Dispersion Relationship for the Slant Frames Approximation

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In the article on slant frames of the present report, Claerbout, J. F., transformed the wave equation:

$$P_{xx} + P_{zz} = c^{-2} P_{tt} \quad , \tag{1}$$

according to the coordinate transformations:

$$\begin{aligned} x' &= x - c \sin\theta \; t \\ z' &= z \\ t' &= -\frac{1}{c} \sin\theta \; x \pm \frac{1}{c} \cos\theta \; z + t \quad , \end{aligned} \tag{2}$$

and, after defining $P(x,z,t) = Q(x',z',t')$ and dropping the $Q_{z'z'}$ term, got the approximated equation:

$$Q_{z't'} = \pm \frac{c}{2} \cos\theta \; Q_{x'x'} \tag{3}$$

where for the downgoing waves we use the minus sign in (3)(the plus sign in (2)).

In order to understand better this approximation, let's transform back equation (3), to compare it with the actual wave equation (1). If we limit our further considerations to the downgoing waves only, the inverse transformations, corresponding to (2), can be written as:

$$\begin{aligned} x &= \sec^2\theta \; x' + \tan\theta \; z' + c \sec\theta \tan\theta \; t' \\ z &= z' \\ t &= \frac{1}{c} \sec\theta \tan\theta \; x' + \frac{1}{c} \sec\theta \; z' + \sec^2\theta \; t' \quad , \end{aligned} \tag{4}$$

from where,

$$Q_{x'x'} = (\sec^4\theta)P_{xx} + (\frac{1}{c^2} \sec^2\theta \tan^2\theta)P_{tt} + (\frac{2}{c} \sec^3\theta \tan\theta)P_{xt} \quad (5)$$

and

$$Q_{z't'} = (c \sec\theta\tan^2\theta)P_{xx} + (\frac{1}{c}\sec^3\theta)P_{tt} + (c \sec\theta\tan\theta)P_{xz}$$

$$+ (2 \sec^2\theta \tan\theta)P_{xt} + (\sec^2\theta)P_{zt} \quad (6)$$

Replacing (5) and (6) into (3) (taken with the plus sign), we finally get:

$$\frac{1}{2}(\tan^2\theta - 1)P_{xx} + \frac{1}{2c^2}(\tan^2\theta + 2)P_{tt} + \tan\theta\, P_{xz} + \frac{1}{c}\sec\theta \tan\theta\, P_{xt} + \frac{1}{c}\sec\theta\, P_{zt} = 0 \quad (7)$$

Notice that if we make $\theta = 0$ in equation (7), then we have:

$$\frac{c}{2} P_{xx} - \frac{1}{c} P_{tt} - P_{zt} = 0 , \quad (8)$$

which for a time dependence of the form $\exp(-i\omega t)$, becomes:

$$P_{xx} + 2 \frac{\omega^2}{c^2} + 2 i \frac{\omega}{c} P_z = 0 \quad (9)$$

or, defining $m = \omega/c$,

$$P_{xx} + 2 i m P_z + 2 m^2 P = 0 , \quad \theta = 0 \quad (10)$$

Equation (10) is exactly the well known equation for the parabolic approximation, earlier discussed in the second part of this report, "A Tutorial on Monochromatic Waves."

In order to get the dispersion relationship, corresponding to (7), the usual procedure is to insert a trial solution for a plane wave of unit magnitude, propagating along the k-direction:

$$P = \exp(i k_x x + i k_z z - i \omega t) . \quad (11)$$

After inserting (11) into (7) we get the wanted relation:

$$(1-\tan^2\theta)k_x^2 - (2\tan\theta)k_x k_z + (2m\sec\theta\tan\theta)k_x + (2m\sec\theta)k_z - m^2(2+\tan^2\theta) = 0 \qquad (12)$$

or

$$k_z = \frac{(\tan^2\theta - 1)k_x^2 - 2m\sec\theta\tan\theta\, k_x + m^2(2+\tan^2\theta)}{2\sec\theta\,(m - \sin\theta\, k_x)} \qquad (13)$$

where, as previously, $m = \omega/c$.

Equation (12) is of the general type:

$$Ak_x^2 + Bk_x k_z + Ck_z^2 + Dk_x + Ek_z + F = 0 , \qquad (14)$$

and since in our case $B = -2\tan\theta \neq 0$ (if $\theta \neq 0$!) and $B^2 > 4AC = 0$, we may conclude that equation (12) represents a hyperbola for $\theta \neq 0$. For the case $\theta = 0$, equation (12) becomes

$$k_x^2 + 2mk_z - 2m^2 = 0 , \qquad (15)$$

which corresponds, as we might already have expected from (10), to an inverted parabola with its axis being the k_z-axis. For other values of θ , we have the following situations: $\underline{\theta = 90°}$: In this case, equation (12) gives just a straight line, parallel to the k_z-axis and at a distance equal + m to its right:

$$k_x = m \qquad (16)$$

$\underline{0 < \theta < 90°}$: For these angles, as we said, we get a hyperbola with one of its asymptotes being parallel to the k_z-axis, according to the equation:

$$k_x = m \operatorname{cosec}\theta \qquad (17)$$

so that, as we move (θ) from 0° to 90° , this line moves from $+\infty$ to 1 .

The other asymptote is the straight line:

$$z = \cot(2\theta)\cdot x + \frac{m}{\sin(2\theta)\sin\theta} \tag{18}$$

This second line, contrary to the first one, has a varying direction and as θ goes from 0° to 90° , it rotates 180° . From this consideration it's obvious that the approximation becomes worse as we approach bigger angles.

Next are given the tables and the corresponding graphs for $m = 1$ and values of θ equal 0° , 15°, 45° and 75° . The included relative error refers to how well the hyperbolas (or parabola in the case of $\theta = 0°$) fit a circle of unit radius (exact solution) at each side of the angle of propagation θ .

DISPERSION RELATION FOR M=1 AND DIP= 15.00 DEG

KX	ANG	HYPERBOLA KZ	HYPERBOLA RADVECTOR	ERROR(%) (1.-RADVECTOR)
-0.800000	-46.14	0.768861	1.109570	10.95695
-0.700000	-40.49	0.819952	1.078110	7.81097
-0.600000	-34.73	0.865572	1.053102	5.31021
-0.500000	-28.91	0.905345	1.034230	3.42300
-0.400000	-23.08	0.938858	1.020515	2.05154
-0.300000	-17.26	0.965662	1.011188	1.11876
-0.200000	-11.47	0.985259	1.005352	0.53520
-0.100000	-5.73	0.997107	1.002100	0.21000
0.000000	0.00	1.000000	1.000001	0.00000
0.100000	5.74	0.995079	1.000090	0.00896
0.200000	11.54	0.979790	1.000002	0.00019
0.300000	17.46	0.953941	1.000001	0.00010
0.400000	23.58	0.910588	1.000066	0.00658
0.500000	29.98	0.866710	1.000598	0.05980
0.600000	36.76	0.803175	1.002542	0.25415
0.700000	44.01	0.724666	1.007542	0.75417
0.800000	51.79	0.629726	1.018113	1.81131
0.900000	60.14	0.516691	1.037771	3.77712
1.000000	69.01	0.383664	1.071073	7.10726
1.099999	78.27	0.228479	1.123470	12.34700

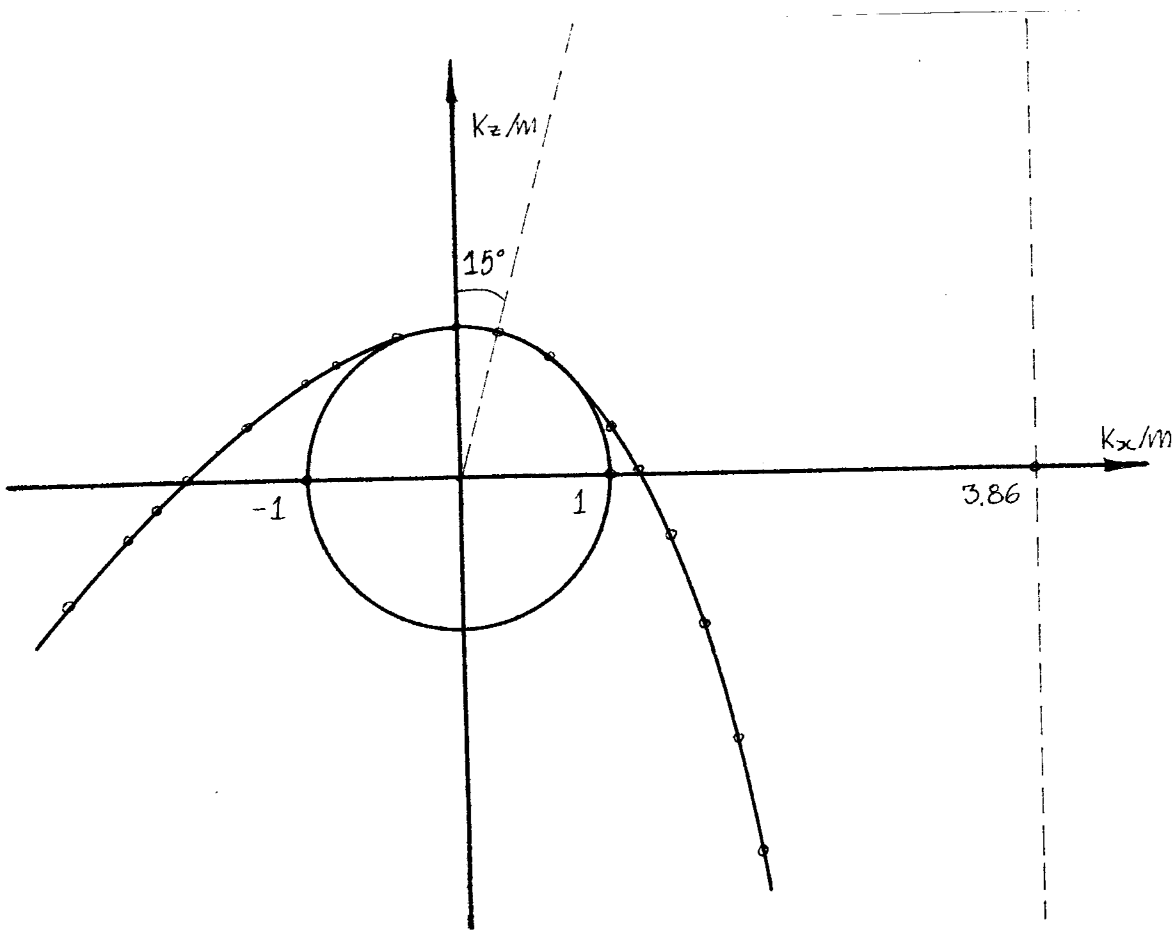


Figure 1.

DISPERSION RELATION FOR M=1 AND DIP= 45.00 DEG

KX	ANG	HYPERBOLA KZ	HYPERBOLA RADVECTOR	ERROR(%) (1.-RADVECTOR)
-0.050000	-2.67	1.072734	1.073897	7.38974
0.000000	0.00	1.060660	1.060659	6.06594
0.050000	2.73	1.047702	1.048893	4.88930
0.100000	5.53	1.033758	1.038583	3.85828
0.150000	8.38	1.018711	1.029695	2.96946
0.200000	11.28	1.002425	1.022182	2.21815
0.250000	14.24	0.984749	1.015978	1.59779
0.300000	17.26	0.965468	1.011003	1.10025
0.350000	20.34	0.944384	1.007154	0.71545
0.400000	23.47	0.921222	1.004315	0.43154
0.450000	26.68	0.895658	1.002349	0.23479
0.500000	29.96	0.867298	1.001101	0.11005
0.550000	33.35	0.835654	1.000408	0.04082
0.699999	44.43	0.714145	1.000000	0.00000
0.899999	63.65	0.441850	1.002615	0.26150
0.940000	70.46	0.337127	1.008043	0.80433
0.990000	78.30	0.207111	1.021221	2.12212
1.049999	87.74	0.041402	1.050814	5.08137

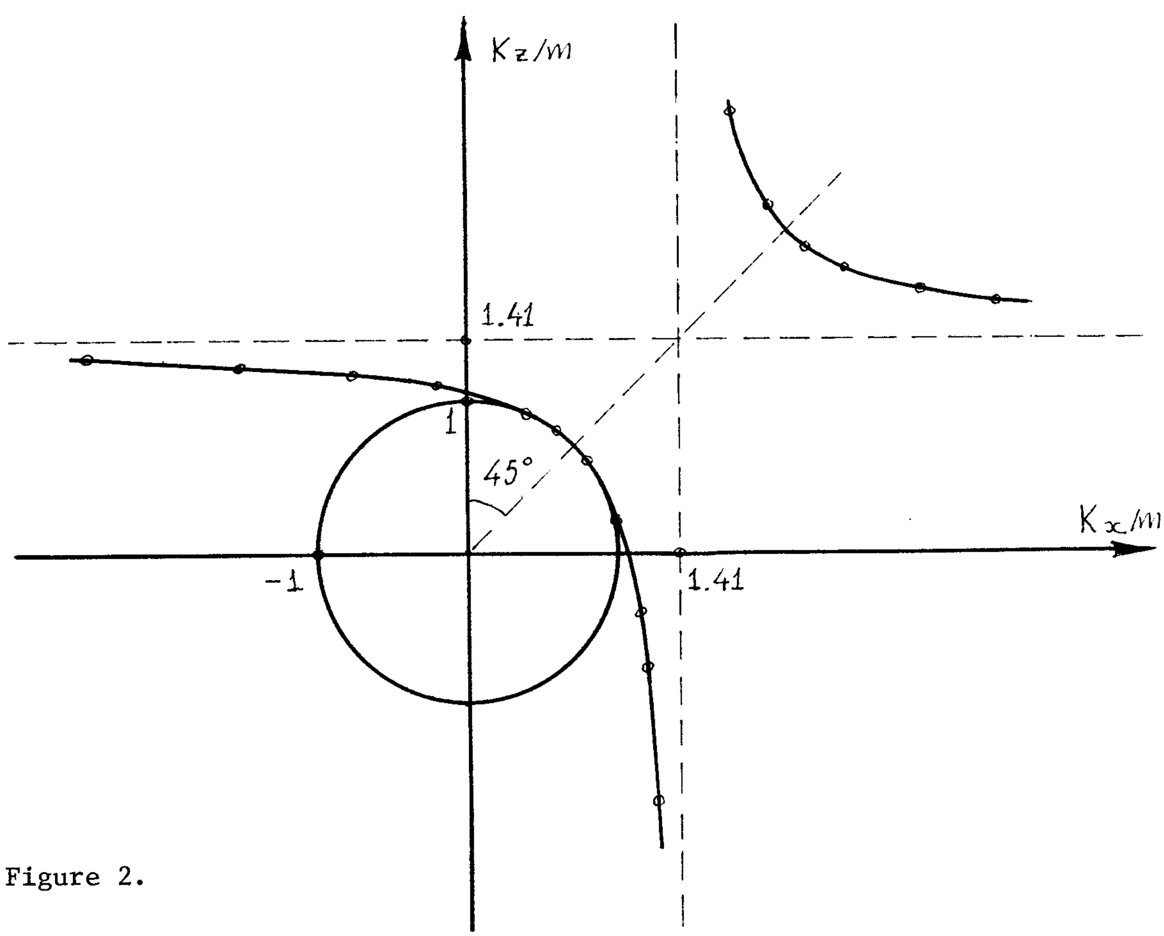


Figure 2.

DISPERSION RELATION FOR M=1 AND DIP= 75.00 DEG

KX	ANG	HYPERBOLA KZ	HYPERBOLA RADVECTOR	ERROR(%) (1.-RADVECTOR)
0.710000	41.10	0.811234	1.078053	7.80525
0.730000	43.30	0.774652	1.064417	6.44169
0.750000	45.47	0.737806	1.052972	5.29716
0.770000	47.70	0.700620	1.041041	4.10414
0.790000	49.90	0.663920	1.031359	3.13597
0.820000	53.55	0.605509	1.019386	1.93863
0.850000	57.27	0.546401	1.010471	1.04713
0.860000	58.54	0.526123	1.008160	0.81602
0.970000	75.93	0.245136	1.000997	0.09967
0.980000	78.51	0.199174	1.000034	0.00343
0.990000	81.76	0.143395	1.000330	0.03309
1.000000	86.23	0.065872	1.002167	0.21667
1.005000	93.36	-0.059317	1.011738	1.17383
1.010000	107.71	-0.325630	1.070717	7.07169
1.020000	146.15	-1.535632	1.849079	84.90795

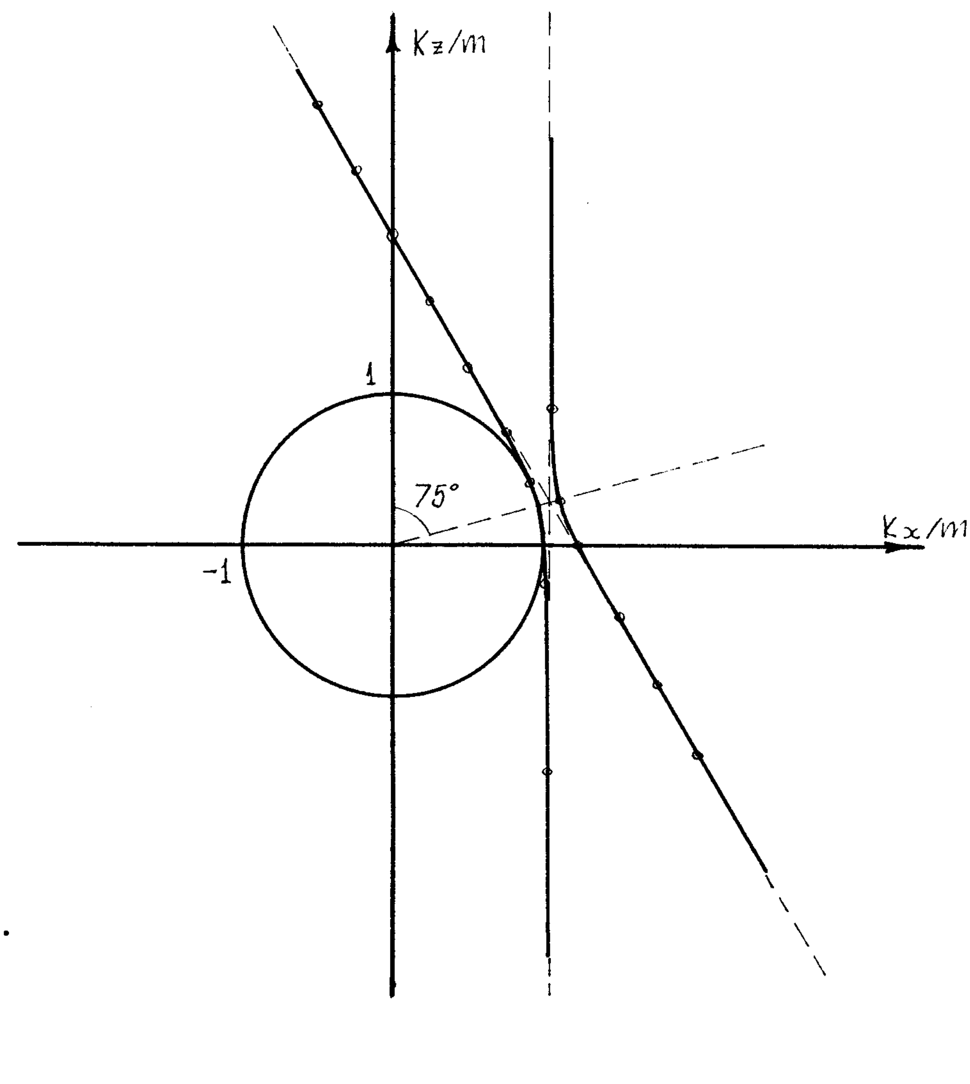
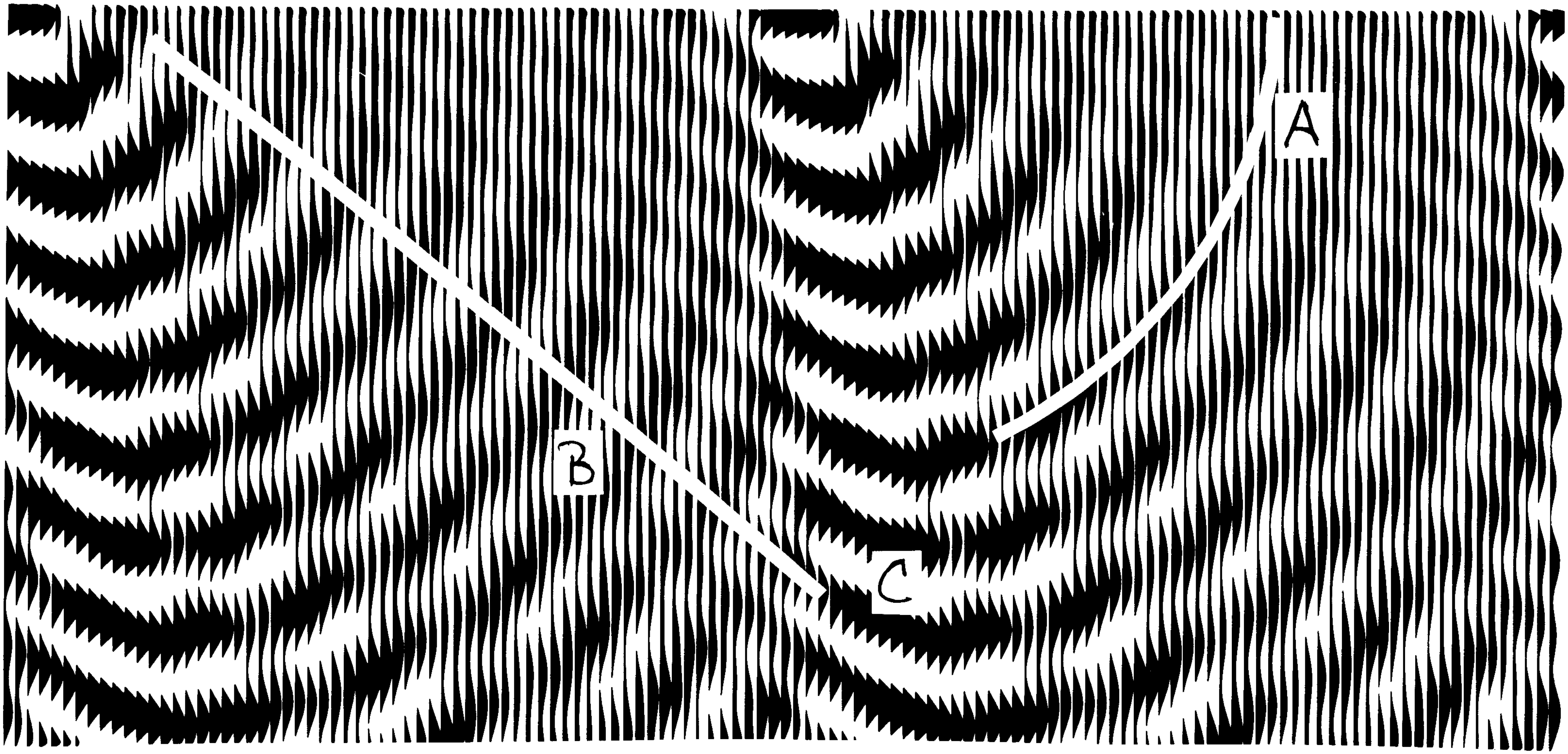


Figure 3.



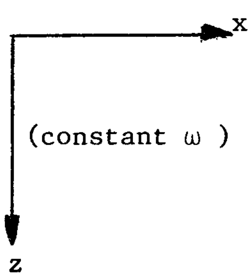


Figure 4. The slant frame approximation at 45 degrees. The figure should be interpreted as the responses of three point apertures (center and both corners of the upper edge of the figure) in the x-direction to an incident monochromatic wave. The circular arc, "A" shows the region where the hyperbolic approximation closely models normal wave propagation (where the circle and hyperbola are tangent in the first three figures). The wavefront "B" demonstrates that the asymptotes of the hyperbola model a plane wave where k_x and k_z are proportional. The region labeled "C" shows a high energy density near $k_x = o$.

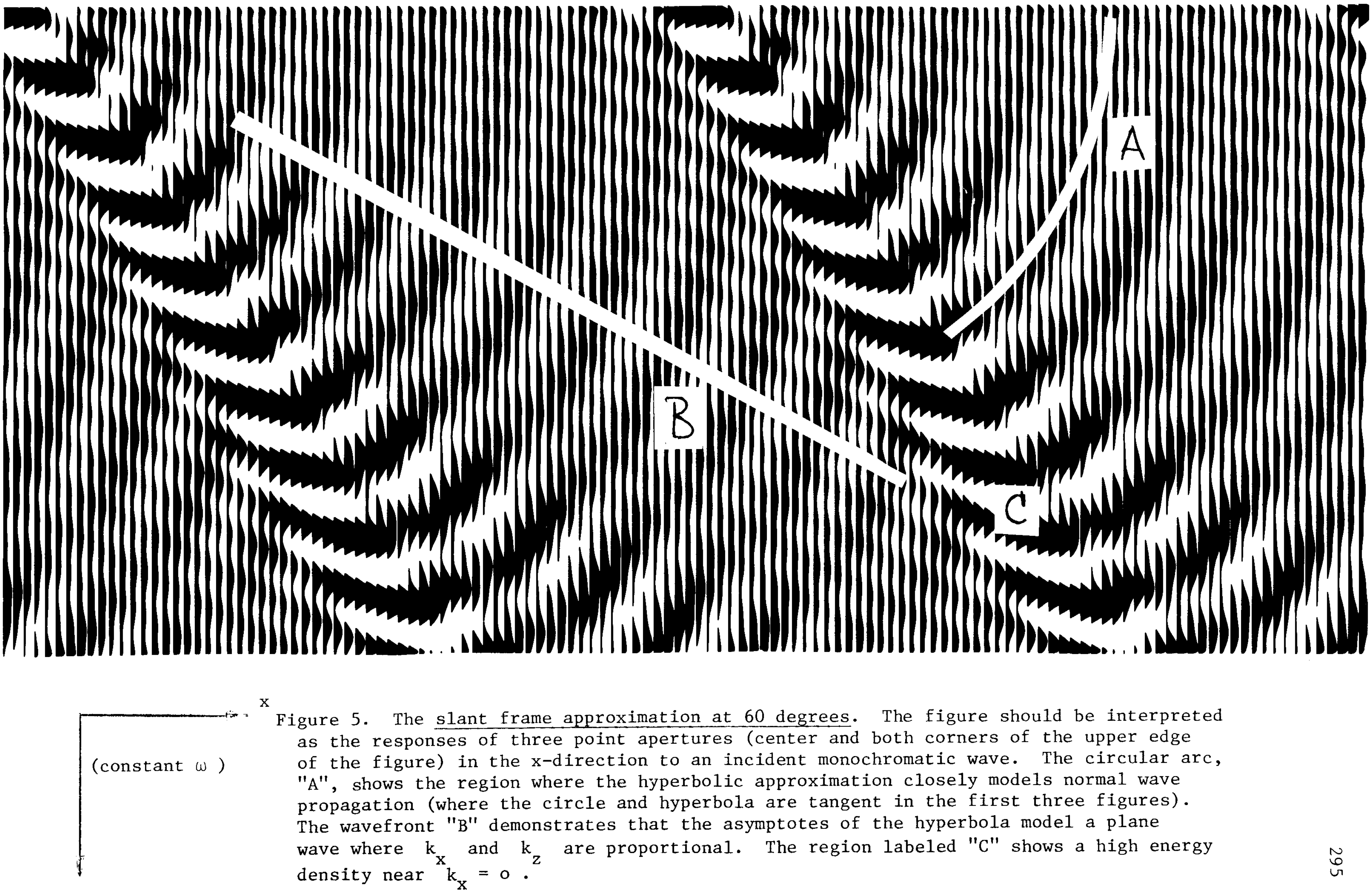


Figure 5. The slant frame approximation at 60 degrees. The figure should be interpreted as the responses of three point apertures (center and both corners of the upper edge of the figure) in the x-direction to an incident monochromatic wave. The circular arc, "A", shows the region where the hyperbolic approximation closely models normal wave propagation (where the circle and hyperbola are tangent in the first three figures). The wavefront "B" demonstrates that the asymptotes of the hyperbola model a plane wave where k_x and k_z are proportional. The region labeled "C" shows a high energy density near $k_x = o$.