Transformations and Migration Equations

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Introduction

The coordinate transformations often used in the derivation of migration equations have three primary purposes. First, transforming the wave equation often makes it easier to see how it can be modified so that it becomes a one-way wave equation. One-way wave equations are important because the cost of computing their solutions is much less than the cost of computing solutions of the whole wave equation. Thus, if the situation of interest is one which can be described by a one-way equation, it is desirable to do so. In another vein, sometimes it is desirable to treat each propagation direction separately. One-way equations offer an easy method of separating the two wavefields.

A second reason for using non-cartesian coordinate frames is to reduce the variability of the wave forms. In order to achieve reasonable accuracy, finite difference algorithms typically require sample rates 4 to 5 times the Nyquist rate. Thus, even simple problems like plane wave propagation may require large numbers of grid points (and thus long computation times) if they are solved in the cartesian coordinate system. However, when viewed in a coordinate system which propagates with the speed and direction of the wave, plane wave propagation is very simple (nothing happens) and very coarse grids can be used. Aside from diminishing the cost of solutions, reducing waveform variation is also important because we wish to apply our finite difference algorithms to field data which is often not sampled at the required rate. A CDP gather is an example of this type of data. Before moveout correction the data is nearly spatially aliased. However, after NMO it is densely sampled spatially.

A third reason for using coordinate transformations is to improve the accuracy of the 'parabolic approximation', which is used to obtain a one-way wave equation. (The 'parabolic approximation' usually consists of assuming a second derivative of the wave field with respect to the propagation direction is zero). The closer the model (transformation equations) follows the actual behavior of the waves, the less the transformed wave forms must change as they are propagated and the better the parabolic approximation becomes.

There are two ways to improve the accuracy of a migration equation in a particular situation. One way is to change the coordinate system so that it more closely models the expected behavior of the waves. Another way is to leave the coordinate system unchanged and estimate the second derivative rather than assuming that it was zero. Both of these approaches appear to achieve the same result. However, the latter approach has the disadvantage of requiring more dense sampling of the wave form in the direction of propagation. The fact that one has to estimate the second derivative means the wave field changes rather rapidly as it propagates. This in turn implies that one must sample rather densely in the propagation direction to maintain the accuracy of the finite difference algorithms.

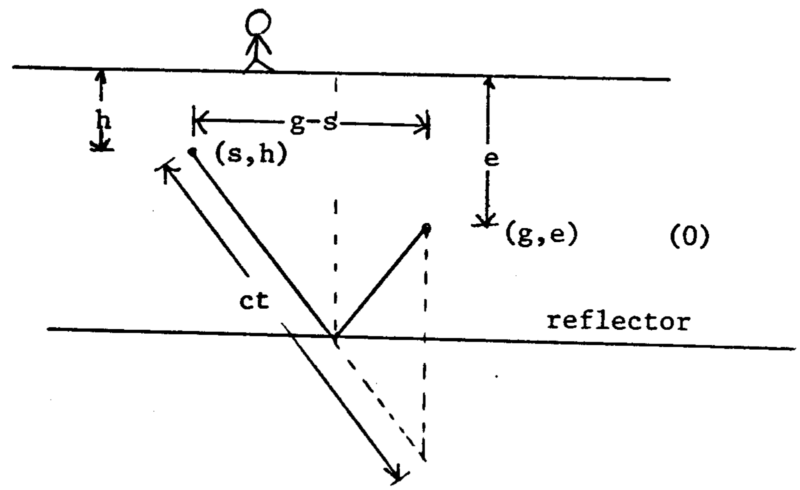
One might then ask: Why not model the expected behavior as closely as possible so that very coarse steps can be used in the propagation direction? Unfortunately, the more closely one models an expected behavior the less well the model fits other situations. In addition, tight fitting models tend to result in elaborate coordinate transformations and complicated migration equations which sometimes are costly to solve. Thus, there is something to be said for crude models and simple equations.

A General Migration Equation

With the previous discussion in mind we will now derive a migration equation in a general coordinate system.. We define migration as the act of downward continuing both shots and receivers.

Our general coordinate system will be defined as follows

$$z = (e + h)/2$$
$$w = (e - h)/2$$
$$y = (g + s)/2$$
$$k = k(g - s, t)$$
$$d = d(g-s, e+h, t)$$



We have a relationship between cartesian system on the right and the transformed system on the left. The t is the travel time of the reflected phase traveling from the shot to the geophone. z is the average depth of the shot and receiver, y is the midpoint coordinate. (If $e = h$, y is the horizontal coordinate of the reflection point.) Special attention needs to be given to the as yet undefined coordinates k, and d. Note that k is specifically dependent on shot receiver offset, $g-s$. In some cases this will indeed be the definition, in others we will use k to describe angle of emergence. d will be used as a measure of reflector depth. As such it may have units of feet, or if it is scaled by a velocity, it may have units of seconds. d could be directly proportional to t or it may also be dependent on offset, in which case the transformation from t to d is a type of normal correction.

The reason that the coordinate system has 5 coordinates is that we wish to downward continue both the shots and the receivers. Thus we need

4 spatial coordinates to describe the location of the shots and receivers. Since we have 5 coordinates we need to find a wave equation which has 5 variables. We can write the wave equation in terms of the shot coordinates

$$P_{hh} + P_{ss} + \frac{1}{\tilde{c}^2} P_{tt} = \delta((g-s)\ ,\ (e-h)) \qquad (1a)$$

By reciprocity we have

$$P_{ee} + P_{gg} + \frac{1}{\tilde{c}^2} P_{tt} = \delta((s-g)\ ,\ (h-e)) \qquad (1b)$$

Adding (1a) and (1b) we find that the wave equation for a source-free region is:

$$1/2(P_{ee} + P_{hh}) + 1/2(P_{gg} + P_{ss}) + \frac{1}{\tilde{c}^2} P_{tt} = 0 \qquad (2)$$

We will need to express the wave equation in the transformed coordinate system in order to find the migration equation. To do this we need to calculate coordinate transformation derivatives and use the chain rule.

Since the disturbance is the same in both systems we have

$$P(e,h,g,s,t) = Q(z,w,y,k,d)$$

where Q is the wave field in the new frame.

Using the chain rule we have

$$P_g = Q_y y_g + Q_d d_g + Q_k k_g + Q_z z_g + Q_w w_g = \frac{1}{2} Q_y + d_g Q_d + k_g Q_k$$

$$P_s = Q_y y_s + Q_d d_s + Q_k k_s + Q_z z_s + Q_w w_s = \frac{1}{2} Q_{yy} + d_s Q_d + k_s Q_k$$

$$P_e = Q_y y_e + Q_d d_e + Q_k k_e + Q_z z_e + Q_w w_e = d_e Q_d + \frac{1}{2} Q_z + \frac{1}{2} Q_w$$

$$P_h = Q_y y_h + Q_d d_h + Q_k k_h + Q_z z_h + Q_w w_h = d_h Q_d + \frac{1}{2} Q_z - \frac{1}{2} Q_w$$

$$P_t = Q_y y_t + Q_d d_t + Q_k k_t + Q_z z_t + Q_w w_t = d_t Q_t + k_t Q_k$$

Forming second derivatives being careful to keep all derivatives of the coordinate transformation we get

$$P_{gg} = \frac{1}{4} Q_{yy} + d_g^2 Q_{dd} + k_g^2 Q_{kk} + d_g Q_{yd} + k_g Q_{yk} + 2d_g k_g Q_{kd}$$

$$+ (d_g d_{gd} + k_g d_{gk}) \quad Q_d + (k_g k_{gk} + d_g k_{gd}) Q_k$$

$$P_{ss} = \frac{1}{4} Q_{yy} + d_s^2 Q_{dd} + k_s^2 Q_{kk} + d_s Q_{yd} + k_s Q_{yk} + 2d_s k_s Q_{kd}$$

$$+ (d_s d_{sd} + k_s d_{sk}) \quad Q_d + (k_s k_{sk} + d_s k_{sd}) Q_k$$

$$P_{ee} = \frac{1}{4} Q_{zz} + \frac{1}{4} Q_{ww} + d_e^2 Q_{dd} + d_e Q_{dz} + d_e Q_{dw} + \frac{1}{2} Q_{wz}$$

$$+ (d_e d_{ed} + \frac{1}{2} d_{ez} + \frac{1}{2} d_{ew}) Q_d$$

$$P_{hh} = \frac{1}{4} Q_{zz} + \frac{1}{4} Q_{ww} + d_h^2 Q_{dd} + d_h Q_{zd} - d_h Q_{dw} - \frac{1}{2} Q_{wz}$$

$$+ (d_h d_{hd} + \frac{1}{2} d_{hz} - \frac{1}{2} d_{hw}) Q_d$$

$$P_{tt} = d_t^2 Q_{dd} + k_t^2 Q_{kk} + 2d_t k_t Q_{kd}$$

$$+ (d_t d_{td} + k_t d_{tk}) Q_d + (k_t k_{tk} + d_t k_{td}) Q_k$$

Since k is explicitly a function of $g - s$ we have

$$k_s = - k_g \tag{3a}$$

Since d is explicitly a function of $g - s$ and $e + h$ we have

$$d_s = - d_g \qquad d_e = d_h \tag{3b}$$

Using (3a) and (3b) and substituting into (2) for the transformed wave equation we get

$$\frac{1}{4} Q_{yy} + (d_g^2 + d_e^2 - \frac{1}{\tilde{c}^2} d_t^2) Q_{dd} + (k_g^2 - \frac{1}{\tilde{c}^2} k_t^2) Q_{kk}$$

$$+ \frac{1}{4} Q_{zz} + \frac{1}{4} Q_{ww} + 2(k_g d_g - \frac{1}{\tilde{c}^2} d_t k_t) Q_{kd} + d_e Q_{dz} \qquad (4)$$

$$+ (d_g d_{gd} + k_g d_{gk} - \frac{1}{\tilde{c}^2}(d_t d_{td} + k_t d_{tk})) Q_d$$

$$+ (k_g k_{gk} + d_g k_{gd} - \frac{1}{\tilde{c}^2}(k_t k_{tk} + d_t k_{td})) Q_k = 0$$

Equation (4) looks very formidable indeed. However, we shall find that for many of the transformations we might think of using most of the coefficients are identically zero. To get an idea of the relative importance of the terms of equation (4) we will evaluate (4) for some specific coordinate systems.

The Simplest Frame - No Moveout Correction

Let's define k and d by equations (5)

$$k = g - s \quad , \quad d = t + (e+h)/\bar{c}$$

Thus k is just the shot receiver offset and d is a two way travel time. d does not depend on $g-s$ and thus the transformation from d to t does not make any moveout correction.

For the partial derivatives needed in equation (4) we get

$$k_g = 1 \, , \; k_t = k_{gk} = k_{gd} = k_{tk} = k_{td} = 0$$

$$d_t = 1, d_e = \frac{1}{\bar{c}} , d_g = d_{tk} = d_{gk} = d_{gd} = 0 \qquad (6)$$

Substituting (5) and (6) into (4) we get

$$Q_{dz} = - \frac{\bar{c}}{4} Q_{yy} - \bar{c} Q_{kk} - \frac{\bar{c}}{4} Q_{zz} - \frac{\bar{c}}{4} Q_{ww} + \bar{c}(\frac{1}{\bar{c}^2} - \frac{1}{\tilde{c}^2}) Q_{dd} \qquad (7)$$

Using the parabolic approximation to delete Q_{ww} and Q_{zz} and assuming the transformation velocity $\bar{c}$, equals the wave velocity $\tilde{c}$ we have for the migration equation

$$Q_{dz} = -\frac{c}{4} Q_{yy} - c\, Q_{kk} \tag{8}$$

Since the travel time of a reflection from a particular interface depends on shot receiver offset, the transformation from d to t (equation (5)) does not follow the actual behavior of the waves we are attempting to model. Because the deviation of the model increases with offset, we should expect the region of accuracy of the parabolic approximation and hence, equation (8) to be limited to small values of k . In fact (8) is quite accurate for waves propagating at angles up to 15° from the vertical, and thus it is valuable for migrating near trace data. Equation (8) has the desirable attribute of simplicity, however, this simplicity was achieved at the expense of accuracy. Equation (8) can be made more accurate by estimating Q_{zz} , however the severe sampling problems (in k and z) of the type noted in the introduction, remain the limiting factor in the usefulness of the coordinate system.

<u>A Moveout Correction Transformation</u>

Let's define k and d by equations (9).

$$k = g - s \qquad d = t\left(1 - (g - s)^2 / c_n^2\, t^2\right)^{1/2} + (e + h)/\bar{c} \tag{9}$$

Thus k is as before the shot receiver offset. d is still a two way travel time. However, d now depends on offset. Examination of the definition of d shows that, for a flat reflector , the arrival times of the transformed data, Q(z,w,d,y,k) , can be made independent of k .

In other words, the transformation from t to d performs a type of normal moveout correction. Notice that if c_n, the moveout velocity, is very large, little moveout correction is done in transforming from t to d. In the limit of large c_n this transformation reduces to the simple one already discussed.

For the partial derivatives of k we have

$$k_g = 1 \qquad k_t = k_{td} = k_{tk} = k_{gd} = k_{gk} = 0 \tag{10}$$

The derivatives of d are more difficult.

$$d_e = \frac{1}{\bar{c}} \qquad d_t = \left(1 - \frac{(g-s)^2}{c_n^2 t^2} \right)^{-1/2} \tag{11}$$

$$d_g = - \frac{(g-s)}{c_n^2 t} \Big/ \left(1 - \frac{(g-s)^2}{c_n^2 t^2} \right)^{1/2}$$

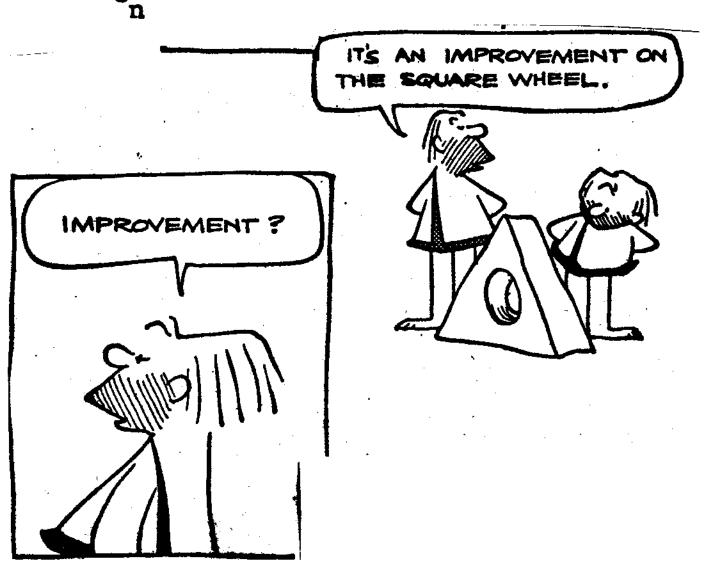
In terms of (z,w,d,y,k) we have

$$d_g = - \frac{k}{c_n^2 (d-2z/\bar{c})} \qquad d_t = \left(1 + \frac{1 + k^2}{c_n^2 (d-2z/\bar{c})^2} \right)^{1/2} \tag{12}$$

For the second partials we get

$$d_{gk} = \frac{1}{c_n^2 \ell} \;, \quad d_{gd} = \frac{k}{c_n^2 \ell^2} \quad d_{td} = - \frac{k^2}{c_n^2 \ell^3} \left(1 + k^2/\bar{c}^2 \ell^2 \right)^{-1/2} \tag{13}$$

where $\ell = (d - 2z/\bar{c})$



Substituting (10), (11), (12), and (13) into the migration equation (4) and then using the parabolic approximation we get

$$Q_{dz} = -\frac{\bar{c}}{4} Q_{yy} - \bar{c}\, Q_{kk} + \frac{2\,\bar{c}\,k}{c_n^2\,\ell}\, Q_{kd} + A\, Q_{dd} + B\, Q_d$$

$$A = \left\{ \left(\frac{1}{\bar{c}^2} - \frac{1}{\tilde{c}^2} \right) + \frac{k^2}{c_n^2 \ell^2} \left(\frac{1}{c_n^2} - \frac{1}{\tilde{c}^2} \right) \right\} \bar{c} \qquad (14)$$

$$B = \bar{c} \left\{ \frac{-k^2}{c_n^4 \ell^3} + \frac{1}{c_n^2 \ell} + \frac{1}{\tilde{c}^2}\, \frac{k^2}{c_n^2 \ell^3} \right\} = \left\{ \frac{+1}{c_n^2 \ell} - \frac{k^2}{c_n^2 \ell^3} \left(\frac{1}{c_n^2} - \frac{1}{\tilde{c}^2} \right) \right\} \bar{c}$$

Equation (14) reduces to (8) if $c_n = \infty$ as it should.

If $\tilde{c} = c_n = \bar{c}$ equation (14) becomes

$$Q_{dz} = -\frac{c}{4} Q_{yy} - c\, Q_{kk} + \frac{2k}{c\ell} Q_{kd} + \frac{1}{c\ell} Q_d \qquad (15)$$

Let's examine the new terms in (15). Notice that the coefficient of Q_{kd} is just $2 \tan(\theta)$ where θ is the propagation angle of a particular ray path.

Consider the equation

$$Q_{dz} = R\, Q_{dk} \qquad R > 0 \qquad (16)$$

where R is independent of d and k. Equation (16) has the solution

$$Q(z,k,d) = Q\left(z + \frac{k}{R},\, d \right) \qquad (17)$$

Thus equation (16) is just a shifter. It shifts Q toward $(-)k$ as it propagates Q in the $(+)\ z$ direction. Similarly the Q_{dk} term can be thought of (approximately) as a shifter which shifts data toward $k = 0$ with a velocity proportional to $\tan\theta$. Thus the Q_{dk} term appears to do the major work in reducing the extent (in the k direction) of properly moveout corrected data during downward continuation. The data must focus around $k = 0$ when it is continued to the depth of the reflectors.

Now let's consider the Q_{dd} term. Assume that there is no dip and that we have done the moveout correction properly. In this case $Q_{yy} = 0$ and $Q_{kk} \approx 0$. Ignoring the Q_{kd} term already discussed (15) becomes

$$Q_{dz} = \frac{1}{c(d-2z/c)} Q_d \qquad (18)$$

Assume either that we are far from the reflector or that we are working at high frequencies, then $\frac{1}{d-2z/c}$ is slowly variable compared to Q and equation (18) becomes

$$Q_z = \frac{1}{c\ell} Q \quad \text{with solution} \quad Q = Q_o \exp\left(\frac{z}{c\ell}\right) \qquad (19)$$

In view of (19) we shall interpret the Q_d term as a geometrical spreading term. If we do a geometrical spreading correction to the data before migration we can neglect the Q_d term. With this in mind (15) becomes

$$Q_{dz} = -\frac{c}{4} Q_{yy} - c\ Q_{kk} + \frac{2k}{c(d-2z/c)} Q_{kd} \qquad (20)$$

In trying to analyze the function of the various terms in (15) we have ignored coupling and examined each term separately. We have also neglected variations in some coefficients. These approximations are probably the cause of such things as the exponential (instead of (distance)$^{-1}$) in equation (19).

Since we have done moveout correction we should expect (20) to be useful at large propagation angles. The price we paid for this improved accuracy was generation of the geometrical spreading term Q_d, and the shifting term Q_{kd}. In addition we have generated a pole at (d-2z/c). This pole occurs because the coordinate system is designed to focus the waves to a point when the shot and receiver are located at the reflector (when $z = \frac{cd}{2}$).

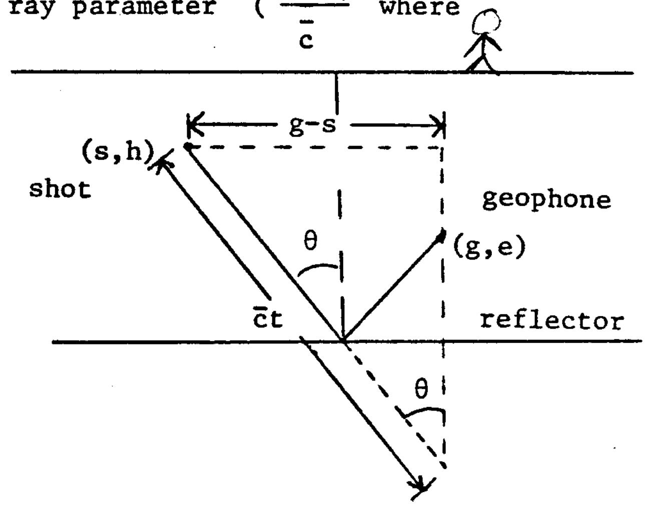
Equation (20) has another undesirable property which limits its usefulness rather markedly. Since this transformation does moveout correction, the data will be a slowly variable (in k) at the earth's surface. Thus we will be able to sample it very coarsely in k at z = 0. However, as z approaches $\frac{cd}{2}$ (as we get near the reflector) energy begins to focus near k = 0. Since the finite difference algorithms require about 8 points per wavelength, for accuracy, the data must be rather densely sampled at z = 0 if we wish to describe the waveforms near the reflector. This sampling problem can be overcome if we devise a transformation for which the grid points get closer together as the reflector is approached. We shall see that a coordinate system expressed in terms of emergence angle has this property.

The Emergence Angle Frame

Let's define k and d by equations (21)

$$k = (g - s)/\bar{c}^2\, t \qquad d = t(1 - (g-s)^2/c^2t^2)^{1/2} + (e+h)/\bar{c} \qquad (21)$$

The definition of d is the same as it was for the previous transformation. Thus, the transformation from t to d performs normal move correction. k is defined as the ray parameter ($\frac{\sin\theta}{\bar{c}}$ where θ is the propagation angle of a ray)



From the figure at the right we see that $\sin\theta = \frac{g-s}{\bar{c}\,t}$

and thus we have

ray parameter $= k = \frac{\sin\theta}{\bar{c}} = \frac{g-s}{\bar{c}^2 t}$

If the data is expressed in terms of ray parameter, the sampling problem due to focusing at the reflector mentioned previously is unimportant. This is because the coordinate system collapses as the distance to the reflector is decreased. To put it another way, it's because energy propagates at the same angle at the earth's surface and at the reflector, so if we grid in terms of angle, sampling remains equally dense at all stages of migration.

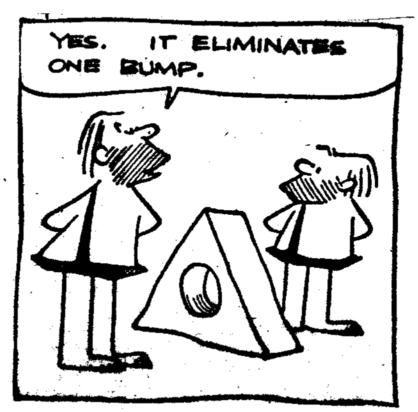
The partial derivatives of k needed for substitution into the general migration equation are

$$k_g = \frac{1}{\bar{c}^2 t} \qquad k_t = -\frac{g-s}{\bar{c}^2 t^2} \tag{22}$$

Rewriting equation (11), for the d derivatives we have

$$d_e = \frac{1}{\bar{c}} \qquad d_t = \left(1 - \frac{(g-s)^2}{\bar{c}^2\, t^2}\right)^{-1/2} \tag{23}$$

$$d_g = \frac{-(g-s)}{\bar{c}^2\, t} \left(1 - \frac{(g-s)^2}{\bar{c}^2 t^2}\right)^{-1/2}$$



In terms of (y,k,z,w,d) we have

$$k_g = \frac{(1-k^2\bar{c}^2)^{1/2}}{\bar{c}^2(d-2z/\bar{c})} \qquad k_t = \frac{-k(1-k^2\bar{c}^2)^{1/2}}{(d-2z/\bar{c})} \qquad d_e = \frac{1}{\bar{c}} \tag{24}$$

$$d_g = -k(1-k^2\bar{c}^2)^{-1/2} \qquad d_t = (1-k^2\bar{c}^2)^{-1/2}$$

For the second partials we have

$$k_{gk} = \frac{-k}{s\,\ell} \qquad k_{gd} = \frac{-s}{\bar{c}^2\ell^2} \qquad k_{td} = \frac{ks}{\ell^2} \qquad k_{tk} = \frac{(1-2k^2\bar{c}^2)}{s\,\ell}$$

$$d_{gk} = -s^{-3} \qquad d_{gd} = 0 \qquad d_{td} = 0 \qquad d_{tk} = +k\bar{c}^2 s^{-3} \tag{25}$$

where $s = (1-k^2\bar{c}^2)^{1/2}$ and $\ell = (d-2z/\bar{c})$

Substituting (24) and (25) into the migration equation (4) , using the parabolic approximation and setting $\bar{c} = \tilde{c}$ we get (see the appendix for some intermediate steps)

$$Q_{dz} = -\frac{c}{4}Q_{yy} - \frac{(1-k^2c^2)^2}{c^3\ell^2}Q_{kk} + \frac{1}{c\ell}Q_d + \frac{2k(1-k^2c^2)}{c\,\ell^2}Q_k \tag{26}$$

Comparing equation (26) to equation (15) we see that by defining k as the ray parameter we have generated a variable coefficient for Q_{kk} and have introduced the Q_k term. Notice that the Q_{kd} term has vanished. This occurs because we are working in emergence angle coordinates and thus the data does not have to collapse toward $k = 0$ as the reflector is approached. As before we can interpret the Q_d term as a geometrical spreading term.

Equation (26) has poles at $z = \frac{cd}{2}$ in the coefficients of Q_k and Q_{kk} . Computationally these poles are not a great problem. This is because the derivatives, Q_k and Q_{kk}, also become small at $z = \frac{cd}{2}$. To see that the solutions to (26) can't have radical behavior at the reflector we need only to remember that we can transform the data back to cartesian coordinates at any receiver depth. Since the solutions are smooth functions in cartesian coordinates they must also be smooth functions in the frame we are discussing. The poles can be a problem if one is trying to use equation (26) as the migration equation in a velocity estimation algorithm. Picking a velocity to use in a migration equation implicitly estimates a depth for all events on the record. If the migration velocity is not correct, Q_{kk} and Q_k, need not be small when the data has been continued to the estimated depth of the reflectors. In this case the effect of the poles in (26) will be large and focusing will be forced to occur no deeper than $z = \frac{cd}{2}$.

This problem can be reduced somewhat by choosing a migration velocity which is always too large. In that case, focusing should occur before the data is continued down to the estimated reflector depth.

Migration in this coordinate system has a disadvantage which may be important when field data is being migrated. Since the coordinate system collapses to a point at the reflector there is no way to describe the focus of data which should focus at a location slightly different than the point to which the coordinate system collapses. Again this problem can be reduced by choosing a large migration velocity so that focusing occurs before the coordinate system collapses.

Because of these problems, coordinate systems which completely collapse at the reflector seem to be sub-optimal if the migration is to be used as part of a velocity analysis algorithm. The type of coordinate system necessary for this application is one which is a hybrid of the emergence angle system and the simple system that was described first. One would like a system which collapses, but not completely, as the data is continued to the reflector depth.

Appendix

Evaluation of the Coefficients for Equation (26)

First we calculate the coefficient of Q_{dd}. Substituting the partials from (24) into the coefficient of Q_{dd} in equation (3) we have

$$d_g^2 + d_e^2 - \frac{1}{\tilde{c}^2} d_t = k^2(1-k^2\bar{c}^2)^{-1} + \frac{1}{\bar{c}^2} - \frac{1}{\tilde{c}^2}(1-k^2\bar{c}^2)^{-1} \qquad \text{(A1)}$$

$$= \frac{(1-k^2\bar{c}^2)}{\bar{c}^2}\left(k^2\bar{c}^2 + 1 - k^2\bar{c}^2 - \frac{\bar{c}^2}{\tilde{c}^2}\right)$$

$$= \frac{(1-k^2\bar{c}^2)^{-1}}{\bar{c}^2}\left(1 - \bar{c}^2/\tilde{c}^2\right)$$

$$= 0 \text{ if } \bar{c} = \tilde{c}$$

Now we will evaluate the coefficient of Q_d. From (25) we have $d_{gd} = d_{td} = 0$. Thus $d_g d_{gd} + k_g d_{gk} - \frac{1}{\tilde{c}^2}(d_t d_{td} + k_t d_{tk}) = k_g d_{gk} - \frac{1}{\tilde{c}^2} k_t d_{tk}$

Substituting from (24) and (25) we have

$$k_g d_{gk} - \frac{1}{\tilde{c}^2} k_t d_{tk} = \frac{(1-k^2\bar{c}^2)^{1/2}}{\bar{c}^2\,\ell} \cdot \frac{-1}{(1-k^2\bar{c}^2)^{3/2}} - \frac{1}{\tilde{c}^2}\,\frac{(-k)(1-k^2\bar{c}^2)^{1/2}}{\ell} \cdot \frac{k\bar{c}^2}{(1-k^2\bar{c}^2)^{3/2}}$$

$$= \frac{-1}{\bar{c}^2\ell(1-k^2\bar{c}^2)} + \frac{1}{\tilde{c}^2}\,\frac{k^2c^2}{\ell(1-k^2\bar{c}^2)} \qquad \text{(A2)}$$

$$= \frac{1}{\bar{c}^2\ell} \quad \text{if } \tilde{c} = \bar{c}$$

Now we will evaluate the Q_k coefficient. The coefficient Q_k in equation (31) is

$$d_g\, k_{gd} + k_g\, k_{gk} - \frac{1}{\tilde{c}^2}(d_t\, k_{td} + k_t\, k_{tk}) = R \qquad \text{(A3)}$$

Substituting from equation (24) and (25) we have

$$- k\, s^{-1} \frac{(-s)}{\bar{c}^2 \ell^2} + \frac{s}{\bar{c}^2 \ell} \frac{(-k)}{s\ell} - \frac{1}{\tilde{c}^2} \left\{ s^{-1} \frac{ks}{\ell^2} + \frac{(-ks)}{\ell} \frac{(1-2k^2\bar{c}^2)}{-s\,\ell} \right\} = R \qquad (A4)$$

where $s = (1-k^2\bar{c}^2)^{1/2}$ $\quad \ell = (d-2z/\bar{c})$

Simplifying (A4) we have

$$+ \frac{k}{\bar{c}^2\ell^2} - \frac{k}{\bar{c}^2\ell^2} - \frac{1}{\tilde{c}^2} \left\{ \frac{k}{\ell^2} + \frac{k}{\ell^2} (1-2k^2\bar{c}^2) \right\} = R$$

$$R = - \frac{2k(1-k^2\bar{c}^2)}{\tilde{c}^2\ell^2} = - \frac{2k(1-k^2c^2)}{c^2\ell^2} \quad \text{for } \tilde{c} = \bar{c} \qquad (A5)$$

Lastly we shall evaluate the coefficient of Q_{kk}.

Substituting to (3) from (24) and (25) we have for the coefficient of Q_{kk}

$$k_g^2 - \frac{1}{\tilde{c}^2} k_t^2 = \frac{(1-k^2\bar{c}^2)}{\bar{c}^4(d-2z/\bar{c})^2} - \frac{1}{\tilde{c}^2} \frac{k^2(1-k^2\bar{c}^2)}{(d-2z/\bar{c})}$$

$$= \frac{(1-k^2\bar{c}^2)}{\bar{c}^2(d-2z/\bar{c})^2} \left(\frac{1}{\bar{c}^2} - \frac{1}{\tilde{c}^2} k^2\bar{c}^2 \right)$$

$$= \frac{(1-k^2c^2)^2}{c^4(d-2z/c)^2} \quad \text{if } \bar{c} = \tilde{c}$$