4.1 Extrapolation of Magnetotelluric Wave Fields

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4.1.I. Electromagnetic Waves: Frequency Domain Finite Difference Techniques

We will first consider transverse electric waves, then the transverse magnetic case. Maxwell's equations are (in MKS):

$$\nabla \times E = - \dot{B} \tag{1}$$

$$\nabla \times H = J + \dot{D} \tag{2}$$

We assume $\dot{D} \approx 0$ in equation (2) and remember Ohm's law, $J = \sigma E$, and that $B = \mu H$. Equation (2) by components is (using superscripts for components):

$$\partial_y (B^z/\mu) = 0 \tag{3a}$$

$$- \partial_y (B^x/\mu) = 0 \tag{3b}$$

$$- \partial_x (B^z/\mu) + \partial_z (B^x/\mu) = \sigma E^y \tag{3c}$$

Take B to have time dependence $B_o e^{-i\omega t}$; eq. (1) by components is then:

$$- \partial_z E^y = i \omega B^x \tag{4a}$$

$$\partial_x E^y = i \omega B^z \tag{4b}$$

Substitute equations (4) into (3c):

$$- \partial_x [(1/\mu) \partial_x (E^y/i\omega)] + \partial_z [(1/\mu) \partial_z (E^y/i\omega)] = \sigma E^y$$

For μ as a function of x only, we have (using subscripts for derivatives):

$$E^y_{xx} + E^y_{zz} - (\mu_x/\mu) E^y_x = - \ell^2 E^y \quad ; \quad \ell = (i \omega \mu \sigma)^{1/2} \tag{5}$$

Now, let's consider a perturbation technique:

$$E^y = Q (x, z) e^{i\ell z}$$

Then:

$$E^y_x = Q_x \, e^{i\ell z} \tag{7a}$$

$$E^y_{xx} = Q_{xx} \, e^{i\ell z} \tag{7b}$$

$$E^y_{zz} = (Q_{zz} + 2 \, i \, \ell \, Q_z - \ell^2 \, Q) \quad e^{i\ell z} \tag{7c}$$

Substituting equations (7) into (5) and dropping Q_{zz} (parabolic approximation) yields:

$$Q_{xx} - (\mu_x/\mu) \, Q_x = - 2 \, i \, \ell \, Q_z \tag{8}$$

This equation can be solved by Crank Nicolson technique. A sample program to solve (8) where $\mu_x = 0$ is given in Example A.

Before going on, let's consider the parallel development for the transverse magnetic problem. Writing equation (1) by components:

$$\partial_y \, E^z = 0 \tag{9a}$$

$$- \partial_y \, E^x = 0 \tag{9b}$$

$$- \partial_x \, E^z + \partial_z \, E^x = - i \, \omega \, B^y \tag{9c}$$

where B has $e^{-i\omega t}$ time dependence. Similarly, equation (2) by components is:

$$- \partial_z \, (B^y/\mu) = \sigma \, E^x \tag{10a}$$

$$\partial_x \, (B^y/\mu) = \sigma \, E^z \tag{10b}$$

Substituting equations (10) into equation (9c) gives:

$$\partial_x \, [\, (1/\sigma) \, \partial_x \, (B^y/\mu) \,] + \partial_z \, [\, (1/\sigma) \, \partial_z \, (B^y/\mu) \,] = - i \, \omega \, B^y \tag{11}$$

Now, let's take σ , μ to be functions of x only:

$$B^y_{zz} + B^y_{xx} + B^y_x \; (- 2 \; (\mu_x/\mu) - (\sigma_x/\sigma) \;)$$

$$+ B \; (- \; (\mu_{xx}/\mu) + 2 \; (\mu_x/\mu)^2 + (\mu_x \sigma_x/\mu\sigma) + \ell^2) \;\; = \;\; 0 \qquad (12)$$

The equation can now be solved by perturbation:

$$B^y \;=\; Q \; (\; x, \; z \;) \; e^{i\ell z} \quad \text{with} \quad \ell = \ell(x)$$

Thus

$$B^y_x \;=\; (\; Q_x + i \; \ell_x \; z \;\; Q \;) \;\; e^{i \, \ell \, z}$$

$$B^y_{xx} = \; (\; Q_{xx} + 2 \; i \; \ell_x \; z \; Q_x - \ell^2_x \;\; z^2 \;\; Q \;) \;\; e^{i \, \ell \, z}$$

$$B^y_{zz} \;=\; (\; Q_{zz} + 2 \; i \; \ell \;\; Q_z - \ell^2 \; Q \;) \;\; e^{i \, \ell \, z}$$

Substituting into (12), dropping Q_{zz} , and cancelling the exponential:

$$- \; 2 \; i \; \ell \; Q_z \;\; = \;\; Q_{xx} + Q_x \; (\; 2 \; i\ell_x z + C \;)$$

$$+ \; Q \; (- \; \ell^2_x \; z^2 + i\ell_x \; C + D \;) \qquad (13)$$

where

$$C = (- \; 2 \; (\; \mu_x/\mu) \; - \; (\sigma_x/\sigma) \;)$$

$$D = (- \; C \; (\mu_x/\mu) \; - \; (\mu_{xx}/\mu) \;)$$

A more practical problem is to reconsider (from equation (11)):

$$\partial_x \; [\; (1/\sigma) \; \partial_x \; B^y \;] \;\; + \;\; \partial_z \; [\; (1/\sigma) \; \partial_z \; B^y \;] \;\; = \;\; i \; \mu \; \omega \;\; B^y \qquad (14)$$

with the realistic assumptions that μ is constant and σ is a function of x and z. We then have:

$$B_{zz} + B_{xx} - (\sigma_x/\sigma)\, B_x - (\sigma_z/\sigma)\, B_z = - i\, \mu\, \sigma\, \omega\, B \;= -\ell^2\, B \qquad (15)$$

Now, we use a perturbation:

$$B = Q\, e^{i\phi}, \text{ where } \phi = \int_0^z \ell \; dz \qquad (16)$$

$$B_x = (Q_x + i\, \phi_x\, Q)\, e^{i\phi} \qquad (16a)$$

$$B_{xx} = (Q_{xx} + 2\, i\, \phi_x\, Q_x + Q\, [i\, \phi_{xx} + (i\, \phi_x)^2])\, e^{i\phi} \qquad (16b)$$

$$B_z = (Q_z + i\, \ell\, Q)\, e^{i\phi} \qquad (16c)$$

$$B_{zz} = (Q_{zz} + 2\, i\, \ell\, Q_z + Q\, [i\, \ell_z + (i\, \ell)^2]\,)\, e^{i\phi} \qquad (16d)$$

Substituting equations (16) into equation (15) and cancelling the exponential:

$$(- 2\, i\, \ell + (\sigma_z/\sigma))\, Q_z = Q_{xx} + Q_x\, (\, 2\, i\, \phi_x - (\sigma_x/\sigma))$$

$$+ Q\, (\, i\, \ell_z + i\, \phi_{xx} - (\phi_x)^2 - (\sigma_x/\sigma)\, i\, \phi_x - (\sigma_z/\sigma)\, i\, \ell\,) \qquad (17)$$

A program to solve this is given in Example B. The use of equation (16) in B is not the ideal; Part III deals with further frequency domain techniques which appear to work better.

4.1.II. Electromagnetic Waves: Time Domain Recursions

First we note that all the solutions in Section I were in the frequency domain. In this section we want to consider similar problems in the time domain. Initially let's consider how to solve the one dimensional equation:

$$Q_z = (- i \omega \mu \sigma)^{1/2} Q \tag{1}$$

First we change notation: drop superscripts for components and subscripts for derivatives. Then let's use superscripts for z coordinate and subscripts for time, t, coordinate. Using finite differences with this notation we can write:

$$(Q^{N+1} - Q^{N}) = \underbrace{- \frac{\Delta z}{2} \left(\frac{\mu\sigma}{\Delta t} \right)^{1/2}}_{\text{denote as A, a constant}} \underbrace{(-i \omega \Delta t)^{1/2}}_{\text{denote as S}} (Q^{N+1} + Q^{N})$$

or,

$$(1 + A S) Q^{N+1} = (1 - A S) Q^{N} \tag{2}$$

There are two ways to proceed from here. Both will require the development of time domain coefficients for the half order derivative operator S. Example C gives this development for both half order derivative and integral operators.

The first way to solve equation (2) is the more obvious, but is relatively expensive. Rewrite equation (2) in time domain as:

$$P_t^{N+1} + A \sum_{\tau=0}^{t} s_\tau P_{t-\tau}^{N+1} = P_t^{N} - A \sum_{\tau=0}^{t} s_\tau P_{t-\tau}^{N}$$

Rearranging we get a recurrence relation for increasing time and depth

$$[\ 1 + A\ s_0\ ]\ P_t^{N+1}$$

$$= [\ 1 - A\ s_0\ ]\ P_t^{N} - A \sum_{\tau=1}^{t} s_\tau\ [\ P_{t-\tau}^{N+1} + P_{t-\tau}^{N}\ ]$$

Appendix D contains a program for this equation.

However, there is an alternate way to solve equation (2). First we observe from equation (2) that to get, for example Q^1, from Q^0 we have:

$$Q^1 = \frac{1 - AS}{1 + AS}\ Q^0$$

Hence, we might then postulate Q^N to be of the form indicated for Q^1, i.e.,

$$Q^N = \frac{U^N - A\ S\ V^N}{U^N + A\ S\ V^N} \tag{3}$$

Substituting equation (3) into equation (2):

$$\frac{U^{N+1} - A\ S\ V^{N+1}}{U^{N+1} + A\ S\ V^{N+1}} = \frac{1 - A\ S}{1 + A\ S}\ \frac{U^N - A\ S\ V^N}{U^N + A\ S\ V^N}$$

Matching even and odd powers of S in either the numerator or denominator (note: same recursion for both numerator and denominator):

even: $$U^{N+1} = U^N + (A\ S)^2\ V^N$$

odd: $$V^{N+1} = U^N + V^N$$

This recursion is still in the frequency domain, as is equation (2). We use the bilinear approximation:

$$S^2 \approx 2\ \frac{1-z}{1-z}$$

where z is the z transform variable to write:

$$U_t^{N+1} = - U_{t-1}^{N+1} + U_t^N + U_{t-1}^N + 2 A^2 [V_t^N - V_{t-1}^N]$$

and (4 a,b)

$$V_t^{N+1} = U_t^N + V_t^N$$

At this point we see to solve equation (2) for any given z level, we use the recursions of equations (4 a,b) to obtain U, V at the desired level, then find Q by equation (3). Note that in equation (3) we have the half order operator S for which we obtained the time domain representation in Example C.

If we desire to solve equation (2) for all z then there is no advantage to the technique just discussed. However, to solve for a deep z level only, we have only to perform the recursions of equations (4 a,b) for intermediate z levels. A program is given in Example E.

Having outlined a one dimensional time domain solution that was based on assuming a solution of the form of equation (3), let's now consider the two dimensional problem:

$$Q_z = \frac{-1}{2(-i\omega\mu\sigma)^{1/2}} Q_{xx} \tag{5}$$

Examples are equation (8) or equation (11) of section I with μ , σ constant. Proceeding as in the one dimensional case:

$$Q^{N+1} - Q^N = \underbrace{-\frac{\Delta z}{4} \left(\frac{\Delta t}{\mu\sigma} \right)^{1/2}}_{\text{denote as } A_2} \underbrace{\frac{1}{(-i\omega\Delta t)^{1/2}}}_{\text{denote as } S} \underbrace{[T]}_{\text{second difference operator}} (Q^N + Q^{N+1})$$

$$(I + A_2 S T) Q^{N+1} = (I - A_2 S T) Q^N \tag{6}$$

Rewriting in time domain:

$$(I + A_2 \; s_o \; T) \; Q_t^{N+1} = Q_t^N - A_2 \; T \; (s_o \; Q_t^N + \sum_{\tau = 0}^{t} s_\tau \; (Q_{t-\tau}^{N+1} + Q_{t-\tau}^N)) \qquad (7)$$

This recursion in z and t then requires a solution for the tridiagonal operator T on the left hand side. A program for this "expensive" solution is given in Example F.

The extension of the one dimensional technique to give a recursion on U and V would be desirable. However, there apparently are problems in this approach. Consider how we might want to get from Q^0 to Q^2 by equation (6):

$$Q^1 = (I + A_2 \; S \; T)^{-1} (I - A_2 \; S \; T) \; Q^0$$

then,

$$Q^2 = (I + A_2 \; S \; T)^{-1} (I - A_2 \; S \; T) (I + A_2 \; S \; T)^{-1} (I - A_2 \; S \; T) \; Q^0 \qquad (8)$$

Now, if $(I + A_2 \; S \; T)^{-1}$ and $(I - A_2 \; S \; T)$ were to commute, then we could write:

$$(I + A_2 \; S \; T)^N \; Q^N = (I - A_2 \; S \; T)^N \; Q^0$$

If this were the case, then we could develop an algorithm as for the one dimensional case: however, the two terms in question do not commute. We might be lead to try to expand $(I + A_2 \; S \; T)^{-1}$ in a power series:

$$(I + A_2 \; S \; T)^{-1} = I - (A_2 \; S \; T) + (A_2 \; S \; T)^2 - (A_2 \; S \; T)^3 + \cdots$$

Disregarding questions of convergence, etc., we see to substitute into equation (8) so that we have products like:

$$[I - (A_2 S \; T) + (A_2 S \; T)^2 - (A_2 S \; T)^3 + \cdots] \; [I - A_2 S \; T]$$

All the terms now commute, but we have terms like T^k with large k : there is no gain in efficiency, even if the power series is valid.

We leave economic techniques for our two dimensional problem as an open question.

4.1.III. Electromagnetic Waves: Further Frequency Domain Techniques

Equation (15) of part I is: (TM; gradient in x,z for σ ; μ constant):

$$B_{zz} + B_{xx} - \frac{\sigma_x}{\sigma} B_x - \frac{\sigma_z}{\sigma} B_z = - \ell^2 B \tag{1}$$

$$\ell = (i \omega \mu \sigma)^{1/2}$$

let

$$B = Q e^{i\bar{\ell}z}$$

$$B_x = Q_x e^{i\bar{\ell}z}$$

$$B_{xx} = Q_{xx} e^{i\bar{\ell}z}$$

$$B_z = (Q_z + i \bar{\ell} Q) e^{i\bar{\ell}z}$$

$$B_{zz} = (Q_{zz} + 2 i \bar{\ell} Q_z - \bar{\ell}^2 Q) e^{i\bar{\ell}z}$$

Substituting into (1):

$$(Q_{zz} + 2 i \bar{\ell} Q_z - \bar{\ell}^2 Q) + Q_{xx} - \frac{\sigma_x}{\sigma} Q_x - \frac{\sigma_z}{\sigma} (Q_z + i \bar{\ell} Q) + \ell^2 Q = 0$$

Dropping Q_{zz} and rearranging:

$$(2 i \bar{\ell} - \frac{\sigma_z}{\sigma}) Q_z + Q_{xx} - \frac{\sigma_x}{\sigma} Q_x + (\ell^2 - \bar{\ell}^2 - i \bar{\ell} \frac{\sigma_z}{\sigma}) Q = 0 \tag{2}$$

Now

$$Q = B e^{-i\bar{\ell}z}$$

$$Q_x = B_x e^{-i\bar{\ell}z}$$

$$Q_{xx} = B_{xx} e^{-i\bar{\ell}z}$$

$$Q_z = (B_z - i \bar{\ell} B) e^{-i\bar{\ell}z}$$

Substituting into equation (2):

$$\left(2\, i\bar{\ell} - \frac{\sigma_z}{\sigma} \right)\left(B_z - i\bar{\ell} B \right) + B_{xx} - \frac{\sigma_x}{\sigma} B_x + \left(\ell^2 - \bar{\ell}^2 - i\bar{\ell}\frac{\sigma_z}{\sigma}\right)B = 0$$

Rearranging:

$$\left(\frac{\sigma_z}{\sigma} - 2\, i\bar{\ell} \right) B_z = B_{xx} - \frac{\sigma_x}{\sigma} B_x + \left(\ell^2 + \bar{\ell}^2 \right) B \tag{3}$$

Finally, we take the important step of replacing $\bar{\ell}$ by ℓ. This is justified because dropping Q_{zz} was a "local" approximation, i.e. it depended on the "goodness" of $e^{i\bar{\ell}z}$ locally to make Q_{zz} small. Using ℓ instead of $\bar{\ell}$ in (3) should locally improve the approximation:

$$\left(\frac{\sigma_z}{\sigma} - 2\, i\ell \right) B_z = B_{xx} - \frac{\sigma_x}{\sigma} B_x + 2\,\ell^2 B \tag{4}$$

Example G gives a program for equation (4).

By dropping Q_{zz} to get equation (2) we are limiting the validity of our calculations to propagation in directions that are close to the z-axis. To increase this range of angles we may solve (2) for Q_z, then differentiate to find an approximation to the Q_{zz} term that was dropped. For simplicity let:

$$\begin{aligned} A &= \frac{\sigma_z}{\sigma} - 2\, i\,\bar{\ell} \\ E &= \ell^2 - \bar{\ell}^2 - i\,\bar{\ell}\,\frac{\sigma_z}{\sigma} \\ C &= \frac{\sigma\sigma_{xz} - \sigma_x\sigma_z}{\sigma^2} \\ D &= \frac{\sigma\sigma_{zz} - \sigma_z\sigma_z}{\sigma^2} \end{aligned} \tag{5}$$

Then we substitute the approximation to Q_{zz} into equation (4); using equations (5) and rearranging, we get:

$$Q_z\left(A - \frac{E}{A}\right) + Q_{xz}\left(\frac{\frac{\sigma_x}{\sigma}}{A}\right) + Q_{xxz}\left(-\frac{1}{A}\right) = Q_{xx}\left(1 - \frac{D}{A^2}\right) + Q_x\left(\frac{\sigma_x}{\sigma}\left\{\frac{D}{A^2} - 1\right\} - \frac{C}{A}\right)$$

$$+ Q\left(E + \frac{2\ell\ell_z - i\bar{\ell}D}{A} - \frac{ED}{A^2}\right) \qquad (6)$$

This equation could be solved efficiently. However, we choose to follow through with the development at the beginning of Part III in order to increase accuracy. We will go back to an equation in B and then set the $\bar{\ell}$'s to ℓ's . First we write:

$$Q = B e^{-i\bar{\ell}z}$$

$$Q_{xz} = (B_{xz} - i\bar{\ell} B_x)e^{-i\bar{\ell}z}$$

$$Q_{xxz} = (B_{xxz} - i\bar{\ell} B_{xx})e^{-i\bar{\ell}z}$$

Q_x , Q_{xx} , and Q_z are as before.

Substituting into equation (6) gives:

$$\left(A - \frac{E}{A}\right)(B_z - i\bar{\ell} B) + \left(\frac{\frac{\sigma_x}{\sigma}}{A}\right)(B_{xz} - i\bar{\ell} B_x) + \left(-\frac{1}{A}\right)(B_{xxz} - i\bar{\ell} B_{xx}) =$$

$$\left(1 - \frac{D}{A^2}\right)(B_{xx}) + \left(\frac{\sigma_x}{\sigma}\left\{\frac{D}{A^2} - 1\right\} - \frac{C}{A}\right) (B_x) + \left(E + \frac{2\ell\ell_z - i\bar{\ell}D}{A} - \frac{ED}{A^2}\right) B$$

Rearranging and setting $\bar{\ell}$ to ℓ we get:

$$B_z\left(A - \frac{E}{A} \right) + B_{xz}\left(\frac{\frac{\sigma_x}{\sigma}}{A} \right) + B_{xxz}\left(-\frac{1}{A} \right) = B_{xx}\left(1 - \frac{D}{A^2} - \frac{i\ell}{A} \right)$$

$$+ B_x\left(\frac{\sigma_x}{\sigma} \left[\frac{D}{A^2} - 1 + \frac{i\ell}{A} \right] - \frac{C}{A} \right) + B\left(E + \frac{\ell}{A} \left[2\ell_z - iD \right] - \frac{ED}{A^2} + i\ell \left[A - \frac{E}{A} \right] \right) \qquad (7)$$

Now let's consider the problem of multiply reflected waves as in a layered structure. We can first solve for downgoing propagation; then we consider:

$$B = B^+ + B^-$$

$$= Q^+ e^{i\bar{\ell}z} + Q^- e^{-i\bar{\ell}z}$$

We are writing B as the sum of its upgoing and downgoing parts, each as a perturbation about a complex exponential. Next:

$$B_z = (Q^+_z + i \bar{\ell} Q^+) e^{i\bar{\ell}z} + (Q^-_z - i\bar{\ell} \bar{Q}) e^{-i\bar{\ell}z}$$

$$B_{zz} = (Q^+_{zz} + 2 i\bar{\ell} Q^+_z - \bar{\ell}^2 Q^+) e^{i\bar{\ell}z} + (Q^-_{zz} - 2i\bar{\ell} Q^-_z - \bar{\ell}^2 Q^-) e^{-i\bar{\ell}z}$$

$$B_x = Q^+_x e^{i\bar{\ell}z} + Q^-_x e^{-i\bar{\ell}z}$$

$$B_{xx} = Q^+_{xx} e^{i\bar{\ell}z} + Q^-_{xx} e^{-i\bar{\ell}z}$$

Substituting into equation 15 of Part I,

$$[(Q^+_{zz} + 2 i\bar{\ell} Q^+_z - \bar{\ell}^2 Q^+) e^{i2\bar{\ell}z} + (Q^-_{zz} - 2 i \bar{\ell} Q^-_z - \bar{\ell}^2 Q^-)]$$

$$+ [Q^+_{xx} e^{i2\bar{\ell}z} + Q^-_{xx}] - \frac{\sigma_x}{\sigma} [Q^+_x e^{i2\bar{\ell}z} + Q^-_x]$$

$$- \frac{\sigma_z}{\sigma}[(Q^+_z + i\bar{\ell} Q^+)e^{i2\bar{\ell}z} + (Q^-_z - i\bar{\ell} Q^-)] = - \ell^2[Q^+ e^{i2\bar{\ell}z} + Q^-]$$

Dropping Q^-_{zz} and all terms in Q^+ except Q^+_{zz} (in other words we drop those terms in Q^+ for which we have solved when finding the downgoing wave) :

$$Q^+_{zz}\ e^{i2\bar{\ell}z} + Q^-_z\left(- 2i\bar{\ell} - \frac{\sigma_z}{\sigma} \right) + Q^-_{xx} + Q^-_x\ \left(- \frac{\sigma_x}{\sigma} \right)$$

$$+ Q^-\left(\ell^2 - \bar{\ell}^2 + i\bar{\ell}\,\frac{\sigma_z}{\sigma} \right) = 0$$

This is then an equation for the upward propagation with a source term that involves Q^+_{zz} .

Now

$$Q^- = B^-\ e^{i\bar{\ell}z}$$

$$Q^-_z = \left(B^-_z + i\bar{\ell}\ B^- \right)\ e^{i\bar{\ell}z}$$

$$Q^-_{zz} = \left(B^-_{zz} + 2\ i\bar{\ell}\ B^- - \bar{\ell}^2\ B^- \right)\ e^{i\bar{\ell}z}$$

also

$$Q^+ = B^+\ e^{-i\bar{\ell}z}$$

$$Q^+_{zz} = \left(B^+_{zz} - 2\ i\bar{\ell}\ B^+_z - \bar{\ell}^2\ B^+ \right)\ e^{-i\bar{\ell}z}$$

Substituting into equation (8) gives:

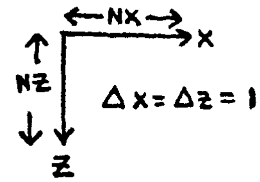
$$\left(2\ i\ \bar{\ell} + \frac{\sigma_z}{\sigma} \right)\ B^-_z = B^-_{xx} - \frac{\sigma_x}{\sigma}\ B^-_x + B^-\left(\ell^2 + \bar{\ell}^2 \right)$$

$$+ \left(B^+_{zz} - 2\ i\bar{\ell}\ B^+_z - \bar{\ell}^2\ B^+ \right) \qquad (9)$$

Finally, we again set the $\bar{\ell}$'s to ℓ . A program and some comments on the problems encountered in this approach are found in Example I.

Example A: Frequency Domain Finite Difference Technique for T.E. or T.M. with Constant μ and σ .

The program uses an implicit scheme with 2 levels in z (Crank Nicolson):




```
      IMPLICIT COMPLEX (C)
      DIMENSION CD ( 40), CE ( 40), CF ( 40), CQ ( 40), IPLOT ( 40)
      READ (5,1000) NX, NZ, OMEGA, RMU, SIGMA
      WRITE (6,1000) NX, NZ, OMEGA, RMU, SIGMA
 1000 FORMAT (1X, I9, I10, 3F10.5)
      PROD= OMEGA*RMU*SIGMA
      CA= -CMPLX(.3535535,.3535535) / SQRT(PROD)
      CC = CA
      CB = 1.0-2.0*CA
      N1 = NX-1
         DO 20 I=1,15
         CQ(I) = 99.0
   20    IPLOT(I) = 99               }  Initial conditions
         DO 21 I=16,NX
         CQ(I) = 0.0
   21    IPLOT(I) = 0
      WRITE (6,2000) (IPLOT(I),I=1,NX)
 2000 FORMAT (/1X,40I3)
      CXX= CMPLX(-.707107,.707107)*SQRT(PROD)
         DO 10 J=1,NZ
         CD(1) = -CA*CQ(2)+(1.0+2.0*CA)*CQ(1)
         CD(NX) = (1.0+2.0*CA)*CQ(NX)-CA*CQ(N1)
            DO 30 I=2,N1
   30       CD(I) = -CA*CQ(I+1)+(1.0+2.0*CA)*CQ(I)-CA*CQ(I-1)
         CALL TRI2 (CA,CB,CC,NX,CQ,CD,CE,CF)
         CWMS = J*CXX
            DO 40 K=1,NX
   40       IPLOT(K) = REAL(CQ(K)*CEXP(CWMS) )  }  Plot real part
   10    WRITE (6,3000) (IPLOT(I),I=1,NX)           of solution
 3000 FORMAT (1X,40I3)
      RETURN
      END
      SUBROUTINE TRI2 (A,B,C,N,T,D,E,F)
C
C     TRIDIAGONAL SOLUTION WITH ZERO SLOPE BOUNDARY CONDITIONS.
C
      COMPLEX A, B, C, T, D, E, F, DEN
      DIMENSION T ( N), D ( N), E ( N), F ( N)
      N1 = N-1
      E(1) = 1.0
      F(1) = 0.0
         DO 10 I=2,N1
         DEN = B + C*E(I-1)
         E(I) = -A/DEN
   10    F(I) = (D(I) - C*F(I-1) ) / DEN
      T(N) = F(N1) / (1.0-E(N1) )
         DO 20 J=1,N1
         I = N-J
   20    T(I) = E(I)*T(I+1) + F(I)
      RETURN
      END
```

Sample output from program above:

This output is a plot of amplitudes of a transverse E or H component (in and out of plane of paper). It represents the solution for one frequency over the x , z plane. The top row represents the boundary conditions.

```
NX= 30     NZ= 30 ω=1.00000 μ=1.00000 σ=0.01000

99 99 99 99 99 99 99 99 99 99 99 99 99 99 99  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
92 92 92 92 92 92 91 91 89 87 82 75 63 45 19 72 46 28 16  9  4  2  0  0  0  0  0  0  0  0
86 86 86 86 85 84 81 78 74 68 62 55 51 53 67 17 32 33 29 23 16 10  6  3  1  0 -1 -1 -1 -1
80 80 79 78 75 73 69 66 62 59 57 55 53 46 25 53 32 24 22 21 19 15 12  8  5  2  0 -1 -1 -1
70 70 69 68 66 64 62 60 57 55 51 46 41 38 49 22 33 30 24 19 16 13 11  9  7  5  3  2  1  1
60 60 60 59 59 57 56 53 51 47 44 42 42 40 26 39 24 22 22 20 17 14 11  9  7  6  5  4  4  4
53 53 53 52 51 50 48 47 45 43 42 39 35 30 36 22 28 23 19 16 15 13 11 10  8  7  6  5  5  5
47 47 46 46 45 44 43 42 40 38 35 33 32 32 24 28 20 20 19 17 14 12 10  9  8  7  7  6  6  6
40 40 40 40 39 39 37 36 35 33 32 31 28 24 26 21 23 18 16 14 13 12 10  9  8  7  6  6  6  6
35 35 35 35 34 33 33 32 31 29 28 25 24 25 21 20 16 17 16 14 12 10  9  9  8  7  6  6  6  6
31 31 30 30 30 29 28 27 26 25 24 23 22 19 18 18 17 14 13 12 11 10  9  8  7  6  6  6  6  6
26 26 26 26 26 25 24 24 23 22 21 19 18 18 17 14 13 14 12 10  9  8  8  7  6  6  5  5  5  5
23 23 22 22 22 21 21 20 20 19 18 17 17 15 13 14 12 10 10  9  8  7  7  6  6  5  5  5  4  4
19 19 19 19 19 18 18 17 17 16 15 14 13 13 14  9 10 10  9  7  7  6  6  5  5  4  4  4  4  4
16 16 16 16 16 15 15 15 14 13 13 12 12 11  8 11  8  7  7  7  6  5  5  4  4  4  3  3  3  3
13 13 13 13 13 13 12 12 11 11 11 10  9  8 10  6  7  7  6  5  5  4  4  3  3  3  3  2  2  2
11 11 11 11 11 10 10 10 10  9  8  8  8  8  5  8  5  5  5  4  4  3  3  2  2  2  2  2  2  2
 9  9  9  9  9  8  8  8  7  7  7  7  6  5  7  3  5  4  3  3  3  2  2  2  1  1  1  1  1  1
 7  7  7  7  7  7  6  6  6  6  5  5  5  5  2  5  2  2  2  2  2  1  1  1  1  0  0  0  0  0
 5  5  5  5  5  5  5  5  4  4  4  4  3  2  4  1  2  2  1  1  1  1  0  0  0  0  0  0  0  0
 4  4  4  4  4  3  3  3  3  3  3  2  2  2  1  2  0  1  1  0  0  0  0  0  0  0  0  0  0  0
 3  3  2  2  2  2  2  2  2  2  1  1  1  0  1  0  1  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1
 1  1  1  1  1  1  1  1  1  1  1  0  0  0  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1 -1 -1 -1
 0  0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1  0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -2 -2
 0  0  0  0  0  0  0  0  0  0  0  0  0  0 -1  0 -1 -1 -1 -1 -1 -1 -2 -2 -2 -2 -2 -2 -2 -2
 0  0  0  0  0  0  0  0  0 -1 -1 -1 -1 -1 -1 -2 -1 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
-1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -2 -1 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2
-1 -1 -1 -1 -1 -1 -1 -1 -1 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -3 -3 -3 -3 -3 -3 -3
-2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3
-2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -3 -2 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3
-2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -2 -3 -2 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3 -3
```

Example B: Frequency Domain Finite Difference Technique for T.M. with μ constant and σ a function of x, z .

Equation (17) of section I in finite difference notation with superscripts on Q for z-coordinate and subscripts on Q for x-coordinate:

$$\underbrace{\left[- \frac{2i\ell}{\Delta z} + \frac{\partial_z \sigma}{\sigma \Delta z} \right]}_{C1} \left(Q_x^{N+1} - Q_x^{N} \right) = \underbrace{\left[\frac{1}{2\Delta x^2} \right]}_{C2} \left(Q_{x+1}^{N} - 2\, Q_x^{N} + Q_{x-1}^{N} + Q_{x+1}^{N+1} - 2\, Q_x^{N+1} + Q_{x-1}^{N+1} \right)$$

$$+ \underbrace{\left[\frac{i}{2\Delta x} \int \partial_x \ell \, dz - \frac{\partial_x \sigma}{4\sigma\Delta x} \right]}_{C3} \left(Q_{x+1}^{N} - Q_{x-1}^{N} + Q_{x+1}^{N+1} - Q_{x-1}^{N+1} \right)$$

$$+ 1/2 \underbrace{\left[i\, \partial_z \ell + i \int \partial_{xx} \ell \, dz - \left\{ \int \partial_x \ell \, dz \right\}^2 - \frac{\partial_x \sigma}{\sigma}\, i \int \partial_x \ell \, dz - \frac{\partial_z \sigma}{\sigma}\, i\ell \right]}_{C4} \left(Q_x^{N} + Q_x^{N+1} \right)$$

C1, ..., C4 are as used in the program; rearranging, we get the tridiagonal system:

$$[- C2 + C3]\, Q_{x-1}^{N+1} + [C1 + 2 * C2 - C4]\, Q_x^{N+1} + [- C2 - C3]\, Q_{x+1}^{N+1}$$

$$= [C2 - C3]\, Q_{x-1}^{N} + [C1 - 2 * C2 + C4]\, Q_x^{N} + [C2 + C3]\, Q_{x+1}^{N}$$

There is one final detail to be taken care of: we have a one way wave equation and we have "secondary sources" at the contrasts in σ . It is necessary to halve these secondary sources since we are modeling only that secondary energy propagated in one direction.

```
$WATFIV
      IMPLICIT COMPLEX (C)
      REAL CABS
      DIMENSION SIG ( 30, 40), SIGX ( 30), SIGZ( 30), CL ( 30, 2),
     * CLZ ( 30), CLS ( 30), CLXS ( 30), CLXXS ( 30), CQ ( 30),
     * CP ( 30, 40), CA ( 30), CB ( 30), CC ( 30), CD ( 30), CE ( 30),
     * CF ( 30), IPLOT ( 30), CLX ( 30), CLXX ( 30)
      READ (5,1000) NX, NZ, DX, DZ, RMU, OMEG
      WRITE (6,1000) NX, NZ, DX, DZ, RMU, OMEG
 1000 FORMAT (1X,G9.4, 7G10.4)
      CALL SIGIN (NX,NZ,SIG)
      NX1 = NX-1
      DDX = 1.0/DX
      DDDX = DDX*DDX
      DDZ = 1.0/DZ
      RMOG = RMU*OMEG
      CI = CMPLX(0.0,1.0)
         DO 5 IX=1,NX
         CQ(IX) = 99.0
         CP(IX,1) = CQ(IX)
         CLS(IX) = (0.0,0.0)
         CLXS(IX) = (0.0,0.0)
         CLXXS(IX) = (0.0,0.0)
         DUM = RMOG*SIG(IX,1)
    5    CL(IX,2) = CSQRT(CMPLX(0.0,DUM) )
         DO 10 IZ=2,NZ
            DO 20 IX=2,NX1
   20       SIGX(IX) = (SIG(IX+1,IZ)+SIG(IX+1,IZ-1)-SIG(IX-1,IZ)
     *      -SIG(IX-1,IZ-1) ) *.25*DDX
         SIGX(1) = 0.0
         SIGX(NX) = 0.0
            DO 30 IX=1,NX
            SIGZ(IX) =(SIG(IX,IZ)-SIG(IX,IZ-1))*DDZ
            CL(IX,1) = CL(IX,2)
            DUM = RMOG*SIG(IX,IZ)
            CL(IX,2) = CSQRT(CMPLX(0.0,DUM) )
   30       CLZ(IX) = CL(IX,2)-CL(IX,1)
            DO 40 IX=2,NX1
            CLX(IX) = (CL(IX+1,1)+CL(IX+1,2)-CL(IX-1,1)-CL(IX-1,2) )
     *      *.25*DDX
   40       CLXX(IX) = (CL(IX+1,1)+CL(IX+1,2)-CL(IX,1)-CL(IX,1)-CL(IX,2)
     *      -CL(IX,2)+CL(IX-1,1)+CL(IX-1,2) ) *.5*DDDX
         CLX(1) = (0.0,0.0)
         CLX(NX) = (0.0,0.0)
         CLXX(1) = (0.0,0.0)
         CLXX(NX) = (0.0,0.0)
            DO 50 IX=1,NX
            CLS(IX) = CLS(IX)+.5*(CL(IX,1)+CL(IX,2) )*DZ
            CLXS(IX) = CLXS(IX)+CLX(IX)*DZ
            CLXXS(IX) = CLXXS(IX)+CLXX(IX)*DZ
C     ******** IN COL 73-80 DENOTES TERMS WITH .5 FACTORS FOR HALF SOURCE
            SIGAV1 = 2.0 / (SIG(IX,IZ)+SIG(IX,IZ-1) )
            C1 = -CI*(CL(IX,1)+CL(IX,2))*DDZ + SIGZ(IX)*DDZ*.5*SIGAV1   *****
            C2 = 0.5*DDDX
            C3 = .5*DDX*(CI*CLXS(IX)-.5*SIGX(IX)*SIGAV1)
            C4 = .5*CI*(CLZ(IX)+CLXXS(IX)-(SIGX(IX)*SIGAV1 )
     *      *CLXS(IX)-(SIGZ(IX) * SIGAV1    )*.5*(CL(IX,1)+CL(IX,2) ) )
     *      -.5*CLXS(IX)**2
            C3 = .5*C3                                                  *****
            C4 = .5*C4                                                  *****
```

```
               CA(IX) = -C2+C3
               CB(IX) = C1+C2+C2-C4
               CC(IX) = -C2-C3
               IF(IX.EQ.1 .OR. IX.EQ.NX) GO TO 60
               CD(IX) = (C2-C3)*CQ(IX-1)      +(C1-C2-C2+C4)*CQ(IX)
     *         +(C2+C3)*CQ(IX+1)
   60          IF(IX.EQ.1) CD(1) = (C1-C2-C2+C4)*CQ(IX)+(C2+C3)*CQ(IX+1)
   50          IF (IX.EQ.NX) CD(NX) = (C2-C3)*CQ(IX-1)+(C1-C2-C2+C4)*CQ(IX)
            CALL TRI3(CA,CB,CC,NX,CQ,CD,CE,CF)
               DO 10 IX=1,NX
   10          CP(IX,IZ) = CQ(IX)*CEXP(CI*CLS(IX) )
            DO 70 IZ=1,NZ
               DO 75 IX=1,NX
   75          IPLOT(IX) = CABS(CP(IX,IZ) )
   70       WRITE (6,3000)(IPLOT(IX), IX=1,NX)
 3000 FORMAT (1X,40I3)
      RETURN
      END
      SUBROUTINE SIGIN (NX,NZ,SIG)
      DIMENSION SIG ( 1, 1)
      READ (5,1000) S1,S2
      WRITE (6,1000) S1, S2
 1000 FORMAT (1X,F9.6,7F10.6)
         DO 10 IZ=1,NZ
            DO 20 IX=1,15
   20       SIG(IX,IZ) = S1
            DO 10 IX=16,NX
   10       SIG(IX,IZ) = S2
      RETURN
      END
      SUBROUTINE TRI3 (A,B,C,N,T,D,E,F)
C
C     TRIDIAGONAL SOLUTION WITH ZERO SLOPE BOUNDARY CONDITIONS AND
C     NONCONSTANT ENTRIES ALONG THE DIAGONALS.
C
      COMPLEX A, B, C, T, D, E, F, DEN
      DIMENSION T ( N), D ( N), E ( N), F ( N), A ( N), B ( N), C ( N)
      N1 = N-1
      E(1) = 1.0
      F(1) = 0.0
         DO 10 I=2,N1
         DEN = B(I) + C(I)*E(I-1)
         E(I) = -A(I)/DEN
   10    F(I) = (D(I) - C(I)*F(I-1) ) / DEN
      T(N) = F(N1) / (1.0-E(N1) )
         DO 20 J=1,N1
         I = N-J
   20    T(I) = E(I)*T(I+1) + F(I)
      RETURN
      END
```

The values here are complex amplitudes over the x , z plane for two abutting quarter spaces of conductivities .01 on left and .015 on right. The contours should decrease monotonically from right to left. The excessive "droop" of the contours at, and to the left of the vertical contact appears to be due to "incorrect" propagation of energy from secondary sources along the contact. Essentially, the method in this Example doesn't work.

NX=30 NZ=40 1.000 1.000 1.000 1.000
0.010000 0.015000 DX DZ μ ω

99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99
92	92	92	92	92	92	92	92	92	92	92	92	92	92	91	91	91	91	90	90	90	90	90	90	90	90	90	90	90	90
85	85	85	85	85	85	85	85	85	85	85	85	85	85	85	84	84	83	83	83	83	83	83	83	83	83	83	83	83	83
80	80	80	80	80	80	80	79	79	79	79	79	79	79	79	78	77	77	77	76	76	76	76	76	76	76	76	76	76	76
74	74	74	74	74	74	74	74	74	74	74	74	74	74	74	72	71	71	71	70	70	70	70	70	70	70	70	70	69	69
69	69	69	69	69	69	69	69	69	69	69	69	69	69	69	67	66	66	65	65	65	64	64	64	64	64	64	64	64	64
64	64	64	64	64	64	64	64	64	64	64	64	65	65	65	62	61	61	60	60	60	59	59	59	59	59	58	58	58	58
60	60	60	60	60	60	60	60	60	60	60	60	60	61	61	57	57	56	56	55	55	55	54	54	54	54	54	54	54	54
56	56	56	56	56	56	56	56	56	56	56	56	57	57	57	53	53	52	52	51	51	50	50	50	50	49	49	49	49	49
52	52	52	52	52	52	52	52	52	52	53	53	53	53	53	50	49	48	48	47	47	47	46	46	46	46	45	45	45	45
48	48	48	48	48	48	49	49	49	49	49	49	50	50	50	46	46	45	44	44	43	43	43	42	42	42	42	42	42	42
45	45	45	45	45	45	45	46	46	46	46	46	47	47	47	43	42	42	41	41	40	40	39	39	39	39	38	38	38	38
42	42	42	42	42	42	42	43	43	43	43	44	44	44	44	40	40	39	38	38	37	37	37	36	36	36	35	35	35	35
39	39	39	39	39	40	40	40	40	40	41	41	41	42	42	38	37	36	36	35	35	34	34	33	33	33	33	33	33	33
37	37	37	37	37	37	37	37	38	38	38	39	39	39	39	35	35	34	33	33	32	32	31	31	31	30	30	30	30	30
34	34	34	34	35	35	35	35	35	36	36	36	37	37	37	33	32	32	31	31	30	30	29	29	28	28	28	28	28	28
32	32	32	32	32	33	33	33	33	34	34	34	35	35	35	31	30	30	29	28	28	28	27	27	26	26	26	26	26	26
30	30	30	30	30	31	31	31	31	32	32	32	33	33	33	29	28	28	27	27	26	26	25	25	25	24	24	24	24	24
28	28	28	29	29	29	29	29	30	30	30	31	31	31	32	27	27	26	25	25	24	24	24	23	23	23	22	22	22	22
27	27	27	27	27	27	27	28	28	28	29	29	29	30	30	26	25	24	24	23	23	22	22	22	21	21	21	21	21	21
25	25	25	25	25	26	26	26	26	27	27	27	28	28	28	24	23	23	22	22	21	21	21	20	20	20	20	19	19	19
24	24	24	24	24	24	24	25	25	25	26	26	26	27	27	23	22	21	21	20	20	20	19	19	19	18	18	18	18	18
23	23	23	23	23	23	23	23	24	24	24	25	25	25	26	21	21	20	20	19	19	18	18	18	17	17	17	17	17	17
21	21	21	21	22	22	22	22	22	23	23	23	24	24	24	20	19	19	18	18	18	17	17	17	16	16	16	16	16	16
20	20	20	20	21	21	21	21	21	22	22	22	22	23	23	19	18	18	17	17	17	16	16	16	15	15	15	15	15	15
19	19	19	19	20	20	20	20	20	21	21	21	21	22	22	18	17	17	16	16	16	15	15	15	14	14	14	14	14	14
18	18	18	19	19	19	19	19	19	20	20	20	20	21	21	17	16	16	15	15	15	14	14	14	14	13	13	13	13	13
18	18	18	18	18	18	18	18	18	19	19	19	19	20	20	16	15	15	14	14	14	13	13	13	13	13	13	12	12	12
17	17	17	17	17	17	17	17	18	18	18	18	18	19	19	15	14	14	14	13	13	13	12	12	12	12	12	12	12	12
16	16	16	16	16	16	16	17	17	17	17	17	17	18	18	14	14	13	13	12	12	12	12	12	11	11	11	11	11	11
15	15	15	15	16	16	16	16	16	16	16	16	17	17	17	13	13	12	12	12	11	11	11	11	11	11	11	11	11	11
15	15	15	15	15	15	15	15	15	15	15	16	16	16	16	12	12	12	11	11	11	11	10	10	10	10	10	10	10	10
14	14	14	14	14	14	14	14	14	14	15	15	15	15	15	12	11	11	10	10	10	10	10	10	10	10	10	10	10	10
14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	11	10	10	10	10	9	9	9	9	9	9	9	9	9	9
13	13	13	13	13	13	13	13	13	13	13	13	13	13	14	10	10	9	9	9	9	9	9	9	9	9	9	9	9	9
13	13	13	13	13	13	12	12	12	12	12	12	12	12	13	9	9	9	8	8	8	8	8	8	8	8	8	8	8	8
12	12	12	12	12	12	12	12	12	12	11	11	11	12	12	9	8	8	8	8	8	8	8	8	8	8	8	8	8	8
12	12	12	12	12	11	11	11	11	11	11	11	11	11	11	8	8	7	7	7	7	7	7	7	7	7	7	7	7	7
11	11	11	11	11	11	11	10	10	10	10	10	10	10	10	7	7	7	6	6	6	6	7	7	7	7	7	7	7	7
11	11	11	11	11	10	10	10	10	9	9	9	9	9	9	7	6	6	6	6	6	6	6	6	6	6	7	7	7	7

Example C: Time Domain Coefficients for Half Order Derivative and Integral.

The time domain representation of the half order operators is obtained by matching coefficients of various powers of the complex variable z ; e.g.:

$$\left(\frac{1}{(-i\omega\Delta t)^{1/2}} \right)^2 = \frac{1+z}{2(1-z)}$$

letting $SI(z)$ be the half order integral operator:

$$2\,(1-z)\,[(1 + SI_1\, z + SI_2\, z^2 + \cdots)^2] = 1 + z$$

or, letting $b_i = \frac{1}{\sqrt{2}} SI_i$,

$$(1 - z)\,[\sum_{k=0}^{\infty} z^k \,(\sum_{i=0}^{k} b_{k-i}\; b_i)] = 1 + z$$

Matching coefficients of z^0 and z gives $b_0 = b_1 = 1$
For z^k , $k > 1$, we have:

$$\sum_{i=0}^{k} b_{k-i}\; b_i - \sum_{i=0}^{k-1} b_{k-1-i}\; b_i = 0$$

Rearranging:

$$b_k\, b_0 + (b_k - b_{k-1})\, b_0 + \sum_{i=1}^{k-1} (b_{k-i} - b_{k-1-i})\, b_i = 0$$

With b_0 known we have:

$$b_k = \frac{1}{2}\,[\, b_{k-1} - \sum_{i=1}^{k-1} (b_{k-i} - b_{k-1-i})\, b_i \,]$$

Given the coefficients, SI_i , for the half order integral operator, we can find the half order derivative:

$$\frac{1}{(-i\omega\Delta t)^{1/2}}\,(-i\,\omega\,\Delta t) = (i\,\omega\,\Delta t)^{1/2}$$

or, in terms of z transforms:

$$(SI_0 + SI_1 z + SI_2 z^2 + \cdots)\, 2 \frac{1-z}{1+z} = (SD_0 + SD_1 z + SD_2 z^2 + \cdots)$$

Matching coefficient of z^0 after multiplying through by $(1 + z)$ gives

$$SD_0 = \sqrt{2}$$

The coefficients of z^k, $k \geqslant 1$ give:

$$2\,(SI_k - SI_{k-1}) = SD_k + SD_{k-1}$$

thus:

$$SD_k = -SD_{k-1} + 2\,(SI_k - SI_{k-1})$$

Program is:

```
      DIMENSION SI(20),SD(20),SSI(39),SSD(39)
      DATA SSI,SSD/78*0./
      LS=10
      LSS=2*LS-1
      CALL ROUGH(LS,SI,SD)
      DO 10 I=1,LS
      DO 10 J=1,LS
      K=I+J-1
      SSI(K)=SSI(K)+SI(I)*SI(J)
   10 SSD(K)=SSD(K)+SD(I)*SI(J)
      PRINT 20,(SI(I),SSI(I),SD(I),SSD(I),I=1,LS)
   20 FORMAT(4F9.4)
      STOP
      END
      SUBROUTINE ROUGH(N,SI,SD)
C     COEFFICIENTS OF SI(Z)*SI(Z)=.5,1.,1.,1.,1...
C     COEFFICIENTS OF SD(Z)*SD(Z) = 2*(1-Z)/(1+Z)
      DIMENSION SI(N),SD(N)
      SI(1)=1.
      SI(2)=1.
      DO 20 K=3,N
      NSUM=K-1
      SUM=0.
      DO 10 I=2,NSUM
   10 SUM=SUM+(SI(K-I+1)-SI(K-I))*SI(I)
   20 SI(K)=(SI(K-1)-SUM)/2.
      SQRT2=SQRT(2.)
```

TEST PROGRAM

```
      DO 30 K=1,N
   30 SI(K)=SI(K)/SQRT2
      SD(1)=SQRT2
      DO 40 K=2,N
   40 SD(K)=-SD(K-1)+2*(SI(K)-SI(K-1))
      RETURN
      END
```

SI	$DATA	SD	
0.7071	0.5000	1.4142	1.0000
0.7071	1.0000	-1.4142	0.0000
0.3536	1.0000	0.7071	-0.0000
0.3536	1.0000	-0.7071	0.0000
0.2652	1.0000	0.5303	-0.0000
0.2652	1.0000	-0.5303	0.0000
0.2210	1.0000	0.4419	-0.0000
0.2210	1.0000	-0.4419	0.0000
0.1933	1.0000	0.3867	-0.0000
0.1933	1.0000	-0.3867	0.0000

Example D: "Expensive" Time Domain Technique for $Q_z = i \ell Q$ and Comparison to Analytic Green's Functions

The analytic Green's function (Morse & Feshbach, p. 894) is given by the statement function, G , and the "expensive" solution for the same one dimensional diffusion problem is given in the subroutine G 1. However, a direct compariosn is not possible (due to differences in definition for Green's function).

Thus we compare Feshbach's function, G_F , at $z = 2$ with the result of propagating Feshbach's function at $z= 1$ across the interval $z=1$ to $z=2$ by our method:

$$G_F\ (z=2) \ = \ [G_F(z=1)] * [G_1(z=1)]$$

where G_1 is our function. The comparison is good as shown by the example. Feshbach's function is defined such that

$$[\ G_F(z=1)\ ] * [\ G_F(z=1)\ ] \neq [\ G_F(z=2)\ ]$$

as we discovered by a numerical convolution and as undoubtedly can be verified theoretically.

Program:

```
      DIMENSION T1(100),TN(100),P(100),PN(100),SI(100),SD(100)
      G(IX,IT)=SQRT(3.1416/IT)*2/A0*EXP(-A0*A0*IX*IX/(4*IT))
      A0=3.1416
      NT=30
      NZ=4
      T1(1)=0.
      TN(1)=0.
      P(1)=0.
      DO 10 IT=2,NT
      T1(IT)=G(1,IT-1)
      TN(IT)=G(2,IT-1)
   10 CONTINUE
      DZ=1./NZ
      CALL G1(NT,P,NZ,DZ,A0)
      DO 70 IT=1,NT
      DOT=0.
```

```
      DO 60 IS=1,IT
60    DOT=DOT+P(IS)*T1(IT-IS+1)
70    PN(IT)=DOT
      PRINT 40,(T1(IT),P(IT),TN(IT),PN(IT),IT=1,NT)
40    FORMAT(4F12.5)
      STOP
      END
      SUBROUTINE G1(NT,P,NZ,DZ,A0)
      DIMENSION P(NT)
      DIMENSION PN(50),SD(50),SI(50)
      CALL ROUGH(NT,SI,SD)
      DT=1.
      DO 10 IT=1,NT
10    P(IT)=0.
      P(1)=1.
      A1=.5*DZ*A0/SQRT(DT)
      DO 50 IZ=1,NZ
      PN(1)=P(1)*(1.-A1*SD(1))/(1.+A1*SD(1))
      DO 30 IT=2,NT
      SUM=0.
      DO 20 I=2,IT
      J=IT-I+1
20    SUM=SUM+SD(I)*(PN(J)+P(J))
30    PN(IT)=(P(IT)*(1.-A1*SD(1))-A1*SUM)/(1+A1*SD(1))
      DO 50 IT=1,NT
50    P(IT)=PN(IT)
      RETURN
      END
```

Output: (comparison is between columns 3 and 4)

$G_F(z=1)$	$G_1(z=1)$	$G_F(z=2)$	$[G_F(z=1)]*[G_1(z=1)]$
0.00000	0.00668	0.00000	0.00000
0.09569	0.04291	0.00006	0.00064
0.23235	0.09724	0.00574	0.00565
0.28622	0.09273	0.02427	0.02119
0.30446	0.05131	0.04785	0.04578
0.30807	0.05384	0.07010	0.06946
0.30534	0.03964	0.08892	0.08859
0.29979	0.03125	0.10413	0.10442
0.29306	0.03367	0.11618	0.11646
0.28594	0.01842	0.12562	0.12613
0.27880	0.02963	0.13299	0.13343
0.27186	0.01080	0.13870	0.13921
0.26519	0.02650	0.14311	0.14355
0.25885	0.00607	0.14648	0.14692
0.25284	0.02391	0.14901	0.14942
0.24716	0.00305	0.15089	0.15126
0.24178	0.02173	0.15223	0.15259
0.23670	0.00106	0.15314	0.15346
0.23189	0.01986	0.15371	0.15402

(continued)

0.22734	-0.00026	0.15399	0.15425
0.22303	0.01823	0.15404	0.15431
0.21894	-0.00116	0.15390	0.15413
0.21505	0.01681	0.15361	0.15384
0.21135	-0.00177	0.15319	0.15338
0.20783	0.01557	0.15267	0.15287
0.20447	-0.00218	0.15207	0.15223
0.20126	0.01446	0.15139	0.15157
0.19819	-0.00245	0.15067	0.15081
0.19526	0.01348	0.14990	0.15006
0.19244	-0.00262	0.14909	0.14922

Example E: Time Domain Recursion for $Q_z = i \ell Q$

The program:

```
      DIMENSION U ( 30, 9), V ( 30, 9), Q ( 59), SI ( 30), SD ( 30),
     * DUM ( 59), RNUM ( 30), RDEN ( 30), UU ( 30), VV ( 30)
      READ (5,1000) NZ,NT,DZ,DT,SIGMA,RMU
 1000 FORMAT (8G10.4)
      A1 = .5 * DZ * SQRT(RMU*SIGMA / DT)
      A1SQ = A1 * A1
      WRITE (6,1000) NZ,NT,DZ,DT,SIGMA,RMU,A1
         DO 10 IT=1,NT
         U(IT,1) = 0.0
   10    V(IT,1) = 0.0
         DO 20 IZ=1,NZ
         U(1,IZ) = 0.0
   20    V(1,IZ) = 0.0
      U(2,1) = 1.0
      CALL ROUGH (NT, SI, SD)
         DO 30 IZ=2,NZ
            DO 30 IT=2,NT
            U(IT,IZ) = -U(IT-1,IZ) + U(IT,IZ-1) + U(IT-1,IZ-1)
     *      + 2.0*A1SQ*(V(IT,IZ-1)-V(IT-1,IZ-1) )
   30       V(IT,IZ) = U(IT,IZ-1) + V(IT,IZ-1)
         DO 40 IT=1,NT
         DUM(IT) = 0.0
            DO 40 IS=1,IT
            K = IT-IS+1
   40       DUM(IT) = DUM(IT) + SD(IS)*V(K,NZ)*A1
         DO 100 IT=1,NT
         RNUM(IT) = U(IT,NZ)-DUM(IT)
  100    RDEN(IT) = U(IT,NZ)+DUM(IT)
      Q(1) = 0.0
      Q(2) = RNUM(2) / RDEN(2)
         DO 110 IT=3,NT
         TEMP = 0.0
            DO 120 IS=3,IT
  120       TEMP = TEMP+RDEN(IS)*Q(IT-IS+2)
  110    Q(IT) = (RNUM(IT)-TEMP)/RDEN(2)
         DO 50 IT=1,NT
   50    WRITE (6,2000) Q(IT)
 2000 FORMAT (20X, F15.5)
      RETURN
      END
```

Boundary conditions and initial conditions

recursion

Find Q from u,v at desired z level

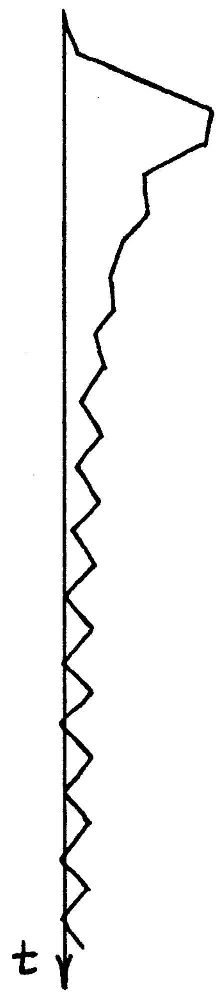
The results here for z=1 may be compared to column 2 of the output in Example D.

NZ = 5 NT= 30 0.2500 DT= 1.000 σ= 3.142 μ= 3.142 A= 0.3927

DZ ↗

z = (skin depth)

0.00000
0.00668
0.04291
0.09724
0.09273
0.05181
0.05384
0.03964
0.03125
0.03367
0.01842
0.02963
0.01080
0.02650
0.00607
0.02392
0.00305
0.02173
0.00106
0.01986
-0.00026
0.01823
-0.00116
0.01681
-0.00177
0.01556
-0.00218
0.01446
-0.00245
0.01348



Example F: "Expensive" Two Dimensional Time Domain Technique

The program uses equation (7) of section II. The comparison is similar to that for the one dimensional case. We compare Feshbach's Green's function, G_F, at $z = 2$ with the result of propagating G_F from $z = 1$ to $z = 2$ by our technique:

$$G_F\ (z=2) = [\ G_F\ (z=1)\ ] * [\ G\ (z=1)\ ] \qquad (1)$$

G_F is given in the statement function G 2 (z) . In the comparison, the left and right columns correspond to the left and right sides of the comparison above. The impulse is at the left side ; x axis to right, t axis down. The numbers along the top are degrees off vertical from the source impulse. The fairly good agreement up to 45° may be spurious in that there is a periodicity to the sources, i.e. for x_{max} on the right hand side there are sources at $x=0$ and at $x=4x_{max}$.

```
      DIMENSION E(17),F(17),D(17),SJM(17),DEG(17)
      DIMENSION Q(17,60),QN(17,60),T1(17,60),TN(17,60)
      DIMENSION SI(60),SD(60),THEO(60)
      G2(Z)=100.*EXP(-A0*A0*(X*X+Z*Z)*.25/IT)/IT
      NX=17
      XMAX=4.
      DX=XMAX/(NX-1.)
      NT=60
      DT=1.
      A0=4.
      DO 10 IT=1,NT
      DO 10 IX=1,NX
      X=(IX-1.)*DX
      T1(IX,IT)=G2(1.)
      TN(IX,IT)=G2(2.)
   10 Q(IX,IT)=T1(IX,IT)
      CALL G1(NT,THEO,4,.25,A0)
      PRINT 20, ((T1(IX,IT),IX=1,NX),IT=1,NT)
   20 FORMAT(' INPUTS'/(17F6.3))
      CALL ROUGH(NT,SI,SD)
      NZ=4
      DZ=1./NZ
```

```
      DO 60 IZ=1,NZ
      A1=.25*DZ/DX/DX*SQRT(DT)/A0
      DO 30 IX=1,NX
   30 QN(IX,1)=0.
      DO 55 IT=2,NT
      DO 40 IX=1,NX
      SUM(IX)=SI(1)*Q(IX,IT)
      DO 40 IS=2,IT
      JS=IT-IS+1
   40 SUM(IX)=SUM(IX)+SI(IS)*(QN(IX,JS)+Q(IX,JS))
      D(1)=Q(1,IT)-A1*(2*SUM(1)-2*SUM(2))
      NX1=NX-1
      DO 50 IX=2,NX1
   50 D(IX)=Q(IX,IT)-A1*(-SUM(IX-1)+2*SUM(IX)-SUM(IX+1))
      B=1+A1*SI(1)*2.
      A=-A1*SI(1)
      C=-A1*SI(1)
      CALL TRI(A,B,C,NX,QN(1,IT),D,E,F)
   55 CONTINUE
      DO 60 IX=1,NX
      DO 60 IT=1,NT
   60 Q(IX,IT)=QN(IX,IT)
      DO 80 IX=1,NX
      DO 80 IT=1,NT
      DOT=0.
      DO 70 IS=1,IT
   70 DOT=DOT+THEO(IS)*Q(IX,IT-IS+1)
   80 QN(IX,IT)=DOT
      DO 88 IX=1,NX
   88 DEG(IX)=ATAN((IX-1)*DX/2.)*180./3.1416
      PRINT 89, (DEG(IX),IX=1,NX)
   89 FORMAT(9F13.0)
      PRINT 90,((TN(J,I),QN(J,I),J=1,9 ),I=1,NT)
   90 FORMAT(' COMPARISON'/9(1X,2F6.3))
      STOP
      END
```

COMPARISON		7°		14°		21°		27°		32°		37°		41°		45°	
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.017	0.004	0.015	0.004	0.010	0.003	0.005	0.002	0.002	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000
0.161	0.056	0.148	0.052	0.115	0.043	0.076	0.030	0.042	0.019	0.020	0.010	0.008	0.005	0.003	0.002	0.001	0.001
0.458	0.277	0.430	0.260	0.357	0.216	0.261	0.158	0.168	0.102	0.096	0.058	0.048	0.029	0.021	0.013	0.008	0.005
0.815	0.704	0.775	0.668	0.667	0.570	0.520	0.437	0.366	0.300	0.234	0.185	0.135	0.102	0.070	0.051	0.033	0.023
1.158	1.084	1.111	1.042	0.930	0.924	0.796	0.751	0.595	0.558	0.409	0.376	0.258	0.229	0.150	0.125	0.080	0.064
1.453	1.357	1.402	1.315	1.259	1.195	1.053	1.014	0.820	0.798	0.595	0.578	0.402	0.384	0.252	0.233	0.149	0.129
1.692	1.654	1.640	1.607	1.493	1.474	1.277	1.272	1.026	1.031	0.775	0.779	0.549	0.547	0.366	0.355	0.229	0.213
1.878	1.774	1.826	1.736	1.680	1.624	1.453	1.445	1.204	1.214	0.938	0.958	0.691	0.705	0.481	0.483	0.317	0.308
2.019	1.980	1.969	1.937	1.827	1.812	1.612	1.619	1.353	1.379	1.081	1.115	0.821	0.850	0.593	0.608	0.408	0.407
2.123	2.004	2.075	1.973	1.938	1.876	1.730	1.716	1.476	1.500	1.203	1.246	0.937	0.979	0.597	0.725	0.495	0.506
2.197	2.144	2.151	2.106	2.021	1.997	1.821	1.826	1.574	1.607	1.305	1.355	1.038	1.090	0.791	0.831	0.579	0.599
2.247	2.114	2.204	2.088	2.080	2.038	1.890	1.871	1.652	1.679	1.389	1.444	1.124	1.185	0.876	0.926	0.656	0.687
2.278	2.210	2.238	2.177	2.121	2.083	1.940	1.935	1.712	1.743	1.458	1.515	1.198	1.265	0.950	1.009	0.725	0.767
2.294	2.149	2.256	2.128	2.146	2.063	1.975	1.948	1.757	1.781	1.513	1.571	1.259	1.332	1.014	1.081	0.790	0.839
2.299	2.217	2.264	2.189	2.160	2.109	1.998	1.983	1.791	1.816	1.556	1.614	1.310	1.386	1.069	1.143	0.846	0.903
2.295	2.141	2.262	2.124	2.164	2.070	2.011	1.974	1.814	1.832	1.589	1.647	1.352	1.430	1.116	1.195	0.895	0.960
2.284	2.189	2.252	2.166	2.161	2.098	2.016	1.990	1.829	1.846	1.614	1.670	1.385	1.464	1.156	1.240	0.939	1.010
2.267	2.106	2.238	2.092	2.151	2.049	2.014	1.969	1.837	1.847	1.632	1.686	1.412	1.492	1.190	1.276	0.977	1.053
2.247	2.141	2.219	2.121	2.137	2.054	2.008	1.972	1.839	1.849	1.644	1.695	1.433	1.512	1.218	1.307	1.009	1.091
2.223	2.056	2.196	2.045	2.119	2.010	1.997	1.943	1.837	1.839	1.651	1.699	1.448	1.527	1.240	1.331	1.038	1.123
2.196	2.083	2.172	2.066	2.099	2.016	1.983	1.938	1.831	1.832	1.653	1.698	1.459	1.536	1.259	1.351	1.061	1.150
2.168	1.998	2.145	1.990	2.076	1.961	1.956	1.905	1.822	1.817	1.652	1.695	1.466	1.542	1.273	1.366	1.082	1.173
2.139	2.019	2.117	2.004	2.052	1.962	1.948	1.895	1.811	1.804	1.649	1.688	1.470	1.544	1.284	1.377	1.098	1.192
2.109	1.937	2.088	1.930	2.026	1.905	1.928	1.860	1.797	1.785	1.643	1.678	1.472	1.544	1.292	1.385	1.112	1.208
2.079	1.953	2.059	1.940	2.000	1.904	1.906	1.847	1.782	1.768	1.634	1.667	1.470	1.540	1.298	1.390	1.123	1.221
2.048	1.874	2.029	1.868	1.973	1.849	1.884	1.811	1.766	1.746	1.625	1.654	1.467	1.535	1.301	1.392	1.132	1.231
2.017	1.887	1.999	1.876	1.946	1.845	1.861	1.795	1.748	1.728	1.613	1.639	1.462	1.528	1.302	1.392	1.139	1.238
1.986	1.812	1.969	1.807	1.919	1.792	1.838	1.769	1.730	1.705	1.601	1.624	1.456	1.519	1.302	1.391	1.144	1.244
1.955	1.822	1.939	1.813	1.891	1.786	1.814	1.743	1.711	1.685	1.588	1.608	1.449	1.509	1.300	1.388	1.147	1.247
1.925	1.751	1.910	1.747	1.864	1.735	1.790	1.708	1.692	1.661	1.574	1.591	1.440	1.498	1.297	1.383	1.149	1.249
1.895	1.760	1.881	1.751	1.837	1.728	1.767	1.691	1.673	1.641	1.559	1.573	1.431	1.486	1.293	1.377	1.150	1.249
1.866	1.692	1.852	1.689	1.810	1.679	1.743	1.657	1.653	1.617	1.544	1.556	1.421	1.473	1.287	1.370	1.149	1.249
1.837	1.700	1.824	1.692	1.784	1.672	1.720	1.640	1.633	1.596	1.529	1.537	1.410	1.460	1.281	1.363	1.148	1.247
1.809	1.635	1.796	1.633	1.758	1.625	1.696	1.607	1.613	1.573	1.513	1.519	1.399	1.447	1.275	1.354	1.145	1.244
1.781	1.642	1.769	1.636	1.732	1.618	1.673	1.590	1.594	1.552	1.497	1.501	1.387	1.432	1.267	1.345	1.142	1.240
1.754	1.581	1.742	1.580	1.707	1.574	1.650	1.559	1.574	1.529	1.481	1.483	1.375	1.418	1.260	1.335	1.138	1.235
1.727	1.587	1.716	1.582	1.682	1.556	1.628	1.542	1.555	1.509	1.465	1.464	1.363	1.404	1.251	1.325	1.134	1.230
1.701	1.530	1.690	1.529	1.658	1.524	1.606	1.512	1.535	1.487	1.449	1.446	1.351	1.389	1.243	1.315	1.129	1.224
1.676	1.535	1.665	1.530	1.634	1.517	1.584	1.496	1.516	1.468	1.433	1.428	1.338	1.374	1.234	1.304	1.123	1.218
1.651	1.481	1.641	1.480	1.611	1.477	1.563	1.467	1.497	1.446	1.418	1.410	1.326	1.360	1.225	1.293	1.118	1.211
1.627	1.486	1.617	1.482	1.588	1.470	1.542	1.452	1.479	1.427	1.402	1.392	1.313	1.345	1.215	1.282	1.111	1.204
1.603	1.434	1.594	1.434	1.566	1.432	1.521	1.424	1.461	1.406	1.386	1.375	1.300	1.330	1.206	1.271	1.105	1.196
1.580	1.439	1.571	1.435	1.544	1.425	1.501	1.409	1.443	1.388	1.371	1.358	1.288	1.316	1.196	1.259	1.098	1.188
1.557	1.390	1.549	1.390	1.523	1.389	1.481	1.383	1.425	1.368	1.355	1.341	1.275	1.301	1.186	1.248	1.091	1.180
1.535	1.395	1.527	1.391	1.502	1.382	1.462	1.369	1.407	1.350	1.340	1.324	1.262	1.287	1.176	1.236	1.084	1.172
1.514	1.348	1.506	1.348	1.482	1.348	1.443	1.344	1.390	1.331	1.325	1.308	1.250	1.273	1.166	1.225	1.077	1.164
1.493	1.353	1.485	1.349	1.462	1.341	1.424	1.330	1.373	1.314	1.311	1.291	1.238	1.258	1.157	1.213	1.070	1.156
1.472	1.308	1.465	1.308	1.443	1.309	1.406	1.306	1.357	1.296	1.296	1.275	1.225	1.244	1.147	1.202	1.062	1.147
1.452	1.313	1.445	1.310	1.424	1.303	1.388	1.293	1.341	1.279	1.282	1.260	1.213	1.230	1.137	1.190	1.055	1.138
1.433	1.270	1.426	1.271	1.405	1.271	1.371	1.270	1.325	1.262	1.268	1.244	1.201	1.217	1.127	1.179	1.047	1.130
1.414	1.275	1.407	1.272	1.387	1.266	1.354	1.257	1.309	1.246	1.254	1.229	1.189	1.203	1.117	1.168	1.039	1.121
1.395	1.234	1.389	1.235	1.369	1.236	1.337	1.236	1.294	1.229	1.240	1.214	1.177	1.190	1.107	1.156	1.032	1.112
1.377	1.239	1.371	1.236	1.352	1.231	1.321	1.224	1.279	1.214	1.226	1.199	1.166	1.177	1.098	1.145	1.024	1.104
1.359	1.200	1.353	1.201	1.335	1.202	1.305	1.203	1.264	1.198	1.213	1.195	1.154	1.164	1.088	1.134	1.016	1.095

Example G: Program for Strong Contrast Techniques

Consider

$$B = B_x^N \frac{\Delta z}{\Delta x}$$

Then equation (4) of part III becomes:

$$\left(\frac{\sigma_z}{\sigma} - 2\, i\ell \right)\left(B_x^{N+1} - B_x^N \right) = \frac{\Delta z}{2\Delta x^2}\left[\left(B_{x+1} - 2B_x + B_{x-1} \right)^{N+1} + \left(B_{x+1} - 2B_x + B_{x-1} \right)^N \right]$$

$$- \frac{\Delta z}{4\Delta x}\left(\frac{\sigma_x}{\sigma} \right)\left[\left(B_{x+1} - B_{x-1} \right)^{N+1} + \left(B_{x+1} - B_{x-1} \right)^N \right] + \frac{\Delta z}{2}\left(\ell^2 + \bar{\ell}^2 \right)\left(B_x^N + B_x^{N+1} \right)$$

Rearranging

$$B_{x-1}^{N+1}\left[-\frac{\Delta z}{2\Delta x^2} - \frac{\Delta z}{4\Delta x}\frac{\sigma_x}{\sigma} \right]$$

$$+ \quad B_x^{N+1}\left[\frac{\sigma_z}{\sigma} - 2i\bar{\ell} + \frac{\Delta z}{\Delta x^2} - \frac{\Delta z}{2}\left(\ell^2 + \bar{\ell}^2 \right) \right]$$

$$+ \quad B_{x+1}^{N+1}\left[-\frac{\Delta z}{2\Delta x^2} + \frac{\Delta z}{4\Delta x}\frac{\sigma_x}{\sigma} \right]$$

$$= B_{x-1}^N\left[\frac{\Delta z}{2\Delta x^2} + \frac{\Delta z}{4\Delta x}\frac{\sigma_x}{\sigma} \right]$$

$$+ \; B_x^N\left[\frac{\sigma_z}{\sigma} - 2i\bar{\ell} - \frac{\Delta z}{\Delta x^2} + \frac{\Delta z}{2}\left(\ell^2 + \bar{\ell}^2 \right) \right]$$

$$+ \; B_{x+1}^N\left[\frac{\Delta z}{2\Delta x^2} - \frac{\Delta z}{4\Delta x}\frac{\sigma_x}{\sigma} \right]$$

As done earlier (Example B) we also halve the secondary source terms since we have an equation for only one way propagation in z.; ℓ is also used in place of $\bar{\ell}$.

```
$WATFIV
      IMPLICIT COMPLEX (C)
      REAL CABS
      DIMENSION SIG ( 40, 40), SIGX ( 40), SIGZ ( 40), CP ( 40, 40),
     * CA ( 40), CB ( 40), CC ( 40), CD ( 40), CE ( 40), CF ( 40),
     * IPLOT ( 40), CL ( 40), RHOA ( 40)
      READ (5,1000) NX, NZ, DX, DZ, RMU, OMEG
      WRITE (6,1000) NX, NZ, DX, DZ, RMU, OMEG
 1000 FORMAT (1X,I9,I10,5F10.6)
      CALL SIGIN (NX,NZ,SIG)
      NX1 = NX-1
      NZ1 = NZ-1
      DDX = 1.0/DX
      DDDX = DDX*DDX
      DDZ = 1.0/DZ
      RMOG = RMU*OMEG
      CI= CMPLX(0.0,1.0)
      CLL = CSQRT(CI*RMOG)
         DO 10 IX=1,NX
   10    CP(IX,1) = CMPLX(99.0,0.0)             Initial conditions
      C1 = DZ*.5*DDDX
      C2 = DZ*.25*DDX
      C3 = 2.0*CI
      C4 = DZ*DDDX
      C5 = DZ*.5
      SIGX(1) = 0.0
      SIGX(NX) = 0.0
         DO 20 IZ=1,NZ1
            DO 30 IX=2,NX1
   30       SIGX(IX) = .5*DDX*(SIG(IX+1,IZ)+SIG(IX+1,IZ+1)-SIG(IX-1,IZ)
     *      -SIG(IX-1,IZ+1) ) / (SIG(IX,IZ)+SIG(IX,IZ+1) )
            DO 40 IX=1,NX
            CL(IX) = CLL*SQRT(.5*(SIG(IX,IZ)+SIG(IX,IZ+1) ) )
            SIGZ(IX) = 2.0*DDZ*(SIG(IX,IZ+1)-SIG(IX,IZ) ) / (SIG(IX,IZ)
     *      +SIG(IX,IZ+1) )
            SIGX(IX) = .5*SIGX(IX)
            SIGZ(IX) = .5*SIGZ(IX)
            CA(IX) = -C1-C2*SIGX(IX)
            CB(IX) = SIGZ(IX)-C3*CL(IX)+C4-C5*(2.0*CL(IX)**2)
   40       CC(IX) = -C1+C2*SIGX(IX)
            DO 45 IX=2,NX1
   45       CD(IX) = CP(IX-1,IZ)*(C1+C2*SIGX(IX) ) + CP(IX,IZ)
     *      *(SIGZ(IX)-C3*CL(IX)-C4+C5*(2.0*CL(IX)**2) ) + CP(IX+1,IZ)
     *      *(C1-C2*SIGX(IX) )
         CD(1) = CP( 1,IZ)*(SIGZ( 1)-C3*CL(IX)-C4+C5*(2.0*CL(1)**2) )
     *   + CP(2,IZ) * (C1-C2*SIGX(IX) )
         CD(NX) = CP(NX-1,IZ)*(C1+C2*SIGX(NX) ) + CP(NX,IZ)*(SIGZ(NX)
     *   -C3*CL(IX)-C4+C5*(2.0*CL(NX)**2) )
   20    CALL TRI3 (CA,CB,CC,NX,CP(1,IZ+1),CD,CE,CF)
C                                                Loop for each z level
         DO 60 IZ=1,NZ
            DO 50 IX=1,NX
   50       IPLOT(IX) = CABS(CP(IX,IZ) )         Plot output
   60    WRITE (6,3000) (IPLOT(IX),IX=1,NX)
 3000 FORMAT (1X,40I3)
C
      RM1 = 1.0/RMOG
      IZL = 1
         DO 100  IX = 1,NX
         DIFF = REAL (CP(IX,IZL+1)-CP(IX,IZL) )*DDZ
```

```
               AV = REAL(CP(IX,IZL+1)+CP(IX,IZL) )*.5
               SIGAV = (SIG(IX,IZL+1)+SIG(IX,IZL) )*.5
      100      RHOA(IX) = RM1*DIFF*DIFF/(SIGAV*AV)**2
            WRITE (6,2000)
     2000   FORMAT (1H1)
            WRITE (6,4000) (RHOA(IX),IX=1,NX)
     4000   FORMAT (1X,10F12.5)
            RETURN
            END
            SUBROUTINE SIGIN (NX,NZ,SIG)
            DIMENSION SIG ( 1, 1)
            READ (5,1000) S1,S2
            WRITE (6,1000) S1, S2
     1000   FORMAT (1X,F9.6,7F10.6)
               DO 10 IZ= 1,NZ
                  DO 20 IX=1,20
       20         SIG(IX,IZ) = S1
                  DO 10 IX=21,NX
       10         SIG(IX,IZ) = S2
            RETURN
            END
```

On the next page we have a plot of complex amplitudes on the x , z plane: all appears well, i.e. is continuous, and asymptotically approaches half space solutions at the left and right edges.

NX=40 NZ=40 1.000000 1.000000 1.000000 1.000000

0.010000 0.050000 DX DZ μ ω

99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99	99
92	92	92	92	92	92	92	92	92	92	92	92	92	92	91	91	91	90	88	86	85	85	84	84	84	84	84	84	84	84	84	84	84	84	84	84	84	84	84	84
85	85	85	85	85	85	85	85	85	85	85	85	85	85	84	83	82	80	78	77	75	74	73	72	72	72	72	72	72	72	72	72	72	72	72	72	72	72	72	72
80	80	80	80	80	80	80	80	80	79	79	79	78	77	76	75	73	72	70	68	67	65	64	63	62	62	61	61	61	61	61	61	61	61	61	61	61	61	61	61
74	74	74	74	74	74	74	74	74	73	73	72	71	70	69	68	66	64	62	60	59	57	56	55	54	53	53	52	52	52	52	52	52	52	52	52	52	52	52	52
69	69	69	69	69	69	69	69	68	68	67	66	65	64	62	61	59	57	55	53	52	50	49	48	47	46	45	45	45	45	45	45	45	45	45	45	45	45	45	45
65	65	64	64	64	64	64	63	63	62	61	60	59	58	56	55	53	51	49	47	46	44	43	42	41	40	39	39	38	38	38	38	38	38	38	38	38	38	38	38
60	60	60	60	60	59	59	58	58	57	56	55	54	52	51	49	47	45	44	42	40	39	38	37	36	35	34	33	33	33	33	32	32	32	32	32	32	32	32	32
56	56	56	56	55	55	54	54	53	52	51	50	49	47	46	44	42	40	39	37	35	34	33	32	31	30	30	29	28	28	28	28	28	28	28	28	28	28	28	28
52	52	52	51	51	51	50	49	49	48	47	46	44	43	41	40	38	36	34	32	31	30	29	28	27	26	26	25	25	24	24	24	24	24	23	23	23	23	23	23
48	48	48	48	47	47	46	45	45	44	43	41	40	39	37	36	34	32	30	29	27	26	25	25	24	23	22	22	21	21	21	20	20	20	20	20	20	20	20	20
44	44	44	44	43	43	42	42	41	40	39	38	36	35	33	32	30	29	27	25	24	23	22	22	21	20	19	19	18	18	18	17	17	17	17	17	17	17	17	17
41	41	41	40	40	40	39	38	37	36	35	34	33	32	30	29	27	25	24	23	21	20	20	19	18	18	17	16	16	16	15	15	15	15	15	14	14	14	14	14
37	37	37	37	37	36	36	35	34	33	32	31	30	29	27	26	24	23	21	20	19	18	17	17	16	15	15	14	14	13	13	13	13	13	12	12	12	12	12	12
34	34	34	34	34	33	33	32	31	30	29	28	27	26	24	23	22	20	19	18	17	16	15	14	14	13	13	12	12	12	11	11	11	11	11	10	10	10	10	10
32	32	31	31	31	30	30	29	29	28	27	26	25	23	22	21	19	18	17	15	15	14	13	13	12	12	11	11	10	10	10	10	9	9	9	9	9	9	9	9
29	29	29	29	28	28	27	27	26	25	24	23	22	21	20	19	17	16	15	14	13	12	12	11	11	10	10	9	9	9	8	8	8	8	8	8	8	7	7	7
26	26	26	26	26	25	25	24	24	23	22	21	20	19	18	17	16	14	13	12	11	11	10	10	9	9	9	8	8	8	7	7	7	7	7	6	6	6	6	6
24	24	24	24	24	23	23	22	22	21	20	19	18	17	16	15	14	13	12	11	10	9	9	8	8	8	7	7	7	7	6	6	6	6	6	6	5	5	5	5
22	22	22	22	22	21	21	20	20	19	18	17	16	16	15	14	12	11	10	9	9	8	8	7	7	7	6	6	6	6	5	5	5	5	5	5	5	5	5	5
20	20	20	20	20	19	19	18	18	17	17	16	15	14	13	12	11	10	9	8	8	7	7	6	6	6	6	5	5	5	5	5	4	4	4	4	4	4	4	4
18	18	18	18	18	18	17	17	16	16	15	14	14	13	12	11	10	9	8	7	7	6	6	6	5	5	5	5	4	4	4	4	4	4	3	3	3	3	3	3
17	17	17	17	16	16	16	15	15	14	14	13	12	11	11	10	9	8	7	7	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3	3	3	3	3
15	15	15	15	15	15	14	14	13	13	12	12	11	10	10	9	8	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3	3	3	3	2	2	2	2	2
14	14	14	14	14	13	13	13	12	12	11	11	10	9	9	8	7	6	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2
13	13	13	12	12	12	12	11	11	11	10	10	9	8	8	7	6	6	5	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2	2	2
11	11	11	11	11	11	11	10	10	10	9	9	8	8	7	6	6	5	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1
10	10	10	10	10	10	10	9	9	9	8	8	7	7	6	6	5	5	4	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
9	9	9	9	9	9	9	9	8	8	7	7	7	6	6	5	5	4	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
9	9	9	8	8	8	8	8	7	7	7	6	6	6	5	5	4	4	3	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	8	8	8	8	7	7	7	7	6	6	6	5	5	5	4	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	7	7	7	7	7	7	6	6	6	6	5	5	4	4	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
6	6	6	6	6	6	6	6	5	5	5	5	4	4	4	3	3	3	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
6	6	6	6	6	5	5	5	5	5	4	4	4	4	3	3	3	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
5	5	5	5	5	5	5	5	4	4	4	4	4	3	3	3	2	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	5	5	5	5	4	4	4	4	4	4	3	3	3	3	2	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	4	4	4	4	4	4	4	3	3	3	3	3	2	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	4	4	4	4	4	4	3	3	3	3	3	3	2	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Example I : Program for Up and Down Going Propagation

To model upward and downward propagation, we first utilize the techniques of Example G for the downward propagation, then use equation (9) of Part III for the upward direction. The up-down sequence may be iterated.

Letting $B^- = B_{x\Delta x}^{N\Delta z}$, equation (9) of Part III is

$$\left(2i\bar{\ell} + \frac{\sigma_z}{\sigma}\right) \frac{\left(B_x^{N+1} - B_x^N \right)}{\Delta z} = \frac{\left(B_{x+1}^{N+1} + B_{x+1}^N - 2B_x^{N+1} - 2B_x^N + B_{x-1}^{N+1} + B_{x-1}^N \right)}{2(\Delta x)^2}$$

$$- \frac{\sigma_x}{\sigma} \left(\frac{B_{x+1}^{N+1} + B_{x+1}^N - B_{x-1}^{N+1} - B_{x-1}^N}{4\,\Delta x}\right) + 1/2\left(B_x^{N+1} + B_x^N \right)\left(\ell^2 + \bar{\ell}^2 \right)$$

$$+ \left(\frac{B*_x^{N+2} - B*_x^{N+1} - B*_x^N + B*_x^{N-1}}{2(\Delta z)^2}\right) - 2i\bar{\ell}\left(\frac{B*_x^{N+1} - B*_x^N}{\Delta z}\right) - \frac{\bar{\ell}^2}{2}\left(B*_x^{N+1} + B*_x^N \right)$$

where we write $B*$ for B^+

Rearranging:

$$B_{x-1}^N\left[- \frac{\Delta z}{2(\Delta x)^2} - \frac{\sigma_x}{\sigma}\frac{\Delta z}{4\Delta x} \right]$$

$$+ B_x^N\left[- 2i\bar{\ell} - \frac{\sigma_z}{\sigma} + \frac{\Delta z}{(\Delta x)^2} - \frac{\Delta z}{2}\left(\ell^2 + \bar{\ell}^2 \right)\right]$$

$$+ B_{x+1}^N\left[- \frac{\Delta z}{2(\Delta x)^2} + \frac{\sigma_x}{\sigma}\frac{\Delta z}{4\Delta z} \right]$$

$$= B_{x-1}^{N+1}\left[\frac{\Delta z}{2(\Delta x)^2} + \frac{\sigma_x}{\sigma}\frac{\Delta z}{4\Delta x}\right]$$

$$+ B_x^{N+1}\left[- 2i\bar{\ell} - \frac{\sigma_z}{\sigma} - \frac{\Delta z}{(\Delta x)^2} + \frac{\Delta z}{2}\left(\ell^2 + \bar{\ell}^2 \right)\right]$$

$$+ B_{x+1}^{N+1}\left[\frac{\Delta z}{2(\Delta x)^2} - \frac{\sigma_x}{\sigma}\frac{\Delta z}{4\Delta x} \right] + B*_x^{N+2}\left(\frac{1}{2(\Delta z)^2} \right)$$

$$+ \; B_x^{*N+1}\left(\frac{-1}{2(\Delta z)^2} - \frac{2i\bar{\ell}}{\Delta z} - \frac{\bar{\ell}^2}{2} \right) + B_x^{*N}\left(\frac{-1}{2(\Delta z)^2} + \frac{2i\bar{\ell}}{\Delta z} - \frac{\bar{\ell}^2}{2} \right) + B_x^{*N-1}\left(\frac{1}{2(\Delta z)^2} \right)$$

Again, we set all $\bar{\ell}$'s to ℓ . Also, the secondary source terms in σ_x and σ_z are halved: half upgoing, half downgoing.

The program is given along with a one dimensional problem of a horizontal layer. The conductivity contrast is 5 to 1; thus, there is a large reflection coefficient. The results aren't what we expect: what's the problem? First we note that the Fresnel approximation is not valid for this type of contrast. We start with $B = Q\, e^{i\bar{m}z}$ and drop Q_{zz} as small. If $\bar{m}$ is constant, then Q has to vary considerably in one or both of the mediums forming a strong contrast. However, we have used m rather than $\bar{m}$ in an equation for B that we got after transforming Q to B by $Q = B\, e^{-i\bar{m}z}$. This doesn't solve the problem here though.

When we estimate Q^+_{zz} to find the "source" terms for the upgoing wave our difference operator will be 3 levels in z and thus span across the contact. The desirable fact that Q was like a perturbation about e^{imz} rather than $e^{i\bar{m}z}$ is somewhat clouded by the way we do estimate Q^+_{zz} .

Thus, upgoing propagation requires "slowly" variable conductivity. In fact, for the large conductivity contrasts at the surface and possibly at some horizontal contact(s) we have to model multiples until amplitudes decay after several skin depths of propagation.

This is a rather basic problem: we are looking for a solution.

```
      IMPLICIT COMPLEX (C)
      REAL CABS
      DIMENSION SIG ( 40, 40), SIGX ( 40), SIGZ ( 40), CP ( 40, 40),
     * CA ( 40), CB ( 40), CC ( 40), CD ( 40), CE ( 40), CF ( 40),
     * IPLOT ( 40), CL ( 40), RHOA ( 40), CPU ( 40, 40), CPD ( 40)
      READ (5,1000) NX, NZ, DX, DZ, RMU, OMEG
      WRITE (6,1000) NX, NZ, DX, DZ, RMU, OMEG
 1000 FORMAT (1X,I9,I10,5F10.6)
      CALL SIGIN (NX,NZ,SIG)
      NX1 = NX-1
      NZ1 = NZ-1
      DDX = 1.0/DX
      DDDX = DDX*DDX
      DDZ = 1.0/DZ
      DDDZ = DDZ*DDZ
      RMOG = RMU*OMEG
      CI= CMPLX(0.0,1.0)
      CLL = CSQRT(CI*RMOG)
         DO 10 IX=1,NX
   10    CP(IX,1) = CMPLX(99.0,0.0)
```

Initial conditions for downgoing

```
      C1 = DZ*.5*DDDX
      C2 = DZ*.25*DDX
      C3 = 2.0*CI
      C4 = DZ*DDDX
      C5 = DZ*.5
      SIGX(1) = 0.0
      SIGX(NX) = 0.0
         DO 20 IZ=1,NZ1
            DO 30 IX=2,NX1
   30       SIGX(IX) = .5*DDX*(SIG(IX+1,IZ)+SIG(IX+1,IZ+1)-SIG(IX-1,IZ)
     *      -SIG(IX-1,IZ+1) ) / (SIG(IX,IZ)+SIG(IX,IZ+1) )
            DO 40 IX=1,NX
            CL(IX) = CLL*SQRT(.5*(SIG(IX,IZ)+SIG(IX,IZ+1) ) )
            SIGZ(IX) = 2.0*DDZ*(SIG(IX,IZ+1)-SIG(IX,IZ) ) / (SIG(IX,IZ)
     *      +SIG(IX,IZ+1) )
            SIGX(IX) = .5*SIGX(IX)
            SIGZ(IX) = .5*SIGZ(IX)
            CA(IX) = -C1-C2*SIGX(IX)
            CB(IX) = SIGZ(IX)-C3*CL(IX)+C4-C5*(2.0*CL(IX)**2)
   40       CC(IX) = -C1+C2*SIGX(IX)
            DO 45 IX=2,NX1
   45       CD(IX) = CP(IX-1,IZ)*(C1+C2*SIGX(IX) ) + CP(IX,IZ)
     *      *(SIGZ(IX)-C3*CL(IX)-C4+C5*(2.0*CL(IX)**2) ) + CP(IX+1,IZ)
     *      *(C1-C2*SIGX(IX) )
         CD(1) = CP( 1,IZ)*(SIGZ( 1)-C3*CL( 1)-C4+C5*(2.0*CL(1)**2) )
     *   + CP(2,IZ) * (C1-C2*SIGX( 1) )
         CD(NX) = CP(NX-1,IZ)*(C1+C2*SIGX(NX) ) + CP(NX,IZ)*(SIGZ(NX)
     *   -C3*CL(NX)-C4+C5*(2.0*CL(NX)**2) )
   20    CALL TRI3 (CA,CB,CC,NX,CP(1,IZ+1),CD,CE,CF)
```

Loop for each z level for downgoing

```
C
         DO 60 IZ=1,NZ
            DO 50 IX=1,NX
   50       IPLOT(IX) = CABS(CP(IX,IZ) )
   60    WRITE (6,3000) (IPLOT(IX),IX=1,NX)
 3000 FORMAT (1X,40I3)
```

Plot complex amplitude of downgoing

```
C
         DO 110 IX=1,NX
  110    CPU(IX,40) = CMPLX(0.0,0.0)
         DO 120 IZU=1,NZ1
```

Initial conditions for upgoing

```
              IZ = NZ-IZU
                DO 130 IX=2,NX1
      130       SIGX(IX) = .5*DDX*(SIG(IX+1,IZ)+SIG(IX+1,IZ+1)-SIG(IX-1,IZ)
     *          -SIG(IX-1,IZ+1) ) / (SIG(IX,IZ)+SIG(IX,IZ+1) )
                DO 140 IX=1,NX
                CL(IX) = CLL*SQRT(.5*(SIG(IX,IZ)+SIG(IX,IZ+1) ) )
                IF (IZ.EQ.NZ1 .OR. IZ.EQ.1) GO TO 200
                CPD(IX) = .5*DDDZ*(CP(IX,IZ+2)+CP(IX,IZ-1)) + CP(IX,IZ+1)*
     *          (-.5*DDDZ-C3*CL(IX)*DDZ-.5*CL(IX)**2) + CP(IX,IZ )*(-.5*DDDZ
     *          +C3*CL(IX)*DDZ-.5*CL(IX)**2)
                GO TO 210
      200       IF(IZ.EQ.NZ1) CPD(IX) = CMPLX(0.0,0.0)
                IF (IZ.EQ.1) CPD(IX) = .5*DDDZ*CP(IX,3) + CP(IX,2)*(-   DDDZ
     *          -C3*CL(IX)*DDZ-.5*CL(IX)**2) + CP(IX,1)*( .5*DDDZ+C3*CL(IX)*
     *          DDZ-.5*CL(IX)**2)
      210       SIGZ(IX) = 2.0*DDZ*(SIG(IX,IZ+1)-SIG(IX,IZ) ) / (SIG(IX,IZ)
     *          +SIG(IX,IZ+1) )
                SIGX(IX) = .5 * SIGX(IX)
                SIGZ(IX) = .5 * SIGZ(IX)
                CA(IX) = -C1 -C2*SIGX(IX)
                CB(IX) =-SIGZ(IX) - C3*CL(IX) + C4 - C5*(2.0*CL(IX)**2)
      140       CC(IX) = -C1 + C2*SIGX(IX)
                DO 145 IX=2,NX1
      145       CD(IX) = CPU(IX-1,IZ+1)*(C1+C2*SIGX(IX) ) + CPU(IX,IZ+1)*
     *          (-SIGZ(IX)-C3*CL(IX)-C4+C5*(2.0*CL(IX)**2)) + CPU(IX+1,IZ+1)
     *          *(C1-C2*SIGX(IX) ) + CPD(IX)
              CD(1) = CPU(1,IZ+1)*(-SIGZ( 1)-C3*CL( 1)-C4+C5*(2.0*CL( 1)**2))
     *        + CPU(2,IZ+1)*(C1-C2*SIGX( 1) ) + CPD( 1)
              CD(NX) = CPU(NX1,IZ+1)*(C1+C2*SIGX(NX) ) + CPU(NX,IZ+1)*
     *        (-SIGZ(NX)-C3*CL(NX)-C4+C5*(2.0*CL(NX)**2) ) + CPD(NX)
      120     CALL TRI3 (CA,CB,CC,NX,CPU(1,IZ),CD,CE,CF)
          WRITE (6,2000)
              DO 160 IZ=1,NZ
                 DO 150 IX=1,NX                          Plot upgoing
      150        IPLOT(IX) = CABS(CPU(IX,IZ) )
      160     WRITE (6,3000) (IPLOT(IX),IX=1,NX)
          WRITE (6,2000)
              DO 180 IZ =1,NZ
                 DO 170 IX=1,NX
                 CP(IX,IZ) = CP(IX,IZ)+CPU(IX,IZ)   Plot combined
      170        IPLOT(IX) = CABS(CP(IX,IZ) )       up and down
      180 WRITE (6,3000) (IPLOT(IX),IX=1,NX)
C
          RM1 = 1.0/RMOG
          IZL = 1
             DO 100  IX = 1,NX
             DIFF = REAL (CP(IX,IZL+1)-CP( IX,IZL) )*DDZ
             AV = REAL(CP(IX,IZL+1)+CP(IX,IZL) )*.5
             SIGAV = (SIG(IX,IZL+1)+SIG( IX,IZL) )*.5            apparent
      100    RHOA(IX) = RM1*DIFF*DIFF/(SIGAV*AV)**2              resistivity
          WRITE (6,2000)
     2000 FORMAT (1H1)
          WRITE (6,4000) (RHOA(IX),IX=1,NX)
     4000 FORMAT (1X,10F12.5)
          RETURN
          END
```

Loop for each a level for upgoing

```
      SUBROUTINE SIGIN (NX,NZ,SIG)
      DIMENSION SIG ( 1, 1)
      READ (5,1000) S1,S2
      WRITE (6,1000) S1, S2
 1000 FORMAT (1X,F9.6,7F10.6)
         DO 10 IX=1,NX
            DO 20 IZ=1,6
   20       SIG(IX,IZ) = S1
            DO 10 IZ=7 ,NZ
   10       SIG(IX,IZ) = S2
      RETURN
      END
```

NX = 5 NZ = 40 1.000000 1.000000 1.000000 1.000000

0.020000 0.004000 DX DZ μ ω

$\sigma_1 = .02$

$\sigma_2 = .004$

(axes: x → , z ↓)

Amplitude of downgoing propogation only

99	99	99	99	99
89	89	89	89	89
81	81	81	81	81
73	73	73	73	73
66	66	66	66	66
60	60	60	60	60
59	59	59	59	59
56	56	56	56	56
54	54	54	54	54
51	51	51	51	51
49	49	49	49	49
47	47	47	47	47
45	45	45	45	45
43	43	43	43	43
41	41	41	41	41
39	39	39	39	39
37	37	37	37	37
36	36	36	36	36
34	34	34	34	34
33	33	33	33	33
31	31	31	31	31
30	30	30	30	30
29	29	29	29	29
27	27	27	27	27
26	26	26	26	26
25	25	25	25	25
24	24	24	24	24
23	23	23	23	23
22	22	22	22	22
21	21	21	21	21
20	20	20	20	20
19	19	19	19	19
18	18	18	18	18
17	17	17	17	17
16	15	16	16	16
16	16	16	16	16
15	15	15	15	15
14	14	14	14	14
14	14	14	14	14
13	13	13	13	13

Amplitude of upgoing propogation only

4	4	4	4	4
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
18	18	18	18	18
21	21	21	21	21
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

Amplitude of combined up and downgoing propogation

95	95	95	95	95
88	88	88	88	88
80	80	80	80	80
72	72	72	72	72
65	65	65	65	65
42	42	42	42	42
40	40	40	40	40
56	56	56	56	56
54	54	54	54	54
51	51	51	51	51
49	49	49	49	49
47	47	47	47	47
45	45	45	45	45
43	43	43	43	43
41	41	41	41	41
39	39	39	39	39
37	37	37	37	37
36	36	36	36	36
34	34	34	34	34
33	33	33	33	33
31	31	31	31	31
30	30	30	30	30
29	29	29	29	29
27	27	27	27	27
26	25	26	26	26
25	25	25	25	25
24	24	24	24	24
23	23	23	23	23
22	22	22	22	22
21	21	21	21	21
20	20	20	20	20
19	19	19	19	19
18	18	18	18	18
17	17	17	17	17
16	16	16	16	16
16	16	16	16	16
15	15	15	15	15
14	14	14	14	14
14	14	14	14	14
13	13	13	13	13