Some Speculative Ideas on Velocity Estimation

by Jon F. Claerbout Lecture at Princeton University 3/16/73

A famous book by Chernov deals with the problem of solving for the statistical properties of waves given the statistical properties of the medium through which they have propagated. Some of Chernov's results are geophysically useful (Aki, Capon) in that in some realistic situations the statistical properties of the observed waves may be used to deduce statistical properties of the seismic velocity distribution in the earth. Chernov typically assumes a spatially stationary velocity perturbation. In the present work we attempt to bite off a much bigger piece than Chernov did. Instead of dealing with a statistical medium we will deal with an arbitrary medium. In the March 3 lecture notes it was shown how we do the forward problem, namely, how to use a computer to project waves through an arbitrary medium. (Naturally there are some regularity restrictions.) Now we are trying to do the inverse problem, specifically let us say: A plane wave source (or a point source) carries an unknown transient time function into a region of unknown two or three dimensional velocity inhomogeneity. After the wave emerges from the region of inhomogeneity we observe that the seismic waveform is now variable along the wavefront. Given this spatial variation in the emerging waveform, our objective is to reconstruct the unknown medium to whatever degree of resolution and uniqueness that we can. We are not frightened away by the obvious difficulties because in being able to do the forward problem we have now seen many examples and have some intuitive ideas about how to begin the inverse problem. At present it is mainly a question of trying to systematize some of our intuitive ideas. This in fact will be the body of the present discussion.

Counterbalancing in our mind the difficulty we anticipate on traveling through this new territory is our belief that this problem is a very important one in geophysics. We believe that layered models of the earth have been nearly mined out and that the many researchers who persist in the same vein must suffer the frustration that each new collection of seismic data will demand a new layered model. In fact, the new plate tectonics and hypotheses of mantle plumes, the abundance of reliable deep regional seismic anomalies, the problems of petroleum prospecting, and the power of modern computers are all directing us into this new territory for research. Some new approaches, along with some unfamilar assumptions, must be made.

Our basic approach is the use of partial differential equations to extrapolate observed data. We will defer the practical question of whether existing seismic arrays like LASA actually provide a sufficient density and extent of information. The present use of LASA is principally signal/noise enhancement by summation. There is little present motivation for increasing the size of the array because of the very marginal relative improvement. On the other hand our present effort, if successful, will convert "signal generated noise" into "signal" by means of backwards extrapolation through an appropriate medium. We need to show theoretically that it can be done. The need for petroleum amply justifies sufficient measurements for the use of partial differential equations in reflection seismic work. Justification for more measurements in earthquake seismology will come easier when we learn how to make fuller use of the data we already have.

After studying numerous examples we made the observation that as a wave propagates along in an inhomogeneous medium it becomes more and more complex. In fact, in the example of Figure 4 of Claerbout and Johnson, it is seen that the wave continues to get more complex even after it has emerged from an inhomogeneous region and is propagating along in a homogeneous one. To us it seems worthwhile as a starting assumption to assume that some suitable definition of complexity of a wavefield must be monotonically increasing with time. Now in the March 3 lecture notes we showed how the wave can be extrapolated either forwards or backwards in time. It is no surprise to see a wave field getting more complicated as it is projected forward in time but we were delighted to see that knowing the velocity model, the end result of such a projection can be extrapolated backward in time and despite all the practical approximations made it retraces its steps with astonishing accuracy. (We couldn't see any difference visually in the test cases.) Now if we do not know the velocity in which to back project the wave, why not assume that a good velocity is the one which makes the wave field get simpler? We can take what is called a "dynamic programming" approach. We first define complexity as some quadratic function Q of the wave field. If we expect a plane wave propagating along the z-axis we might define Q as $\sum_{x,t} (\frac{dP}{dx})^2$. The objective is that as Q is projected back in steps from z_N to z_0 it should decrease to zero. Since this problem is too hard we will first try the easier problem of finding a velocity distribution which decreases Q as much as possible on going from $Q(z_j)$ to $Q(z_{j-1})$. If we can solve this easier problem we can use it successively to go from z_N to z_0 although we may not necessarily end out with the best answer. It is just our hunch based on examples that we have seen that it will be a good answer.

Of course if we are lucky and the process of going from z_j to z_{j-1} turns out to be not very difficult then we can try to get a process which finds the best velocity in going from z_j to z_{j-2}. Ultimately of course we might like to find the best velocity for the whole big jump from z_N to z_0 but in practice it might not be much different than a lot of small jumps. Anyway, once you start to get close to the right answer you can always use perturbation theory to home in exactly.

In the March 3 notes we developed the equation

$$P_z = c/2\, P^t_{xx} + s(x,z)\, P_t \tag{1}$$

By choice of suitable units for measuring z we can set $c/2 = 1$ to keep the algebra uncluttered. Thus

$$P_z = P^t_{xx} + s\, P_t \tag{2}$$

We will first establish that certain quadratic functions of P are constants as P propagates along. Let us show that $d/dz \sum_{x,t} P^2$ is zero which shows that $\sum_{x,t} P^2$ is constant in z.

$$.5\ d/dz \sum_{x,t} P^2 = \sum_{x,t} P\, P_z \tag{3}$$

inserting (2)

$$= \sum_{x,t} P\, (P^t_{xx} + s\, P_t)$$

$$= \sum_{x,t} P\, P^t_{xx} + \sum_x s \sum_t P\, P_t$$

$$= \sum_{x,t} P\, P^t_{xx} + \sum_x s \sum_t (P^2)_t$$

The last term vanishes because it is the integral (or sum) of the derivative over time of P^2 and P^2 vanishes at plus and minus infinite time (by the assumption that our waves are time transient). Thus

$$= \sum_{x,t} P \, P^t_{xx}$$

Next let us integrate (or sum) by parts over x and assume that P vanishes at x equals plus and minus infinity.

$$= - \sum_{x,t} P_x \, P^t_x$$

$$= - \sum_{x,t} \tfrac{1}{2}((P^t_x)^2)_t \qquad = \quad 0$$

Again we note that we have an integral of a derivative which vanishes identically proving the result that $\sum_{x,t} P^2$ is constant in z. The reasoning used to get here from (3) will be used again many times with fewer intermediate steps. This result is like an energy conservation theorem and we are delighted that it works exactly in spite of the Fresnel approximation. It turns out that it is also exactly true with the Crank-Nicolson numerical method.

Let us now consider the more interesting quadratic form Q where

$$Q = \sum_{x,t} P_t \, P_z = \sum_{x,t} P_t \, P^t_{xx} + s \, P_t \, P_t \tag{4}$$

Integrate the first term over t by parts

$$Q = \sum_{x,t} - P \, P_{xx} + s \, P_t \, P_t$$

Then integrate the first term over x by parts

$$Q = \sum_{x,t} (P_x)^2 + s(x,z)\ (P_t)^2 \tag{5}$$

In a region of space in which s is constant this quadratic form is quite a bit like the complexity measure we discussed earlier. Let us now see how this quadratic form changes with z.

$$Q_z = \sum_{x,t} 2(P_x\ P_{xz} + s\ P_t\ P_{tz}) + s_z\ (P_t)^2$$

$$= \sum_{x,t} 2(P_x(P^t_{xxx}+sP_{tx}+s_xP_t) + sP_t(P_{xx}+sP_{tt})) + s_z(P_t)^2$$

$$= \sum_{x,t} 2(-P_{xx}P^t_{xx} + \underbrace{(s\ P_tP_x)_x}_{\text{drops}} + \underbrace{s^2((P_t)^2)_t}_{\text{drops}}) + s_z(P_t)^2$$

$$= \sum_{x,t} 2\ \underbrace{(P^t_{xx})_t\ (P^t_{xx})}_{\text{drops}} + s_z\ (P_t)^2$$

$$Q_z = \sum_x s_z(x,z) \sum_t (P_t(x,t))^2 \tag{6}$$

It is rather amazing that so many terms dropped out of Q_z. The end result says that Q will remain constant in any region of space in which s is a function of x only, that is, when $s(x,z)=s(x)$. Now let us rearrange this result a bit

$$d/dz \sum_{x,t} (P_x^2 + s P_t^2) = \sum_x s_z \sum_t P_t^2$$

$$d/dz \sum_{x,t} P_x^2 = - \sum_x s \sum_t d/dz P_t^2$$

Redefining Q to be half the quadratic form on the left

$$Q_z = - \sum_x s \sum_t P_t (P_{xx} + s P_{tt})$$

$$Q_z = - \sum_x s(x,z) \sum_t P_t P_{xx} \qquad (7)$$

This latter result shows that a rapid decline in this new Q can be achieved by taking s(x) to have a big projection on $\sum_t P_t P_{xx}$. That is easy. What this seems to amount to in simple terms is merely that if the objective is to minimize $\sum P_x^2$ then it can be achieved by time shifting each trace (a trace is the time function P(x,t) seen at some fixed x and z) for best alignment (until you get $\sum P_t P_{xx}=0$). In the exploration business this is called statics corrections.

Now rather than merely requiring Q to have maximum negative derivative let us try to get Q to decline some maximum amount at some finite distance away. Say that we are at z=0 and we wish to get Q as small as possible at some larger value of z. Let us take Q to be representable by a power series

$$Q = q_0 + q_1 z + q_2 z^2 \tag{8}$$

We have already calculated q_1 by (7). Since $P(x,t)$ is given at $z=0$ then q_0 is also given and not a function of the unknown velocity. We have to see how q_2 depends upon the unknown velocity. The values of $s(x,z)$ which brought us from negative z up to $z=0$ will now be forgotten. We seek a new $s(x,z)$ to be used from $z=0$ into the positive z direction. To simplify matters we will require this new s to be a function of x only, i.e. $s_z=0$ in the region we focus upon. This is not to say that we believe $s_z=0$ for the best s, but to say that is easier to estimate a constant than a constant plus linear function. Of course after we go a short distance in the z-direction we can then forget the new s and find a newer one.) Now to get q_2 we differentiate (7) with respect to z obtaining

$$Q_{zz} = \sum_x s(x) \sum_t (P_{tz} P_{xx} + P_t P_{xxz})$$

$$Q_{zz} = \sum_x s(x) \sum_t (P_{xx} + sP_{tt}) P_{xx} + P_t (P^{t}_{xxxx} + sP_{txx} + 2s_x P_{tx} + s_{xx} P_t) \tag{9}$$

The only thing you need to note about (9) is that $q_2 = Q_{zz}/2$ is a quadratic function of s, s_x, and s_{xx}. The statement that (8) attains a minimum is

$$0 = q_1 + 2 q_2 z \tag{10}$$

From this we define the z at which a minimum occursas $z_{min} = -q_1/2q_2$. Inserting z_{min} back into (8) we have

$$Q_{min} = q_0 - q_1^2/2q_2 + q_2 q_1^2/4q_2^2$$

$$= q_0 - q_1^2/4q_2 \qquad (11)$$

To obtain the minimum Q_{min} we need to adjust s to maximize the ratio of q_1^2/q_2 . This would normally be done by seeking the biggest eigenvalue λ which extremalizes the quadratic function of s

$$R = q_1^2 - \lambda q_2$$

If R were merely a function of s but not s_x and s_{xx} we would simply have to solve the eigenvalue problem found by taking $\partial R/\partial s = 0$. Since R is a function of s_x and s_{xx} we use the Euler equation method to get the following linear equations for s.

$$0 = \frac{\partial R}{\partial s} - \frac{d}{dx}\frac{\partial R}{\partial s_x} + \frac{d^2}{dx^2}\frac{\partial R}{\partial s_{xx}} \qquad (12)$$

We have not yet followed this through to an algorithm but it appears to be an eigenvalue problem with tri-diagonal matrices. Such problems seem to be solveable quite cheaply and there is every sign that we should proceed.