addition to the one-dimensional ideal layered model, that the data were recorded at the surface due to a deeply buried source beneath the model. We refer to this as the earthquake geometry. The connection with the reflection geometry, that of a surface source and surface receivers, will later be made.

Burg's Algorithm is

$$\begin{cases} U_{j+1}(t) = U_j(t) + \hat{c}_{j+1} D_j(t-j) & j+1 \leqslant t \leqslant N \\ D_{j+1}(t-j) = D_j(t-j) + \hat{c}_{j+1} U_j(t) \end{cases}$$

$$\hat{c}_{j+1} = -2 \sum_{t=j+1}^{N} U_j(t) D_j(t-j) \Big/ \sum_{t=j+1}^{N} \left(U_j^2(t) + D_j^2(t-j)\right)$$

$$j = 0, 1, 2, \ldots, p$$

with the boundary condition

$$U_1(\cdot) = D_1(\cdot) = X(\cdot) \quad \text{the initial data}$$

$$Y(\cdot) \triangleq U_p(\cdot) \text{ is the } p\text{th forward deconvolved output}$$

We wish to formulate the algorithm in slightly different notation so that the parallel between the above filtering equations and downward continuation in layered media may be more easily recognized. We simply have to reset the time origin of the U array such that $U(j) \equiv$ 1st element in the array, and $U(t)$, $t \leqslant j$ are saved as final output. i.e.

$$\left\{ \begin{array}{l} U_{j+1}(t) = U_j(t) - \hat{c}_j D_j(t-1) \\ D_{j+1}(t) = D_j(t-1) - \hat{c}_j U_j(t) \end{array} \right. \qquad j+1 \leqslant t \leqslant N \qquad (1)$$

$$\hat{c}_j = 2 \sum_{t=j+1}^{N} U_j(t)\, D_j(t-1) \Big/ \sum_{t=j+1}^{N} U_j^2(t) + D_j^2(t-1)$$

The relation (1) is identical to the layer matrix of layered media theory (Goupillaud, 1961), (Claerbout, 1968), for propagating upcoming (U) and downgoing (D) waves from layer j to layer $j+1$ if $\hat{c}_j$ is the correct reflection coefficient as defined from the top of the interface.

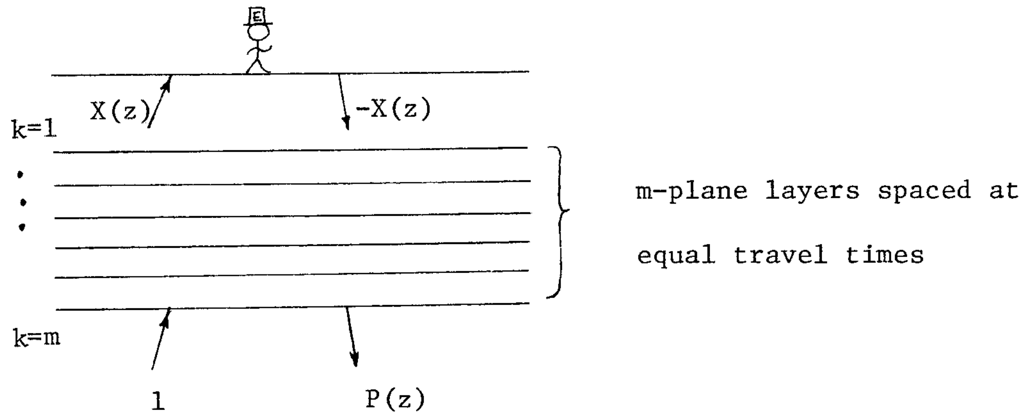
In this form we note that the parameter $\hat{c}_j$ is simply estimated from the 1st lag of the cross-correlation of U on D divided by the averaged autopowers of U and D . We abbreviate this as

$$\hat{c}_j = \frac{r_j(1)}{r_j(0)} = \frac{\sum_{t=1}^{N-j} U_j(t)\, D_j(t-1)}{1/2 \sum_{t=1}^{n-j} U_j^2(t) + D_j^2(t-1)} \qquad (2)$$

What we wish to show is that the Burg Technique correctly estimates the reflection coefficient series for a layered model of the earthquake seismology geometry and downward continues the up and downgoing waves to the source.

For the earthquake geometry below, we have a white input wave from great depths. We record at the surface the seismogram $X(z)$.

$P(z)$ is the back scattered wave. P is all-pass by energy conservation.



Let $U_k(z)$, $D_k(z)$ be the upcoming and downgoing waves seen in the top of the kth layer. The Energy Flux Theorem states that the net flow of energy in one direction is the same in each layer at each frequency in the absence of sources and sinks. Thus, if I_k is the characteristic impedance in the kth layer, we may equate the net upward flux by

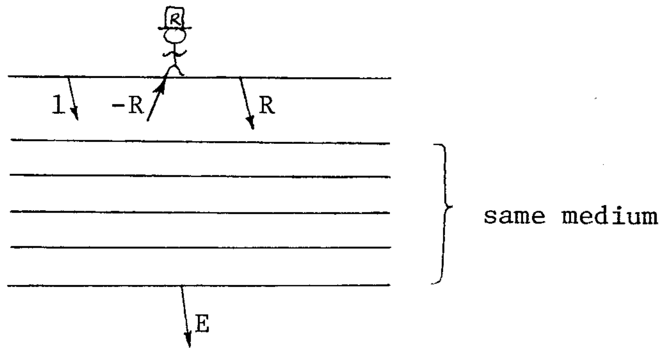
$$I_k \left[U(z)\, U(1/z) - D(z)\, D(1/z) \right]_k = I_{k+j} \left[U(z)\, U(1/z) - D(z)\, D(1/z) \right]_{k+j} \qquad (3)$$

We may immediately use this to determine the form of $P(z)$ by equating fluxes at the top and bottom of the stack

$$I_1 \left[0 \right] = I_m \left[1 - P(z)\, P(1/z) \right] \qquad (4)$$

Thus $P(z)\, P(1/z) = 1$ and $P(z)$ is an all-pass filter.

We shall have a need for the Reflection Seismology geometry with the <u>same</u> medium. Using this we will be able to relate the Burg estimator for $\hat{c}_j$ to the hypothetical reflection coefficients in a layered model.



Relating the waves at the top and bottom of the stack by the Energy Flux Theorem we have the important relation

$$I_1 \left\{ R(z)\, R(1/z) - (1 + R(z))(1+R(1/z)) \right\} = I_m \left\{ - E(z)\, E(1/z) \right\}$$

$$E(z)\, E(1/z) = \frac{I_1}{I_m} [\, 1 + R(z) + R(1/z) \,] \qquad (5)$$

Now we wish to relate the $X(z)$ seismogram to the escaping wave $E(z)$. We can do this easiest by using the principle of reciprocity. By reciprocity we have that if we interchange source and receiver we observe the same waveform. We have

$X(z) = (\pi\, t/t')\, E(z)$ where $\pi\, t/t'$ is a positive scale factor, let $\sqrt{\sigma} = \pi\, t/t'$

and using the relation between $E(z)$ and $R(z)$

$$X(z)\, x(1/z) = \frac{I_1}{I_m} \sigma [\, 1 + R(z) + R(1/z) \,] \qquad (6)$$

We can directly use this to show that the $\hat{c}_1$ estimate is the 1st reflection coefficient in the model. We start the Burg technique by loading-up the $U_1(z)$ and $D_1(z)$ vectors with the data $X(z)$. Then we make the 1st estimate by computing the zero and 1st lags of the autocorrelation of $X(z)$ (or U_1 or D_1). From (6) clearly $r_1(0) = I_1/I_m\, \sigma$. Since $R(z)$ is a reflected wave due to a source directly above the 1st layer $R(z) = c_1 z + ?\, z^2 + ?\, z^3 + \cdots$ Thus

$r_1(1) = I_1/I_m \sigma c_1$ and thus

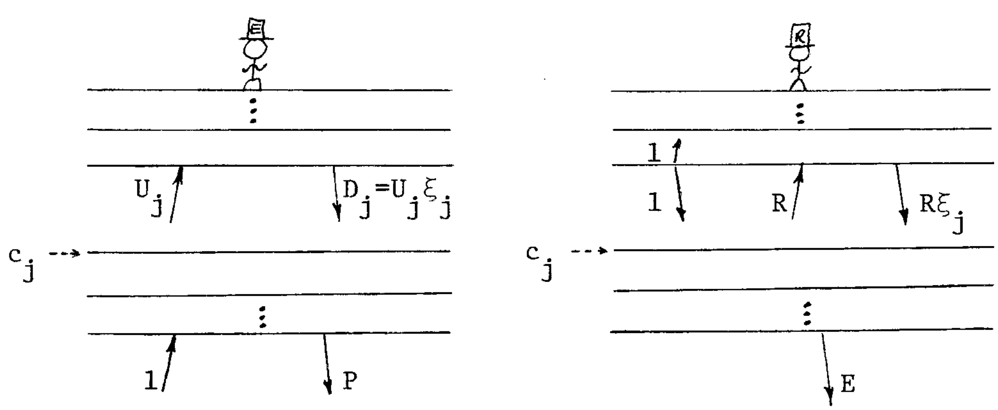
$$\hat{c}_1 = c_1 \quad \text{the "true" reflection coefficient.}$$

To generalize this result, we note that from energy flux $U_j(z)\, U_j(1/z) = D_j(z)\, D_j(1/z)$ for the earthquake geometry. That is, the spectra of U and D are identical for each stage in the Burg Algorithm or layer in the model. U and D can only differ by an all-pass filter, say $\xi(z)$, thus using

$$D_j(z) = U_j(z)\, \xi_j(z) \qquad (7)$$

↖all pass filter

Now we relocate our source in the reflection geometry picture. This allows us to produce a simple early portion of the waveform which then, by reciprocity, and energy flux can be related to the waves at a like depth in the earthquake picture.



In this case we make the same all-pass modification to the downgoing R wave to include all effects of shallow layers. Now (5) takes the form

$$I_m \left\{- E(z)\, E(1/z)\right\} = I_j \left\{R(z)\, R(1/z) - (\,1+R(z)\, \xi_j(z))(\,1+R(1/z)\, \xi_j(1/z))\right\} \tag{8}$$

$$= I_j \left\{R(z)R(1/z) - 1 - R(z)\xi_j(z) - R(1/z)\xi_j(1/z)+R(z)R(1/z)\xi_j(z)\xi_j(1/z)\right\}$$

but $\xi_j(z)\ \xi_j(1/z) = 1$ so that

$$E(z)\ E(1/z) = \frac{I_j}{I_m} \left\{ 1 + R(z)\ \xi(z) + R(1/z)\ \xi(1/z) \right\} \tag{9}$$

and again using reciprocity

$$U_j(z) = \sqrt{\sigma}\ E(z) \qquad \text{different scale factor}$$

Thus we can cross-correlate $U_j(z)$ and $D_j(z)$ and express the result in terms of R .

$$U_j(z)\ D_j(1/z) = U_j(z)\ \xi_j(1/z)\ U_j(1/z)$$

$$= \sigma\ E(z)\ E(1/z)\ \xi_j(1/z)$$

$$= \frac{\sigma I_j}{I_m} \left\{ \xi_j(1/z) + R(z) + R(1/z)\ \xi_j(1/z) \right\} \tag{10}$$

Since we've moved our source down to the jth interface we record c_j uncorrupted. i.e.

$$R(z) = c_j\ z + ?\ z^2 + ?\ z^3 + \cdots$$

$$\therefore\ r_j(1) = \frac{\sigma I_j}{I_m}\ c_j \tag{11}$$

Clearly

$$U_j(z)\ U_j(1/z) = D_j(z)\ D_j(1/z) = \frac{\sigma I_j}{I_m} \left\{ 1 + R(z)\ \xi(z) + R(1/z)\ \xi(1/z) \right\}$$

and $\quad r_j(0) = \dfrac{\sigma I_j}{I_m}$

$$\therefore\ \hat{c} = c_j$$

Thus we are able to, within a scale factor, downward continue the surface earthquake seismograms to the source in a completely stable and robust manner. We end up with a final, whitened U , D whereas was well known, D is an all-pass version of U . We precisely estimate the true model reflection coefficients.

Now what is the interpretation when we put in something that is not a "valid" earthquake. In particular, let's insert a reflection seismogram. From layer matrix results we have that $x(z) = 1/A^{+}(z)$ where $A^{+}(z)$ is a minimum phase waveform. Thus we may expect that inserting any waveform of the form $1/A^{+}(z)$ the Burg algorithm will identify the unique earthquake model for that waveform and invert it. All c's will be less than unity in magnitude and everything will be stable. However, we know also from layered media theory that a reflection seismogram is of the form $1 + 2\,R(z) = A^{-}(z)/A^{+}(z)$. That is, it has a minimum phase denominator plus a numerator contribution. Since we also know that the numerator $A^{-}(z)$ is also minimum phase (same structure as $A^{+}(z)$ except all signs of c reversed) we are in luck. If we follow through the derivation we will find that, since the numerator is minimum phase, its contribution will integrate to zero around the unit circle, except for the first term which can be scaled to unity. Thus the Burg Algorithm goes on its merry way identifying only the reverberatory system $1/A^{+}(z)$. The final output will be whitened in the sense of inverting $1/A^{+}(z)$ and shaped as $A^{-}(z)$. $A^{-}(z)$ will pass through unmodified. If only $A^{-}(z)$ were the reflection coefficient series!

References

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